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Adaptive Control of Wing Rock System in Uncertain Environment Using Contraction Theory

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Abstract— **Contraction theory is used to study stability based on differential and incremental behaviour of trajectories of a system with respect to each other. Application of contraction theory provides a platform to analyze the exponential stability of nonlinear systems. This paper considers the design of a control law for wing rock system using contraction theory principles. An adaptive control design approach based on contraction theory is proposed for controlling the dynamics of such systems for the cases with and without uncertainty in parameters. An adaptive control law along with parameter updation law is derived for uncertain wing rock system to achieve convergence of trajectories of actual system to that of desired system. The analysis made in the paper leads to quite simple results avoiding complexities involved with Lyapunov method. Finally numerical simulations are presented to justify the effectiveness of the proposed controller.**

I. Introduction

Contraction theory is a recent tool for analyzing the convergence behaviour of nonlinear systems in state space form [1]-[3]. It provides a framework to study the exponential stability of nonlinear system trajectories with respect to one another, and therefore belongs to the class of incremental stability methods. Contraction theory is different from the Lyapunov stability based analysis in the sense that it does not require explicit knowledge of a specific attractor [4]-[5]. This theory is having wide range of applications in almost every area as it provides another platform of looking at stability analysis along with widely used Lyapunov approach. Main aspects of contraction theory and its applications can be found in the work cited in [1]-[3],[6]-[9]. Basic results of contraction theory are briefly outlined in the next section.

For the applications where controlled system is too complex and basic physical processes are not fully understood, the adaptive control techniques are used extensively. In the area of aerodynamics, combat aircrafts need to operate at very high speed and at high angle of attack. At sufficiently high angle of attack, these aircrafts becomes unstable due to oscillations, mainly a rolling motion called wing rock [10]-[12]. Wing rock is a highly nonlinear aerodynamic phenomenon in which limit cycle roll oscillations are experienced by aircraft with slender delta wings at high angles of attack. The mechanism of wing rock and related studies can be found in [13]-[14]. However the exact mechanism behind wing rock is still not very clear. Perfect mathematical model of wing rock mechanism in combat aircraft applications is still to be established. Several control strategies are being proposed to tackle this problem and some of them can be found in [15]-[19] and the references there in. In recent years, neural network based techniques have presented an alternative design methodology for identifying and control of dynamic systems [20]-[22]. But feed-forward neural network based techniques need large number of neurons to represent dynamic response of systems in time domain. On the other hand, model free approach based on fuzzy control using linguistic information has been discussed in [23]-[24]. Though fuzzy control has been successfully applied in many applications but due to the lack of formal synthesis techniques that can guarantee the system stability, it has not been viewed as a rigorous technique. An optimal feedback control based design for wing rock is presented in [17]. The results show that an effective way to suppress wing rock is to control the roll rate. Optimal control based on Hamilton-Jacobi-Bellman equation based optimal controller for managing wing rock has been proposed in [25]. However, all these techniques more or less revolve around Lyapunov based stability analysis which ensures asymptotic stability.

In present paper, an adaptive control design approach is proposed for controlling the dynamics of wing rock in combat aircrafts using contraction theory principle. Contraction theory approach offers several significant advantages while analyzing convergence properties of nonlinear systems. In general, nonlinear systems with uncertain parameters could prove quite troublesome for standard Lyapunov methods since the uncertainty can change the equilibrium point of the system in very complicated ways, thus forcing the use of parameter dependent Lyapunov functions in order to prove stability for such systems. However contraction theory framework eliminates many of the restrictions of traditional analysis method while analyzing nonlinear systems. It eliminates the need to know the equilibrium point as it works on incremental analysis of neighbouring trajectories. Also this theory doesn't require selection of a suitable energy function to be a Lyapunov like function for stability analysis. The present paper proposes contraction based control strategy to handle wing rock in environment of uncertainty. Convergence of the system states along with suppression of limit cycle behavior is achieved in presence of uncertainty in system parameters. Explicit updating laws are derived analytically for various uncertain parameters of wing rock system using backstepping like procedure based on contraction principle. The Backstepping based control technique is a recursive procedure that links the choice of a Lyapunov function with

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the design of a controller. This technique yields a wide family of globally asymptotically stabilizing laws, which allows addressing issues of adaptive and robust control [27]-[30]. However, in this paper, we consider different methodology for stability analysis in each step of the backstepping method. As a result, a new adaptive control law is emerged. We make principle of contraction theory as a base to derive the control function instead of Lyapunov technique. The simulation results presented in the end show the effectiveness of the proposed nonlinear controller along with adaptation laws for controlling such behavior.

The paper is outlined as follows: In Section II, basic results of contraction theory are presented. Section III gives basic formulation of wing rock problem. Section IV develops the idea of controller design using backstepping with all parameters of wing rock system known in advance. Section V addresses the case of controller design for the same system in uncertain environment. Here again backstepping procedure along with contraction principle is used to decide the suitable controller and the update laws for uncertain parameters of the wing rock system. Section VI demonstrates the numerical simulations to show the effectiveness of the proposed nonlinear controller along with adaptation laws for controlling the behavior of wing rock dynamic system. Section VII presents the conclusion of the paper.

II. Basics of Contraction Theory

Contraction is a property regarding the convergence between two arbitrary system trajectories. A nonlinear dynamic system is called contracting if initial conditions or temporary disturbances are forgotten exponentially fast i.e., if trajectories of the perturbed system return to their nominal behavior with an exponential convergence rate. Consider a nonlinear system having following description:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{1}$$

where $\{\mathbf{x} \in R^{m\times 1}\}$ is a state vector of the system and $\mathbf{f}$ is an $(m \times 1)$ vector function. Function $\mathbf{f}(\mathbf{x}, t)$ is considered to be a continuously differentiable function. Let $\delta\mathbf{x}$ is the virtual displacement in the state $\mathbf{x}$, which is infinitesimal displacement at fixed time. Introducing the concept of virtual dynamics, first variation of system in (1) will be

$$\delta\dot{\mathbf{x}} = \frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial \mathbf{x}}\delta\mathbf{x} \tag{2}$$

From this equation, we can further write:

$$\frac{d}{dt}\left(\delta\mathbf{x}^T\delta\mathbf{x}\right) = 2\delta\mathbf{x}^T\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\delta\mathbf{x} \le 2\lambda_m(\mathbf{x}, t)\delta\mathbf{x}^T\delta\mathbf{x} \tag{3}$$

Here in above equation, the Jacobian matrix is denoted as $J = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ and the largest eigen value of the symmetric part of Jacobian is represented by $\lambda_m(\mathbf{x}, t)$. If this eigen value $\lambda_m(\mathbf{x}, t)$ is strictly uniformly negative, then any infinitesimal length $\|\delta\mathbf{x}\|$ converges exponentially to zero. Here $\left(\delta\mathbf{x}^T\delta\mathbf{x}\right)$ represents the squared distance between the neighbouring trajectories. By carrying out path integration in (3), it is assured that all the solution trajectories of the system in (1) converge exponentially to single trajectory, independently of the initial conditions.

Definition 1: Given the system equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, a region (open connected space) of state space is called a contracting region if the Jacobian $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ is uniformly negative definite (U.N.D.) in that region.

Definition 2: Uniformly negative definiteness (UND) of Jacobian $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ means that there exists a scalar $\alpha > 0$, $\forall\mathbf{x}$, $\forall t \ge 0$, $\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \le -\alpha I < 0$; or $\frac{1}{2}\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \frac{\partial \mathbf{f}^T}{\partial \mathbf{x}}\right) \le -\alpha I < 0$ because all matrix inequalities will refer to symmetric part of the square matrix involved.

Considering the above definitions, the basic results (without proof) related to exponential convergence of the trajectories can be stated as follows [1]-[3]:

Lemma 1: Given the system equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, any trajectory which starts in a ball of constant radius centered about a given trajectory and contained at all times in a contraction region, remains in that ball and converges exponentially to the given trajectory. Further, global exponential convergence to this given trajectory is guaranteed if the whole state space region is contracting.

The results stated above can also be represented in more general way by using a coordinate transformation

$$\delta\mathbf{z} = \theta\delta\mathbf{x} \tag{4}$$

where $\theta(\mathbf{x}, t)$ is a uniformly invertible matrix. The corresponding results in transformed domain are omitted here and can be found in [1]-[3]. For some systems having representation given in (1), the Jacobian matrix $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ may turn out to be negative semi-definite. By extending the definition (1), such systems are called semi-contracting systems. For such systems asymptotic stability can be ensured using the contraction theory results. Following lemma is defined for analyzing asymptotic stability of semi-contracting systems:

Lemma 2: For the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, let the stable reference system is given by $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t)$. Defining $\mathbf{e} = \mathbf{y} - \mathbf{x}$, for the error system $\dot{\mathbf{e}} = \mathbf{f}_1(\mathbf{e}, \mathbf{x}, t)$, if the Jacobian matrix $J = \frac{\partial \mathbf{f}_1}{\partial \mathbf{e}}$ is uniformly negative semi-definite i.e. in terms of virtual displacement in differential framework, if

$$\begin{bmatrix} \delta\dot{\mathbf{e}}_1 \\ .... \\ \delta\dot{\mathbf{e}}_2 \end{bmatrix} = \begin{bmatrix} J_{11} & : & G(\mathbf{x}, t) \\ .... & .... & .... \\ -G^T(\mathbf{x}, t) & : & \mathbf{0} \end{bmatrix} \begin{bmatrix} \delta\mathbf{e}_1 \\ .... \\ \delta\mathbf{e}_2 \end{bmatrix} \tag{5}$$

where submatrix J_{11} is uniformly negative definite, then system is considered to be semi-contracting. For such system asymptotic stability can be guaranteed.

Proof: As system in (5) is semi-contracting by nature, hence $\delta\mathbf{e}$ is bounded. It leads to the conclusion that any distance between any couple of trajectories is also bounded. It means that $\|\mathbf{y} - \mathbf{x}\|$ is also bounded. As $\mathbf{y}$ is bounded, $\mathbf{x}$ will also be bounded. Here $J_{11} = \frac{\partial \mathbf{f}_1(\mathbf{e}, \mathbf{x}, t)}{\partial \mathbf{e}_1}$ represents the uniformly negative definite submatrix i.e. $\mathbf{f}$ is contracting w. r. t. the vector $\mathbf{e}_1$. Assuming $\mathbf{f}_1$ function which involves $G(\mathbf{x}, t)$ to be smooth, all the quantities in (5) are bounded. So norm of time derivative in (5) is also bounded as all its variables are bounded. Invoking Barbalat's lemma given in [30] implies asymptotic convergence of $\mathbf{e}_1$ to zero. So correponding $\mathbf{x}$

converges to $\mathbf{y}$ asymptotically. Though nothing can be said about the convergence of rest of the states. $\diamond$

Contraction theory results are also extended to various combinations of systems.

Feedback Combination: Consider that two systems possibly of different dimensions are having following dynamics:

$$\begin{aligned} \dot{\mathbf{x}}_1 &= \mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2, t) \\ \dot{\mathbf{x}}_2 &= \mathbf{f}_2(\mathbf{x}_1, \mathbf{x}_2, t) \end{aligned} \quad (6)$$

Let these systems are connected in feedback combination. Then by using the transformation given in (4), we can write virtual displacements in transformed domain as

$$\frac{d}{dt}\begin{bmatrix} \delta\mathbf{z}_1 \\ \delta\mathbf{z}_2 \end{bmatrix} = \begin{bmatrix} F_1 & G \\ -G^T & F_2 \end{bmatrix}\begin{bmatrix} \delta\mathbf{z}_1 \\ \delta\mathbf{z}_2 \end{bmatrix} \quad (7)$$

where coordinate transformation is given as $\delta\mathbf{z} = \theta\delta\mathbf{x}$. Then the augmented system is contracting if and only if the separated plants are contracting. This can be shown easily as symmetric part of the Jacobian turns out to be UND.

Hierarchical Combination: Consider a smooth virtual dynamics of the form

$$\frac{d}{dt}\begin{bmatrix} \delta\mathbf{z}_1 \\ \delta\mathbf{z}_2 \end{bmatrix} = \begin{bmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{bmatrix}\begin{bmatrix} \delta\mathbf{z}_1 \\ \delta\mathbf{z}_2 \end{bmatrix} \quad (8)$$

and assume that F_{21} is bounded. The first equation does not depend on the second, so exponential convergence of $\delta\mathbf{z}_1$ to zero can be concluded for UND F_{11}. In turn, $F_{21}\delta\mathbf{z}_1$ represents an exponentially decaying disturbance in second equation. A UND F_{22} implies the exponential convergence of $\delta\mathbf{z}_2$ to an exponentially decaying ball. Thus, the whole system globally exponentially converges to a single trajectory.

Other aspects of contraction theory and its applications can be found in the work cited in [1],[3]-[6],[8]-[9].

III. Problem Formulation

The approximate dynamic model of the wing rock system as proposed by Nayfeh et al [12] can be written as follows:

$$\ddot{\phi} + \omega^2 = \mu_1\dot{\phi} + b_1\dot{\phi}^3 + \mu_2\phi^2\dot{\phi} + b_2\phi\dot{\phi}^2 + u \quad (9)$$

Here $\ddot{\phi}$, $\dot{\phi}$ and ϕ represents roll acceleration, roll velocity and roll angle respectively. The various coefficients in the equation are dependent on the geometrical constants c_1, c_2 and the variables a_1 to a_5. These variables vary with the angle of attack (AOA). The coefficients of the system μ_1, μ_2, b_1, b_2 and ω^2 can be represented in terms of the geometrical constants and the variables as follows:

$$\begin{aligned} \mu_1 &= c_1a_2 - c_2;\ \mu_2 = c_1a_4;\ b_1 = c_1a_3 \\ b_2 &= c_1a_5;\ \omega^2 = -c_1a_1 \end{aligned} \quad (10)$$

The value of fixed geometrical constants c_1 and c_2 is taken as $c_1 = 0.354$; $c_2 = 0.001$, and the value of variable parameters a_1 to a_5 corresponding to a particular angle of attack are taken from table 1 as given in [26]. By choosing

AOA	a_1	a_2	a_3	a_4	a_5
15^o	-0.01026	-0.02117	-0.14181	0.99735	-0.83478
21.5^o	-0.04207	-0.01456	0.04714	-0.18583	0.24234
22.5^o	-0.04681	0.01966	0.05671	-0.22691	0.59065
25^o	-0.05686	0.03254	0.07334	-0.35970	1.46810

TABLE I

Parameters of wing rock system at different AOA

state variables $x_1 = \phi$ and $x_2 = \dot{\phi}$, state model for wing rock system will be

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\omega_1x_1 + \mu_1x_2 + b_1x_2^3 \\ &\quad +\mu_2x_1^2x_2 + b_2x_1x_2^2 + u + r \end{aligned} \quad (11)$$

For simplicity of notation, it has been assumed that $\omega_1 = \omega^2$. Here r is taken as the additional control input in (11) to meet out the tracking performance. Let us take 2nd order stable reference model as

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= -a_my_1 - b_my_2 + r \end{aligned} \quad (12)$$

By defining the errors in actual state and reference state as $e_i = x_i - y_i,\ i = 1, 2$; the error dynamics will be

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= -a_me_1 - b_me_2 + (a_m - \omega_1)x_1 + (b_m + \mu_1)x_2 \\ &\quad +b_1x_2^3 + \mu_2x_1^2x_2 + b_2x_1x_2^2 + u \end{aligned} \quad (13)$$

To ascertain the stability of the system by designing suitable controller, backstepping technique is used [28]-[29].

Problem Statement: *To design adaptive backstepping based controller for the wing rock system with error dynamics given in (13) so that error convergence is achieved i.e. actual system described in (11) tracks the reference system (12) with and without uncertainty in parameters of the system.*

IV. Controller Design for the System Without Uncertainty

To obtain the control law for the wing rock system, few theorems are defined for considering the case of controller design with and without parametric uncertainty. In each step of backstepping method, we adopt the results of contraction theory to ensure the stability.

Theorem 1: For wing rock system without parametric uncertainty, the convergence of error dynamics (13) i.e. the complete tracking of reference system (12) by actual system (11) is achieved if controller is designed as per following function:

$$\begin{aligned} u &= (a_m - 2)e_1 + (b_m - 2)e_2 - (a_m - \omega_1)x_1 - \\ &\quad (b_m + \mu_1)x_2 - b_1x_2^3 - \mu_2x_1^2x_2 - b_2x_1x_2^2 \end{aligned} \quad (14)$$

Proof: For the system in (13), first subsystem is defined as

$$\dot{e}_1 = e_2 \quad (15)$$

To make it contracting, the virtual control input e_{2d} is to be selected accordingly. The virtual control is designed so as to make the dynamics of first subsystem contracting w. r. t. error variable e_1 i.e. its Jacobian w. r. t. e_1 should be UND. Let this virtual control be $e_{2d} = -e_1$. Defining a new variable $z_1 = e_2 - e_{2d} = e_1 + e_2$, the dynamics of this subsystem becomes

$$\dot{e}_1 = -e_1 + z_1 \tag{16}$$

This system will be contracting if variable z_1 is bounded. Taking derivative of z_1 and using (13) and (16), we get

$$\begin{aligned}\dot{z}_1 &= -e_1 - z_1 + (2 - b_m)e_2 + (2 - a_m)e_1 \\ &\quad +(a_m - \omega_1)x_1 + (b_m + \mu_1)x_2 + b_1 x_2^3 \\ &\quad +\mu_2 x_1^2 x_2 + b_2 x_1 x_2^2 + u \end{aligned}\tag{17}$$

Now it is required to select control input u suitably so as to make the system contracting in nature. Let the control input be selected as per (14). Then, above equation becomes

$$\dot{z}_1 = -e_1 - z_1 \tag{18}$$

So overall transformed system can be represented as

$$\begin{aligned}\dot{e}_1 &= -e_1 + z_1 \\ \dot{z}_1 &= -e_1 - z_1\end{aligned}\tag{19}$$

In general the system in (19) can be written as $\dot{\mathbf{w}} = \mathbf{f}(\mathbf{w}, t)$ where vector $\mathbf{w} = [e_1 \ z_1]^T$. Defining the virtual displacement for this system by $\delta\mathbf{w}$, we get the following;

$$\delta\dot{\mathbf{w}} = \frac{\partial \mathbf{f}(\mathbf{w}, t)}{\partial \mathbf{w}}\delta\mathbf{w} \tag{20}$$

For transformed system, the Jacobian matrix J is defined as

$$J = \frac{\partial \mathbf{f}}{\partial \mathbf{w}} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \tag{21}$$

which is UND, so the error dynamics of the system is contracting as per the results related to feedback combination of systems stated earlier in section II for a contracting system. Hence the trajectories of the actual wing rock system converge to the desired system trajectories. So by contraction theory, exponential convergence is achieved because e_1 and z_1 both converge to zero exponentially as time $t \to \infty$. As $z_1 = e_1 + e_2$ and e_1 as well as z_1 approaches to zero exponentially with $t \to \infty$, so $e_2 \to 0$ exponentially as well. Hence the overall error system becomes contracting.

V. Controller Design for the System With Uncertainty

In actual wing rock dynamical system, parameters involved may have uncertainty. Controller and adaptation laws for different uncertain parameters are derived in the following discussion. Let the parameters of actual system μ_1, μ_2, b_1, b_2 and ω_1 are uncertain and their estimates are represented by $\widehat{\mu}_1, \widehat{\mu}_2, \widehat{b}_1, \widehat{b}_2$ and $\widehat{\omega}_1$, respectively. Defining the error between true parameter and its estimated value as

$$\begin{aligned}\widehat{\mu}_1 - \mu_1 &= \tilde{\mu}_1; \ \widehat{\mu}_2 - \mu_2 = \tilde{\mu}_2; \ \widehat{b}_1 - b_1 = \tilde{b}_1 \\ \widehat{b}_2 - b_2 &= \tilde{b}_2; \ \widehat{\omega}_1 - \omega_1 = \tilde{\omega}_1\end{aligned}\tag{22}$$

The actual state model of wing rock system in uncertain environment can be represented as

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(\widehat{\omega}_1 - \tilde{\omega}_1)x_1 + (\widehat{\mu}_1 - \tilde{\mu}_1)x_2 + (\widehat{b}_1 - \tilde{b}_1)x_2^3 \\ &\quad +(\widehat{\mu}_2 - \tilde{\mu}_2)x_1^2 x_2 + (\widehat{b}_2 - \tilde{b}_2)x_1 x_2^2 + u + r\end{aligned}\tag{23}$$

After simplification, and using the steps proposed in earlier case, the following error dynamics is obtained:

$$\begin{aligned}\dot{e}_1 &= e_2 \\ \dot{e}_2 &= -a_m e_1 - b_m e_2 + (a_m - \widehat{\omega}_1)x_1 + (b_m + \widehat{\mu}_1)x_2 \\ &\quad +\widehat{b}_1 x_2^3 + \widehat{\mu}_2 x_1^2 x_2 + \widehat{b}_2 x_1 x_2^2 + u + \tilde{\omega}_1 x_1 - \tilde{\mu}_1 x_2 \\ &\quad -\tilde{b}_1 x_2^3 - \tilde{\mu}_2 x_1^2 x_2 - \tilde{b}_2 x_1 x_2^2\end{aligned}\tag{24}$$

The problem of deriving the suitable control is stated in the form of following theorem.

Theorem 2: For wing rock system with parametric uncertainty, the convergence of error dynamics (24) i.e. complete tracking of reference system (12) by actual system (23) is achieved if controller is designed as

$$\begin{aligned}u &= (a_m - 2)e_1 + (b_m - 2)e_2 - (a_m - \widehat{\omega}_1)x_1 - \\ &\quad (b_m + \widehat{\mu}_1)x_2 - \widehat{b}_1 x_2^3 - \widehat{\mu}_2 x_1^2 x_2 - \widehat{b}_2 x_1 x_2^2\end{aligned}\tag{25}$$

along with adaptation laws for uncertain parameters as

$$\begin{aligned}\dot{\widehat{\omega}}_1 &= -x_1(e_1 + e_2); \ \ \dot{\widehat{\mu}}_1 = x_2(e_1 + e_2) \\ \dot{\widehat{b}}_1 &= x_2^3(e_1 + e_2); \ \ \dot{\widehat{\mu}}_2 = x_1^2 x_2(e_1 + e_2) \\ \dot{\widehat{b}}_2 &= x_1 x_2^2(e_1 + e_2)\end{aligned}\tag{26}$$

Proof: For the system in (24), first subsystem is defined as

$$\dot{e}_1 = e_2 \tag{27}$$

Selecting virtual control input $e_{2d}(= -e_1)$ and defining a new auxiliary variable z_1 as in last section, we get

$$\dot{e}_1 = -e_1 + z_1 \tag{28}$$

The time derivative of z_1 is simplified as

$$\begin{aligned}\dot{z}_1 &= -e_1 - z_1 + (2 - a_m)e_1 + (2 - b_m)e_2 \\ &\quad +(a_m - \widehat{\omega}_1)x_1 + (b_m + \widehat{\mu}_1)x_2 + \widehat{b}_1 x_2^3 + \widehat{\mu}_2 x_1^2 x_2 \\ &\quad +\widehat{b}_2 x_1 x_2^2 + u + \tilde{\omega}_1 x_1 - \tilde{\mu}_1 x_2 - \tilde{b}_1 x_2^3 - \tilde{\mu}_2 x_1^2 x_2 \\ &\quad -\tilde{b}_2 x_1 x_2^2\end{aligned}\tag{29}$$

To make the system contracting, structure of control law[1] is selected as in (25). Then, the above equation becomes

$$\begin{aligned}\dot{z}_1 &= -e_1 - z_1 + \tilde{\omega}_1 x_1 - \tilde{\mu}_1 x_2 - \tilde{b}_1 x_2^3 \\ &\quad -\tilde{\mu}_2 x_1^2 x_2 - \tilde{b}_2 x_1 x_2^2\end{aligned}\tag{30}$$

So overall dynamics of transformed system will be

$$\begin{aligned}\dot{e}_1 &= -e_1 + z_1 \\ \dot{z}_1 &= -e_1 - z_1 + \tilde{\omega}_1 x_1 - \tilde{\mu}_1 x_2 - \tilde{b}_1 x_2^3 \\ &\quad -\tilde{\mu}_2 x_1^2 x_2 - \tilde{b}_2 x_1 x_2^2\end{aligned}\tag{31}$$

[1]The structure of control law is again similar as desired in (14) except that the unknown parameters are replaced by their estimated values.

In general, this system can be written in compact form as

$$\dot{\mathbf{w}} = \mathbf{f}(\mathbf{w},t) + \mathbf{Q}(\mathbf{x},t)\tilde{\mathbf{p}} \tag{32}$$

where vector $\mathbf{w} = [e_1 \ z_1]^T$, $\mathbf{Q}(\mathbf{x},t)$ is a regression vector with bounded $\mathbf{x}$ and $\tilde{\mathbf{p}} = [\tilde{\omega}_1 \ \tilde{\mu}_1 \ \tilde{b}_1 \ \tilde{\mu}_2 \ \tilde{b}_2]^T$ represents parametric error vector. The regression matrix $\mathbf{Q}(\mathbf{x},t)$ is represented as

$$\mathbf{Q}(\mathbf{x},t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ x_1 & -x_2 & -x_2^3 & -x_1^2 x_2 & -x_1 x_2^2 \end{bmatrix} \tag{33}$$

The system in (32) can be written in matrix form as

$$\begin{bmatrix} \dot{e}_1 \\ \dot{z}_1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} e_1 \\ z_1 \end{bmatrix} + \mathbf{Q}(\mathbf{x},t) \begin{bmatrix} \tilde{\omega}_1 \\ \tilde{\mu}_1 \\ \tilde{b}_1 \\ \tilde{\mu}_2 \\ \tilde{b}_2 \end{bmatrix} \tag{34}$$

Selecting the adaptation laws for parametric error as

$$\dot{\tilde{\mathbf{p}}} = \dot{\hat{\mathbf{p}}} = -\mathbf{Q}^{\mathbf{T}}(\mathbf{x},t)\mathbf{w} \tag{35}$$

The above expression gives adaptation laws for different parameters as in (26). So in compact form, the transformed system in (34) and (35) can be written as

$$\begin{aligned} \dot{\mathbf{w}} &= \mathbf{f}(\mathbf{w},t) + \mathbf{Q}(\mathbf{x},t)\tilde{\mathbf{p}} \\ \dot{\hat{\mathbf{p}}} &= -\mathbf{Q}^{\mathbf{T}}(\mathbf{x},t)\mathbf{w} \end{aligned} \tag{36}$$

Defining the virtual displacement for this system by $\delta\mathbf{w}$ and $\delta\tilde{\mathbf{p}}$, the above system can be represented in differential framework as

$$\begin{bmatrix} \delta\dot{\mathbf{w}} \\ \delta\dot{\hat{\mathbf{p}}} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{f}(\mathbf{w},t)}{\partial \mathbf{w}} & \mathbf{Q}(\mathbf{x},t) \\ -\mathbf{Q}^{\mathbf{T}}(\mathbf{x},t) & 0 \end{bmatrix} \begin{bmatrix} \delta\mathbf{w} \\ \delta\tilde{\mathbf{p}} \end{bmatrix} \tag{37}$$

So complete system in compact form can be represented as

$$\delta\dot{\mathbf{v}} = \frac{\partial \mathbf{f}_1(\mathbf{v},\mathbf{x},t)}{\partial \mathbf{v}}\delta\mathbf{v} \tag{38}$$

where $\mathbf{v} = [\mathbf{w} \ \ \hat{\mathbf{p}}]^T$ is a vector involving transformed variables and parametric error vector. Here Jacobian matrix $J = \frac{\partial \mathbf{f}_1(\mathbf{v},\mathbf{x},t)}{\partial \mathbf{v}}$ can be represented as

$$\mathbf{J} = \begin{bmatrix} \frac{\partial \mathbf{f}(\mathbf{w},t)}{\partial \mathbf{w}} & \mathbf{Q}(\mathbf{x},t) \\ -\mathbf{Q}^{\mathbf{T}}(\mathbf{x},t) & 0 \end{bmatrix} \tag{39}$$

So using lemma 2, virtual dynamics of the system is semi-contracting because $J_{11} = \frac{\partial \mathbf{f}(\mathbf{w},t)}{\partial \mathbf{w}}$ is UND. As above system is semi-contracting, so $\delta\mathbf{v} = [\delta\mathbf{w} \ \delta\hat{\mathbf{p}}]$ are bounded. So any distance between any couple of trajectories is bounded. It implies that $\|x_1 - y_1\|$, $\|(x_1 - y_1) + (x_2 - y_2)\|$ and $\|\hat{\mathbf{p}} - \mathbf{p}\|$ are bounded. As reference system states y_i, for $i = 1, 2$ are bounded and the parameters $\mathbf{p}$ are constant, so consequently x_i for $i = 1, 2$ and $\hat{\mathbf{p}}$ are bounded. Assuming functions $\mathbf{f}$ and **Q**(**x**,t) to be smooth, all quantities in virtual dynamics are bounded. As system is semi-contracting, so the norm of the time derivative in (37) is also bounded as all variables involved are bounded. So using the Barbalat's lemma [30], asymptotic convergence of $\delta\mathbf{w}$ to zero is assured. So system states x_i converges asymptotically to the reference system states y_i,for $i = 1, 2$. Although nothing can be said about the convergence of $\hat{\mathbf{p}}$ to $\mathbf{p}$.

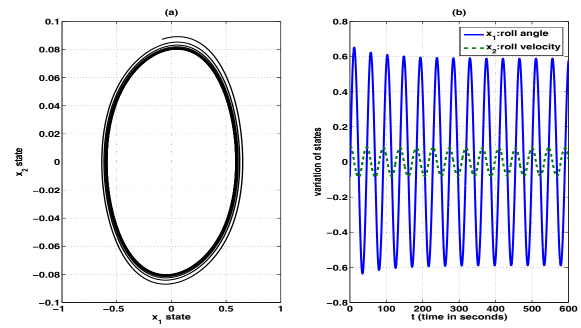

Fig. 1. (a) Phase portrait of uncontrolled wing rock system; (b) Variation of state trajectories.

VI. Numerical Simulations

Numerical simulations are performed using *ode45* MATLAB function with step size 0.01 second. Initial conditions for roll angle and roll velocity of actual system are taken as $\mathbf{x_0} = (-5 \ \ 5)^T$. The chaotic behaviour of uncontrolled wing rock system is shown in fig. 1. In 2nd part of simulation, all the parameters of wing rock system are considered to be known and are selected corresponding to 25-degree angle of attack. For reference model initial conditions are taken as $\mathbf{y_0} = (-5 \quad 0)^T$. For simulation purpose the common input to actual system and reference system is taken as $r = e^{-0.2t}sin(t)$. Various plots for this case are shown in fig.2. In 3rd part of simulation, parameters of the system are assumed to be uncertain. Again initial conditions for states of actual and reference system are taken as that of previous case. The adaptation parameters are initialized as $\mathbf{p_0} = [0 \ 0 \ 0 \ 0 \ 0]^T$. Fig. 3 depicts the various plots for the case with parametric uncertainty.

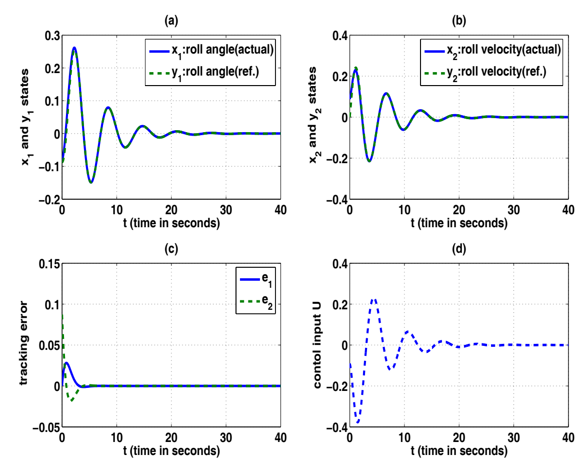

Fig. 2. Response of wing rock system with controller(known parameter case): (a), (b) Comparison of state trajectories of actual system and reference system; (c) Trajectories showing error variation between states and (d) Control input variation with time.

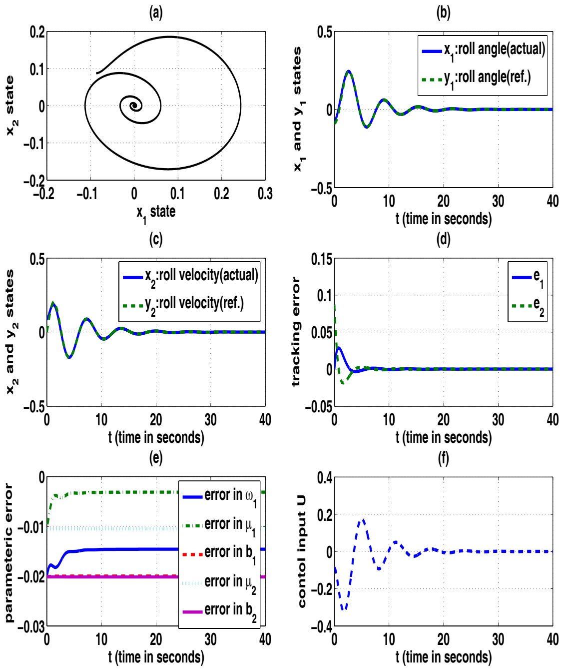

Fig. 3. Response of wing rock system with controller(unknown parameter case): (a) Phase portrait of wing rock system; (b), (c) Comparison of state trajectories; (d) Trajectories showing error variation; (e) Variation of parametric estimation error and (f) Control input variation with time.

VII. Conclusion

In this paper an adaptive backstepping technique is proposed to control wing rock motion system. The adaptive control law is obtained for the case with and without uncertainty in parameters. Step by step design of controller is carried out using contraction theory. Adaptation laws for parameters are also proposed along with the control law. The proposed controller ensures convergence of system states to trajectories of desired system; hence suppression of limit cycle is achieved in presence of uncertainty in parameters. Moreover, use of contraction theory concepts avoid the difficulties associated with Lyapunov approach. The simulation results are shown to describe effectiveness of the proposed approach in controlling the behavior of wing rock dynamic system.

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