2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Optimal Resource Allocation in Conflicts with the Lanchester Linear Law (2,1) Model

P.S. Sheeba and D. Ghose

***Abstract*— This paper presents a detailed analysis of a model for military conflicts where the defending forces have to determine an optimal partitioning of available resources to counter attacks from an adversary in two different fronts in an area fire situation. Lanchester linear law attrition model is used to develop the dynamical equations governing the variation in force strength. Here we address a static resource allocation problem namely, Time-Zero-Allocation (TZA) where the resource allocation is done only at the initial time. Numerical examples are given to support the analytical results.**

I. INTRODUCTION

This paper addresses an optimal resource allocation problem based on Lanchester linear model of attrition. The Lanchester linear law models an unaimed fire situation in which a shooter fires upon an area in which the enemy force is assumed to be, as in the case of mass anti-aircraft barrages, artillery bombardment of enemy positions, etc. Here the fire power remains directed to the whole area [1]. Lanchester models are deterministic differential equations that model attrition to forces in conflict. Lanchester models have been widely used to analyze real wars [2] and determine tactics for deploying forces in war game simulations as they produce reasonably good predictions. Alternative approaches to Lanchester models have been proposed in the literature but till date the Lanchester model still remains a popular paradigm for modeling attrition in conflicts involving military forces. In the modern scenario, with significant advances in technology related to communication and computation, sophisticated decision-making in these situations has become feasible. This has generated renewed interest in formulating decision-making problems in these areas and seeking optimal solutions. Our paper addresses one such problem in which the defending forces need to optimally partition their resources between two attacking forces of differing strengths.

The basic model used in this paper is the Lanchester (2,1) linear model introduced in [3]. The (2,1) model represents a battle between an attacker with two weapon types and a defender with one weapon type. The attacker's weapon types causes attrition to the defenders weapon type at two different attrition rates, each of which is attrited by the defender's weapon at different rates. A resource partitioning problem for the Lanchester square law (n,1) model is developed in [4]. In [5], Taylor analyzed the victor's optimal initial commitment decision as an one-sided static optimization problem for three decision criteria (victor's losses, loss ratio, and loss difference) and for two battle termination conditions. Roberts and Conolly [6] considered an optimization problem of minimizing the attackers's initial resources using the Lanchester square law model. However, they do not address the optimal partitioning of resources by the defending force. Kaup et al. [7] considered a problem in which a heterogeneous force of n different troop types is in conflict with a homogeneous force. They address some limited cases of optimizing the resource partitioning problem based on square law model. Another related paper is by Colegrave and Hyde [8] where a Lanchester (2,2) model, where 2 forces are in conflict with 2 other forces, is analyzed. In [3], we presented only the preliminary results for the Lanchester (2,1) model. Here we present the detailed analysis of the model developed in [3]. In this paper we address a static resource allocation problem where the allocation is done only at the initial time (Time Zero Allocation (TZA)). For the linear law the resources get destroyed completely only at infinite time, hence a situation for redistribution of resources does not arise for this law.

The authors would like to acknowledge the financial support received under the IISc-DRDO Mathematical Engineering Programme.

P.S. Sheeba (Research associate) and D. Ghose (Professor) are with the Department of Aerospace Engineering, Indian Institute of Science, Bangalore 560 012, India sheeba+dghose@aero.iisc.ernet.in

II. LANCHESTER LINEAR LAW ATTRITION MODEL

Consider a military conflict between two opposing forces. Let Y denote the defending force and X denote the attacking force. It is assumed that the defending force consists of only one type of force and the attacking force consists of two types of forces. Let y denote the strength of the defending force and x_1 and x_2 denote the strength of each type of attacking force. Let the initial values of y, x_1, x_2 be N, M_1 and M_2, respectively. The initial strength y is partitioned into two parts, ηN and $(1-\eta)N$ so that ηN interacts with x_1 and $(1-\eta)N$ interacts with x_2 (Figure 1). This paper deals with the problem of optimally choosing η to maximize some objective of the defending forces. Since this is a decision making problem for the defending force Y, we select an objective of maximizing a weighted mean of the surviving resource strength of the Y force and the annihilated resource strength of the X force. Hence, the objective function is defined as,

$$\begin{aligned} J &= \gamma \times \big[\text{Surviving resources of } Y\big] \\ &+ (1-\gamma) \times \big[\text{Destroyed resources of } X_1 \\ &+ \text{Destroyed resources of } X_2\big] \end{aligned} \tag{1}$$

where, $\gamma \in [0,1]$.

The classical *Lanchester Linear Law* is given by,

$$\dot{x}(t) = -\alpha x(t)y(t), \ \dot{y}(t) = -\beta x(t)y(t) \tag{2}$$

978-1-4244-2079-7/08/$25.00 ©2008 AACC.

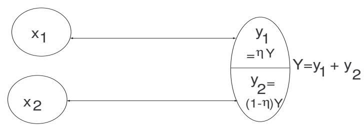


Fig. 1. *Time-Zero-Allocation (TZA)*

where, $x(t)$ and $y(t)$ are the strengths of the two opposing forces surviving at time t in conflict, and α and β are the attrition constants independent of time.

III. Time-Zero-Allocation (TZA)

In this case, the initial strength $y(0)$ of Y is partitioned into $y_1(0)$ and $y_2(0)$ using the decision parameter η. No redistribution of resources takes place when any of the resources is completely destroyed. Thus, allocation is done at the initial time (that is, at time zero) only.

The attrition equations are given by,

$$\dot{x}_i = -\alpha_i x_i y_i, \quad x_i(0) = M_i \tag{3}$$
$$\dot{y}_i = -\beta_i x_i y_i, \quad y_i(0) = \eta_i N, \quad i = 1, 2. \tag{4}$$

where, $\eta_1 = \eta$, $\eta_2 = 1 - \eta$; Also, $\eta \in [0, 1]$ and $(\alpha_i, \beta_i > 0,\ i = 1, 2)$. From (4), after some standard manipulations, we get,

$$\dot{x}_i = -\beta_i x_i^2 + k_i x_i, \quad \dot{y}_i = -\alpha_i y_i^2 - k_i y_i \tag{5}$$
$$\text{where,} \quad k_i = \Big[\beta_i x_i(0) - \alpha_i y_i(0)\Big], \qquad i = 1, 2.$$

Solutions to these equations when $\beta_i x_i(0) \neq \alpha_i y_i(0)$ are given by,

$$x_i(t) = \left[x_i(0)^{-1} e^{-k_i t} + \beta_i k_i^{-1}(1 - e^{-k_i t})\right]^{-1}$$
$$y_i(t) = \left[y_i(0)^{-1} e^{k_i t} + \alpha_i k_i^{-1}(e^{k_i t} - 1)\right]^{-1} \tag{6}$$

When $\beta_i x_i(0) = \alpha_i y_i(0)$, the solutions are given by,

$$x_i(t) = \frac{x_i(0)}{\beta_i x_i(0) t + 1}, \quad y_i(t) = \frac{y_i(0)}{\alpha_i x_i(0) t + 1} \tag{7}$$

With Lanchester linear law, the termination time is always at infinity. Define,

$$\zeta_i = \frac{M_i \beta_i}{N \alpha_i} \tag{8}$$

Now, the following results hold:

(i) If $\eta_i > \zeta_i$, then as $t \to \infty$,

$$\lim_{t\to\infty} x_i(t) = 0; \quad \lim_{t\to\infty} y_i(t) = \eta_i N - \left(\frac{\beta_i}{\alpha_i}\right) M_i \tag{9}$$

(ii) If $\eta_i = \zeta_i$, then as $t \to \infty$,

$$\lim_{t\to\infty} x_i(t) = 0; \quad \lim_{t\to\infty} y_i(t) = 0 \tag{10}$$

(iii) If $\eta_i < \zeta_i$, then as $t \to \infty$,

$$\lim_{t\to\infty} y_i(t) = 0; \quad \lim_{t\to\infty} x_i(t) = M_i - \left(\frac{\alpha_i}{\beta_i}\right) \eta_i N \tag{11}$$

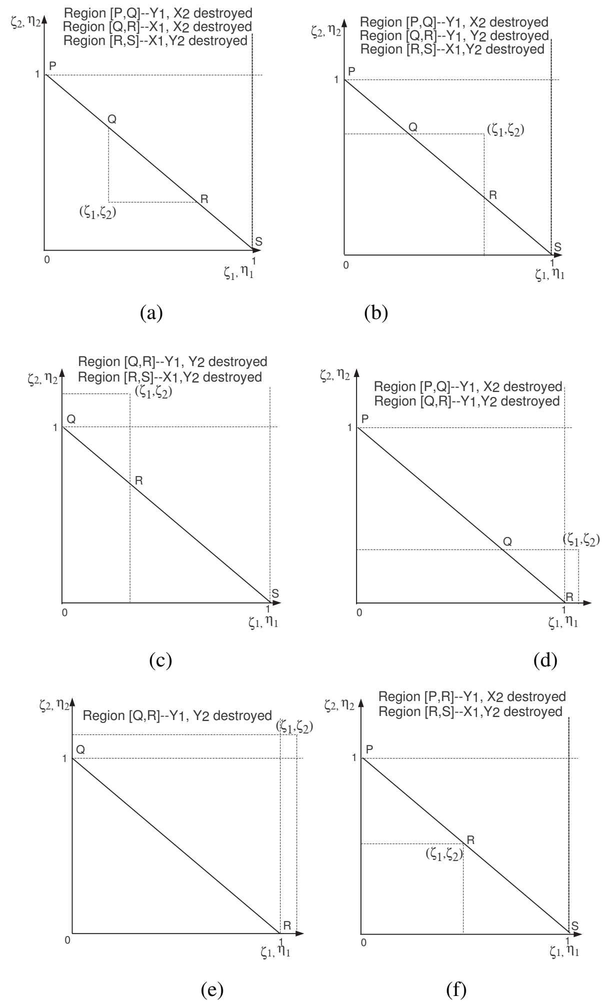


Fig. 2. Time-Zero-Allocation (TZA) events in (ζ_1, ζ_2) and (η_1, η_2) space

When $t \to \infty$, the objective function is given by,

(a) If $\eta_i \geq \zeta_i,\ i = 1, 2.$

$$J = \gamma\left[N - (\beta_1/\alpha_1) M_1 - (\beta_2/\alpha_2) M_2\right] + (1-\gamma)(M_1 + M_2) \tag{12}$$

(b) If $\eta_i \leq \zeta_i,\ i = 1, 2.$

$$J = (1-\gamma)\left[\eta N \left\{(\alpha_1/\beta_1) - (\alpha_2/\beta_2)\right\} + N(\alpha_2/\beta_2)\right] \tag{13}$$

(c) If $\eta_1 \geq \zeta_1$ and $\eta_2 < \zeta_2$,

$$J = \gamma\left[\eta N - \frac{\beta_1}{\alpha_1} M_1\right] + (1-\gamma)\left[M_1 + (1-\eta) N \frac{\alpha_2}{\beta_2}\right] \tag{14}$$

(d) If $\eta_1 \leq \zeta_1$ and $\eta_2 \geq \zeta_2$,

$$J = \gamma\left[(1-\eta) N - \frac{\beta_2}{\alpha_2} M_2\right] + (1-\gamma)\left[M_2 + \eta N \frac{\alpha_1}{\beta_1}\right] \tag{15}$$

Note that in (a)-(d) if $\eta_i = \zeta_i,\ i = 1, 2$, then

$$J = (1-\gamma)(M_1 + M_2) \tag{16}$$

The above cases can be graphically represented as shown in Figure 2. We use the variables P, Q, R and S to divide the regions. For example, $0 < \eta_1 < \zeta_1$ and $1 - \zeta_2 < \eta_2 < 1$ is represented by the region $[P, Q]$. Similarly the other regions. This gives rise to the following cases:

A. Case A: Let $\zeta_1 + \zeta_2 \leq 1$

For this case we get the following three regions as shown in Figure 2(a): (i) In the segment [P,Q], (15) holds, (ii) In the segment [Q,R], (12) holds, (iii) In the segment [R,S], (14) holds.

The objective functions in the region [P,Q], [Q,R] and [R,S] are given by,

$$\begin{aligned} J_{[P,Q]}(\eta) &= \gamma\left[(1-\eta)N - (\beta_2/\alpha_2)M_2\right] \\ &+ (1-\gamma)\left[M_2 + \eta N(\alpha_1/\beta_1)\right] \quad (17) \\ J_{[Q,R]}(\eta) &= \gamma\left[N - (\beta_1/\alpha_1)M_1 - (\beta_2/\alpha_2)M_2\right] \\ &+ (1-\gamma)\left[M_1 + M_2\right] \quad (18) \\ J_{[R,S]}(\eta) &= \gamma\left[\eta N - (\beta_1/\alpha_1)M_1\right] \\ &+ (1-\gamma)\left[M_1 + (1-\eta)N(\alpha_2/\beta_2)\right] \quad (19) \end{aligned}$$

Lemma 1: The function $J_{[P,Q]}(\eta)$, given in (17) and defined for $\eta \in [0, \zeta_1]$, has a maximum at (i) $\eta = 0$, if $\frac{\gamma}{1-\gamma} > \frac{\alpha_1}{\beta_1}$ (ii) at $\eta = \zeta_1$, if $\frac{\gamma}{1-\gamma} < \frac{\alpha_1}{\beta_1}$ and (iii) at all $\eta \in [0, \zeta_1]$, if $\frac{\gamma}{1-\gamma} = \frac{\alpha_1}{\beta_1}$.
Proof. This follows directly from Eqn. (17). □

Lemma 2: The function $J_{[R,S]}(\eta)$, given in (19) and defined for $\eta \in [1-\zeta_2, 1]$, has a maximum at (i) $\eta = 1-\zeta_2$, if $\frac{\gamma}{1-\gamma} < \frac{\alpha_2}{\beta_2}$ (ii) at $\eta = 1$, if $\frac{\gamma}{1-\gamma} > \frac{\alpha_2}{\beta_2}$ and (iii) at all $\eta \in [1-\zeta_2, 1]$, if $\frac{\gamma}{1-\gamma} = \frac{\alpha_2}{\beta_2}$.
Proof. This follows directly from Eqn. (19). □

Theorem 1: If $\zeta_1 + \zeta_2 \leq 1$, the optimum partitioning of the defending force Y will be given by,

$$\eta^* = \begin{cases} 0, & \text{if } \left[\frac{\gamma}{1-\gamma} > \frac{\alpha_1}{\beta_1}\right] \bigwedge \\ & \left[\left\{\frac{\gamma}{1-\gamma} \leq \frac{\alpha_2}{\beta_2}\right\} \bigvee \left\{\frac{\gamma}{1-\gamma} > \frac{\alpha_2}{\beta_2} \bigwedge \right.\right. \\ & \left.\left.\left(\frac{\gamma}{1-\gamma}\right)(\zeta_1 - \zeta_2) > \left(\zeta_1\frac{\alpha_1}{\beta_1} - \zeta_2\frac{\alpha_2}{\beta_2}\right)\right\}\right] \\ [0, 1-\zeta_2], & \text{if } \frac{\gamma}{1-\gamma} = \frac{\alpha_1}{\beta_1} \bigwedge \frac{\gamma}{1-\gamma} < \frac{\alpha_2}{\beta_2} \\ [\zeta_1, 1-\zeta_2], & \text{if } \frac{\gamma}{1-\gamma} < \frac{\alpha_1}{\beta_1} \bigwedge \frac{\gamma}{1-\gamma} < \frac{\alpha_2}{\beta_2} \\ 1, & \text{if } \left[\frac{\gamma}{1-\gamma} > \frac{\alpha_2}{\beta_2}\right] \bigwedge \\ & \left[\left\{\frac{\gamma}{1-\gamma} \leq \frac{\alpha_1}{\beta_1}\right\} \bigvee \left\{\frac{\gamma}{1-\gamma} > \frac{\alpha_1}{\beta_1} \bigwedge \right.\right. \\ & \left.\left.\left(\frac{\gamma}{1-\gamma}\right)(\zeta_1 - \zeta_2) < \left(\zeta_1\frac{\alpha_1}{\beta_1} - \zeta_2\frac{\alpha_2}{\beta_2}\right)\right\}\right] \\ [\zeta_1, 1], & \text{if } \frac{\gamma}{1-\gamma} = \frac{\alpha_2}{\beta_2} \bigwedge \frac{\gamma}{1-\gamma} < \frac{\alpha_1}{\beta_1} \\ [0, 1], & \text{if } \frac{\gamma}{1-\gamma} = \frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2} \end{cases} \quad (20)$$

Here '$\bigwedge$' stands for 'and' and '$\bigvee$' stands for 'or'.
Proof. From Eqn. (18), we know for $\eta = [\zeta_1, 1-\zeta_2]$, the objective function is independent of η. Also, from Lemmas 1 and 2, we can get (20). We omit details. □

The optimal values of η in $(\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2})$ space for this case are shown in Figure 3.

B. Case B: $\zeta_1 + \zeta_2 > 1$, $\zeta_1 < 1$ and $\zeta_2 < 1$

For this case we get the regions as shown in Figure 2(b). Note that (i) In the segment [P,Q], (15) holds. (ii) In the segment [Q,R], (13) holds. (iii) In the segment [R,S], (14) holds.

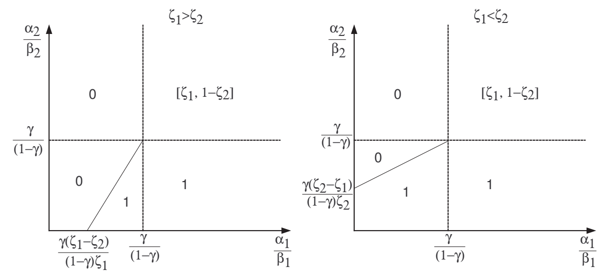


Fig. 3. Optimal values of η for Case A

The expressions for $J_{[P,Q]}$ and $J_{[R,S]}$ are same as Eqns. (17) and (19), respectively, while the objective function in the region $[Q, R]$ is given by,

$$J_{[Q,R]}(\eta) = (1-\gamma)\left[\eta N\left(\alpha_1/\beta_1 - \alpha_2/\beta_2\right) + N\alpha_2/\beta_2\right] \quad (21)$$

Lemma 3: The function $J_{[P,Q]}(\eta)$, given in (17) and defined for $\eta \in [0, 1-\zeta_2]$, has a maximum at (i) $\eta = 0$, if $\frac{\gamma}{1-\gamma} > \frac{\alpha_1}{\beta_1}$ (ii) at $\eta = 1-\zeta_2$, if $\frac{\gamma}{1-\gamma} < \frac{\alpha_1}{\beta_1}$ and (iii) at all $\eta \in [0, 1-\zeta_2]$, if $\frac{\gamma}{1-\gamma} = \frac{\alpha_1}{\beta_1}$.
Proof. This follows directly from Eqn. (17). □

Lemma 4: The function $J_{[Q,R]}(\eta)$, given in (21) and defined for $\eta \in [1-\zeta_2, \zeta_1]$, has a maximum at (i) $\eta = 1-\zeta_2$, if $\alpha_1\beta_2 < \alpha_2\beta_1$ (ii) at $\eta = \zeta_1$, if $\alpha_1\beta_2 > \alpha_2\beta_1$ and (iii) at all $\eta \in [1-\zeta_2, \zeta_1]$, if $\alpha_1\beta_2 = \alpha_2\beta_1$.
Proof. This follows directly from Eqn. (21). □

Lemma 5: The function $J_{[R,S]}(\eta)$, given in (19) and defined for $\eta \in [\zeta_1, 1]$, has a maximum at (i) $\eta = \zeta_1$, if $\frac{\gamma}{1-\gamma} < \frac{\alpha_2}{\beta_2}$ (ii) at $\eta = 1$, if $\frac{\gamma}{1-\gamma} > \frac{\alpha_2}{\beta_2}$ and (iii) at all $\eta \in [\zeta_1, 1]$, if $\frac{\gamma}{1-\gamma} = \frac{\alpha_2}{\beta_2}$.
Proof. This follows directly from Eqn. (19). □

Theorem 2: If $\zeta_1 + \zeta_2 > 1$, $\zeta_1 < 1$ and $\zeta_2 < 1$ the optimum partitioning of the defending force Y will be given by,

$$\eta^* = \begin{cases} 0, & \text{if } \left[\frac{\gamma}{1-\gamma} > \frac{\alpha_1}{\beta_1}\right] \bigwedge \\ & \left[\left\{\frac{\gamma}{1-\gamma} \leq \frac{\alpha_2}{\beta_2}\right\} \bigvee \left\{\frac{\gamma}{1-\gamma} > \frac{\alpha_2}{\beta_2} \bigwedge \right.\right. \\ & \left.\left.\left(\frac{\gamma}{1-\gamma}\right)(\zeta_1 - \zeta_2) > \left(\zeta_1\frac{\alpha_1}{\beta_1} - \zeta_2\frac{\alpha_2}{\beta_2}\right)\right\}\right] \\ 1-\zeta_2, & \text{if } \frac{\gamma}{1-\gamma} < \frac{\alpha_1}{\beta_1} \bigwedge \frac{\alpha_1}{\beta_1} < \frac{\alpha_2}{\beta_2} \\ [0, 1-\zeta_2], & \text{if } \frac{\gamma}{1-\gamma} = \frac{\alpha_1}{\beta_1} \bigwedge \frac{\alpha_1}{\beta_1} < \frac{\alpha_2}{\beta_2} \\ \zeta_1, & \text{if } \frac{\gamma}{1-\gamma} < \frac{\alpha_2}{\beta_2} \bigwedge \frac{\alpha_1}{\beta_1} > \frac{\alpha_2}{\beta_2} \\ [1-\zeta_2, \zeta_1], & \text{if } \frac{\gamma}{1-\gamma} < \frac{\alpha_1}{\beta_1} \bigwedge \frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2} \\ 1, & \text{if } \left[\frac{\gamma}{1-\gamma} > \frac{\alpha_2}{\beta_2}\right] \bigwedge \\ & \left[\left\{\frac{\gamma}{1-\gamma} \leq \frac{\alpha_1}{\beta_1}\right\} \bigvee \left\{\frac{\gamma}{1-\gamma} > \frac{\alpha_1}{\beta_1} \bigwedge \right.\right. \\ & \left.\left.\left(\frac{\gamma}{1-\gamma}\right)(\zeta_1 - \zeta_2) < \left(\zeta_1\frac{\alpha_1}{\beta_1} - \zeta_2\frac{\alpha_2}{\beta_2}\right)\right\}\right] \\ [\zeta_1, 1], & \text{if } \frac{\gamma}{1-\gamma} < \frac{\alpha_1}{\beta_1},\ \frac{\alpha_1}{\beta_1} > \frac{\alpha_2}{\beta_2} \bigwedge \frac{\gamma}{1-\gamma} = \frac{\alpha_2}{\beta_2} \end{cases} \quad (22)$$

Proof. From Lemmas $3-5$ we immediately get (22). □
The optimal values of η in $(\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2})$ space for this case are

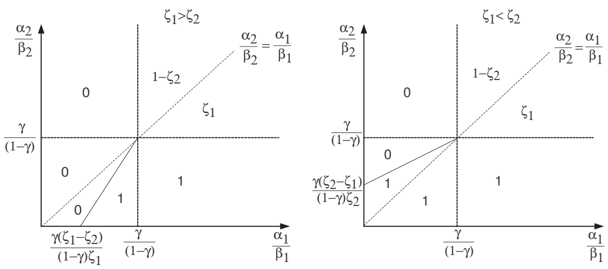


Fig. 4. *Optimal values of η for Case B*

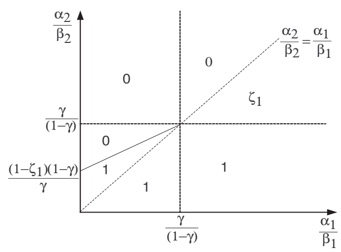


Fig. 5. *Optimal values of η for Case C*

shown in Figure 4.

C. *Case C: Let $\zeta_1 < 1$ and $\zeta_2 \geq 1$*

For this case we get the regions as shown in Figure 2(c). Note that, (i) In the segment [Q,R], (13) holds. (ii) In the segment [R,S], (14) holds.

The expressions for $J_{[Q,R]}$ and $J_{[R,S]}$ are same as Eqns. (21) and (19), respectively.

Lemma 6: The function $J_{[Q,R]}(\eta)$, given in (21) and defined for $\eta \in [0, \zeta_1]$, has a maximum at (i) $\eta = 0$ if $\alpha_1\beta_2 < \alpha_2\beta_1$ (ii) at $\eta = \zeta_1$ if $\alpha_1\beta_2 > \alpha_2\beta_1$ and (iii) at all $\eta \in [0, \zeta_1]$ if $\alpha_1\beta_2 = \alpha_2\beta_1$.

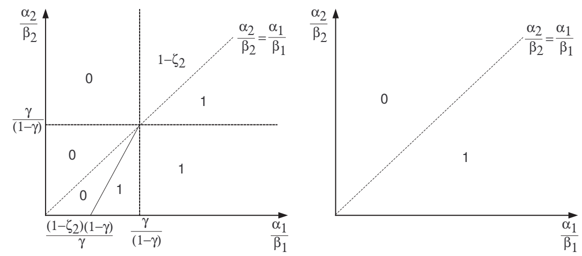
Proof. This follows directly from Eqn. (21). □

Theorem 3: If $\zeta_1 < 1$ and $\zeta_2 \geq 1$ the optimum partitioning of the defending force Y will be given by,

$$\eta^* = \begin{cases} 0, & \text{if } \left[\frac{\alpha_1}{\beta_1} < \frac{\alpha_2}{\beta_2}\right] \bigwedge \\ & \left[\left\{\frac{\gamma}{1-\gamma} \leq \frac{\alpha_2}{\beta_2}\right\} \bigvee \left\{\frac{\gamma}{1-\gamma} > \frac{\alpha_2}{\beta_2} \bigwedge \right.\right. \\ & \left.\left.\left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\alpha_2}{\beta_2} - \zeta_1\frac{\alpha_1}{\beta_1}\right) > 1 - \zeta_1\right\}\right] \\ \zeta_1, & \text{if } \frac{\alpha_1}{\beta_1} > \frac{\alpha_2}{\beta_2} \bigwedge \frac{\gamma}{1-\gamma} < \frac{\alpha_2}{\beta_2} \\ [0, \zeta_1], & \text{if } \frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2} \bigwedge \frac{\gamma}{1-\gamma} < \frac{\alpha_2}{\beta_2} \\ 1, & \text{if } \left[\frac{\gamma}{1-\gamma} > \frac{\alpha_2}{\beta_2}\right] \bigwedge \\ & \left[\left\{\frac{\alpha_1}{\beta_1} \geq \frac{\alpha_2}{\beta_2}\right\} \bigvee \left\{\frac{\alpha_1}{\beta_1} < \frac{\alpha_2}{\beta_2} \bigwedge \right.\right. \\ & \left.\left.\left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\alpha_2}{\beta_2} - \zeta_1\frac{\alpha_1}{\beta_1}\right) < 1 - \zeta_1\right\}\right] \\ [\zeta_1, 1], & \text{if } \frac{\gamma}{1-\gamma} = \frac{\alpha_2}{\beta_2} \bigwedge \frac{\alpha_1}{\beta_1} > \frac{\alpha_2}{\beta_2} \end{cases} \tag{23}$$

Proof. From Lemmas 5 and 6 we can immediately get (23). □

The optimal values of η in $(\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2})$ space for this case are shown in Figure 5.



(a) (b)

Fig. 6. Optimal values of η for (a) Case D and (b) Case E

D. *Case D: $\zeta_1 \geq 1$ and $\zeta_2 < 1$*

For this case we get the regions as shown in Figure 2(d). Note that, (i) In segment [P,Q], (15) holds. (ii) In segment [Q,R], (13) holds.

The expressions for $J_{[P,Q]}$ and $J_{[Q,R]}$ are same as Eqns. (17) and (21), respectively.

Lemma 7: The function $J_{[P,Q]}(\eta)$, given in (17) and defined for $\eta \in [0, 1 - \zeta_2]$, has a maximum at (i) $\eta = 0$, if $\frac{\gamma}{1-\gamma} > \frac{\alpha_1}{\beta_1}$ (ii) at $\eta = 1 - \zeta_2$, if $\frac{\gamma}{1-\gamma} < \frac{\alpha_1}{\beta_1}$ and (iii) at all $\eta \in [0, 1 - \zeta_2]$, if $\frac{\gamma}{1-\gamma} = \frac{\alpha_1}{\beta_1}$.

Proof. This follows directly from Eqn. (17). □

Lemma 8: The function $J_{[Q,R]}(\eta)$, given in (21) and defined for $\eta \in [1 - \zeta_2, 1]$, has a maximum at (i) $\eta = 1 - \zeta_2$ if $\alpha_1\beta_2 < \alpha_2\beta_1$ (ii) at $\eta = 1$ if $\alpha_1\beta_2 > \alpha_2\beta_1$ and (iii) at all $\eta \in [1 - \zeta_2, 1]$ if $\alpha_1\beta_2 = \alpha_2\beta_1$.

Proof. This follows directly from Eqn. (21) . □

Theorem 4: If $\zeta_1 \geq 1$ and $\zeta_2 < 1$ the optimum partitioning of the defending force Y will be given by,

$$\eta^* = \begin{cases} 0, & \text{if } \left[\frac{\gamma}{1-\gamma} > \frac{\alpha_1}{\beta_1}\right] \bigwedge \\ & \left[\left\{\frac{\alpha_1}{\beta_1} \leq \frac{\alpha_2}{\beta_2}\right\} \bigvee \left\{\frac{\alpha_1}{\beta_1} > \frac{\alpha_2}{\beta_2} \bigwedge \right.\right. \\ & \left.\left.\left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\alpha_1}{\beta_1} - \zeta_2\frac{\alpha_2}{\beta_2}\right) < 1 - \zeta_2\right\}\right] \\ 1 - \zeta_2, & \text{if } \frac{\alpha_1}{\beta_1} < \frac{\alpha_2}{\beta_2} \bigwedge \frac{\gamma}{1-\gamma} < \frac{\alpha_1}{\beta_1} \\ [0, 1 - \zeta_2], & \text{if } \frac{\alpha_1}{\beta_1} < \frac{\alpha_2}{\beta_2} \bigwedge \frac{\gamma}{1-\gamma} = \frac{\alpha_1}{\beta_1} \\ 1, & \text{if } \left[\frac{\alpha_1}{\beta_1} > \frac{\alpha_2}{\beta_2}\right] \bigwedge \\ & \left[\left\{\frac{\gamma}{1-\gamma} \leq \frac{\alpha_1}{\beta_1}\right\} \bigvee \left\{\frac{\gamma}{1-\gamma} > \frac{\alpha_1}{\beta_1} \bigwedge \right.\right. \\ & \left.\left.\left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\alpha_1}{\beta_1} - \zeta_2\frac{\alpha_2}{\beta_2}\right) > 1 - \zeta_2\right\}\right] \\ [1 - \zeta_2, 1], & \text{if } \frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2} \bigwedge \frac{\gamma}{1-\gamma} < \frac{\alpha_1}{\beta_1} \end{cases} \tag{24}$$

Proof. From Lemmas 7 and 8 we get (24). □

The optimal values of η in $(\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2})$ space for this case are shown in Figure 6 (a).

E. *Case E: $\zeta_1 + \zeta_2 > 1$, $\zeta_1 \geq 1$ and $\zeta_2 \geq 1$*

For this case we get the region as shown in Figure 2(e). Note that, (i) In the segment [Q,R], (13) holds.

The objective function in the region [Q,R] is same as Eqn. (21).

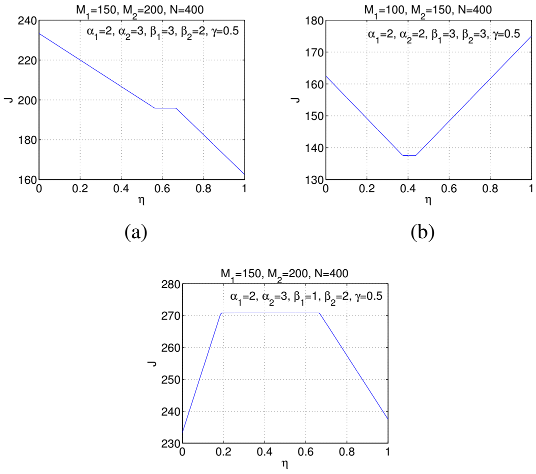


Fig. 7. *Examples to illustrate Theorem 1*

TABLE I
PARAMETERS FOR ILLUSTRATING THEOREM 1, SHOWN IN FIGURE 7, $\gamma = 0.5$

Figure	M_1	M_2	N	α_1	α_2	β_1	β_2	η^*
7(a)	150	200	400	2	3	3	2	0
7(b)	100	150	400	2	2	3	3	1
7(c)	150	200	400	2	3	1	2	$[0.1875, 0.6667]$

Lemma 9: The function $J_{[Q,R]}(\eta)$, given in (21) and defined for $\eta \in [0,1]$, has a maximum at (i) $\eta = 0$, if $\alpha_1\beta_2 < \alpha_2\beta_1$ (ii) at $\eta = 1$, if $\alpha_1\beta_2 > \alpha_2\beta_1$ and (iii) at all $\eta \in [0,1]$, if $\alpha_1\beta_2 = \alpha_2\beta_1$.

Proof. This follows directly from Eqn. (21). □

Theorem 5: If $\zeta_1 \geq 1$ and $\zeta_2 \geq 1$ the optimum partitioning of the defending force Y will be given by,

$$\eta^* = \begin{cases} 1, & \text{if } \frac{\alpha_1}{\beta_1} > \frac{\alpha_2}{\beta_2} \\ 0, & \text{if } \frac{\alpha_1}{\beta_1} < \frac{\alpha_2}{\beta_2} \\ [0,1], & \text{if } \frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2} \end{cases} \tag{25}$$

Proof. From Lemma 9 we can immediately get (25). □

The optimal values of η in $(\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2})$ space for this case are shown in Figure 6 (b).

F. Case F: $\zeta_1 < 1$, $\zeta_2 < 1$, and $\zeta_1 + \zeta_2 = 1$

For this case we get the region as shown in Figure 2(f): (i) In the segment [P,R], (15) holds, (ii) In the segment [R,S], (14) holds.

The objective functions in the region [P,R] and [R,S] are same as (17) and (19), respectively. The limiting case of Theorem 1, with $\zeta_1 + \zeta_2 = 1$ holds for this case.

IV. SOME NUMERICAL RESULTS AND DISCUSSIONS

Simulation results to illustrate Theorem 1 with the parameter values as given in Table I are shown in Figure 7. In Figure 7(a), $\zeta_1 + \zeta_2 \leq 1$, $\frac{\gamma}{1-\gamma} > \frac{\alpha_1}{\beta_1}$, $\frac{\gamma}{1-\gamma} < \frac{\alpha_2}{\beta_2}$ and so according to Theorem 1, $\eta^* = 0$, which is supported by the numerical solution. Similarly, in Figure 7(b), $\frac{\gamma}{1-\gamma} > \frac{\alpha_1}{\beta_1}$, $\frac{\gamma}{1-\gamma} > \frac{\alpha_2}{\beta_2}$, $\frac{\gamma}{1-\gamma}(\zeta_1 - \zeta_2) < \zeta_1\frac{\alpha_1}{\beta_1} - \zeta_2\frac{\alpha_2}{\beta_2}$, and hence $\eta^* = 1$.

TABLE II
PARAMETERS FOR ILLUSTRATING THEOREM 2, SHOWN IN FIGURE 8, $\gamma = 0.5$

Figure	M_1	M_2	N	α_1	α_2	β_1	β_2	η^*
8(a)	320	450	400	3	2	2	1	0.4375
8(b)	320	450	400	2	3	1	2	0.4
8(c)	150	200	400	2	2	3	3	1
8(d)	210	200	350	2	2	3	3	0

Similarly, in Figure 7(c), $\frac{\gamma}{1-\gamma} < \frac{\alpha_1}{\beta_1}$ and $\frac{\gamma}{1-\gamma} < \frac{\alpha_2}{\beta_2}$ and hence $\eta^* = [\zeta_1, 1 - \zeta_2]$.

Simulation results to illustrate Theorem 2 with the parameter values as given in Table II are shown in Figure 8. In Figure 8 (a), $\frac{\gamma}{1-\gamma} < \frac{\alpha_1}{\beta_1}$, $\frac{\gamma}{1-\gamma} < \frac{\alpha_2}{\beta_2}$, $\frac{\alpha_1}{\beta_1} < \frac{\alpha_2}{\beta_2}$ and so, according to Theorem 2, $\eta^* = 1 - \zeta_2 = 0.4375$, which is supported by the numerical solution. Similarly, in Figure 8

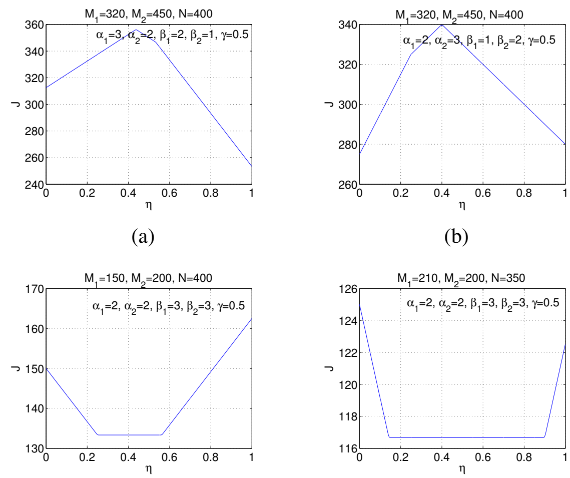


Fig. 8. *Examples to illustrate Theorem 2*

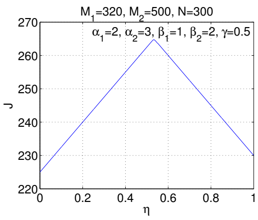
(b), $\frac{\gamma}{1-\gamma} < \frac{\alpha_1}{\beta_1}$, $\frac{\gamma}{1-\gamma} < \frac{\alpha_2}{\beta_2}$, $\frac{\alpha_1}{\beta_1} > \frac{\alpha_2}{\beta_2}$ and hence the optimum value occurs at $\eta^* = \zeta_1 = 0.4$. Similarly, in Figure 8 (c), $\frac{\gamma}{1-\gamma} > \frac{\alpha_1}{\beta_1}$, $\frac{\gamma}{1-\gamma} > \frac{\alpha_2}{\beta_2}$, $\frac{\gamma}{1-\gamma}(\zeta_1 - \zeta_2) < \zeta_1\frac{\alpha_1}{\beta_1} - \zeta_2\frac{\alpha_2}{\beta_2}$ and hence the optimum value occurs at $\eta^* = 1$. In Figure 8 (d), $\frac{\gamma}{1-\gamma} > \frac{\alpha_1}{\beta_1}$, $\frac{\gamma}{1-\gamma} > \frac{\alpha_2}{\beta_2}$, $\frac{\gamma}{1-\gamma}(\zeta_1 - \zeta_2) > \zeta_1\frac{\alpha_1}{\beta_1} - \zeta_2\frac{\alpha_2}{\beta_2}$ hence the optimum objective value occurs at $\eta^* = 0$.

The simulation results to illustrate Theorem 3 with the parameter values as given in Table III are shown in Figure 9. In Figure 9 (a), $\frac{\alpha_1}{\beta_1} > \frac{\alpha_2}{\beta_2}$, $\frac{\gamma}{1-\gamma} < \frac{\alpha_2}{\beta_2}$ and so according to Theorem 3, $\eta^* = \zeta_1 = 0.5333$, which is supported by the numerical solution. In Figure 9 (b), $\frac{\alpha_1}{\beta_1} < \frac{\alpha_2}{\beta_2}$ and $\frac{\gamma}{1-\gamma} < \frac{\alpha_2}{\beta_2}$

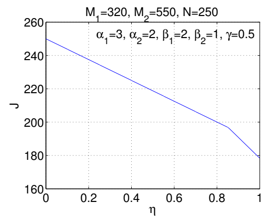
TABLE III
PARAMETERS FOR ILLUSTRATING THEOREM 3, SHOWN IN FIGURE 9, $\gamma = 0.5$

Figure	M_1	M_2	N	α_1	α_2	β_1	β_2	η^*
9(a)	320	500	300	2	3	1	2	0.5333
9(b)	320	550	250	3	2	2	1	0
9(c)	320	550	550	3	1	2	2	1

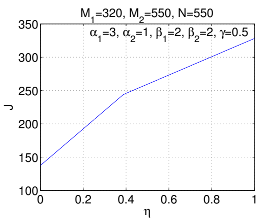
and hence the optimum objective value occurs at $\eta = 0$. In



(a)

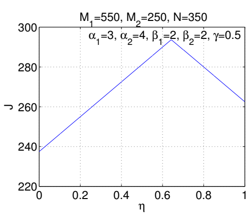


(b)

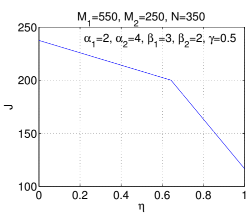


(c)

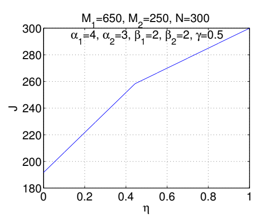
Fig. 9. *Examples to illustrate Theorem 3*



(a)



(b)



(c)

Fig. 10. *Examples to illustrate Theorem 4*

Figure 9 (c), $\frac{\gamma}{1-\gamma} > \frac{\alpha_2}{\beta_2}$ and $\frac{\alpha_1}{\beta_1} > \frac{\alpha_2}{\beta_2}$ and hence the optimum value occurs at $\eta^* = 1$.

The simulation results to illustrate Theorem 4 with the parameter values as given in Table IV are shown in Figure 10. In Figure 10 (a), $\frac{\alpha_1}{\beta_1} < \frac{\alpha_2}{\beta_2}$, $\frac{\gamma}{1-\gamma} < \frac{\alpha_1}{\beta_1}$ and so according to Theorem 4, $\eta^* = 1-\zeta_2 = 0.6429$, which is supported by the numerical solution. Similarly, in Figure 10 (b), $\frac{\gamma}{1-\gamma} > \frac{\alpha_1}{\beta_1}$ and $\frac{\alpha_1}{\beta_1} < \frac{\alpha_2}{\beta_2}$ and hence $\eta^* = 0$. In Figure 10 (c), $\frac{\gamma}{1-\gamma} < \frac{\alpha_1}{\beta_1}$ and $\frac{\alpha_1}{\beta_1} > \frac{\alpha_2}{\beta_2}$ and hence the optimum value occurs at $\eta^* = 1$.

The simulation results to illustrate Theorem 5 with the parameter values as given in Table V are shown in Figure 11. In Figure 11 (a), $\frac{\alpha_1}{\beta_1} > \frac{\alpha_2}{\beta_2}$ and so according to Theorem 5, $\eta^* = 1$, which is supported by the numerical solution. In Figure 11 (b), $\frac{\alpha_1}{\beta_1} < \frac{\alpha_2}{\beta_2}$ and hence the optimum objective value occurs at $\eta^* = 0$.

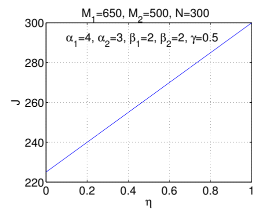
V. Concluding Remarks

In this paper, a battle between a defender with only one type of force and an attacker with two types of forces using

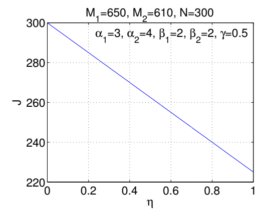
TABLE IV

Parameters for illustrating Theorem 4, shown in Figure 10, $\gamma = 0.5$

Figure	M_1	M_2	N	α_1	α_2	β_1	β_2	η^*
10(a)	550	250	350	3	4	2	2	0.6429
10(b)	550	250	350	2	4	3	2	0
10(c)	650	250	300	4	3	2	2	1



(a)



(b)

Fig. 11. *Examples to illustrate Theorem 5*

TABLE V

Parameters for illustrating Theorem 5, shown in Figure 11, $\gamma = 0.5$

Figure	M_1	M_2	N	α_1	α_2	β_1	β_2	η^*
11(a)	650	500	300	4	3	2	2	1
11(b)	650	610	300	3	4	2	2	0

Lanchester linear model is analyzed in detail. The objective of the defending force is to maximize the destruction of the attacking force and the survival of its own force. With the given initial values, the optimal partitioning of the defending resources for TZA scheme are obtained both analytically and numerically. Hence, by knowing the initial strengths of the opposing forces and the rate at which resources get destroyed, we can optimally partition the defending force accordingly as defined by Theorems 1 to 5 for the TZA scheme.

References

[1] J.G. Taylor, *Lanchester Models of Warfare*, Vols. 1 and 2, Operations Research Society of America, Military Applications Section, Arlington, Virgina, 1983.

[2] Special Issue on Air-Land-Naval Warfare Models, *Naval Research Logistics*, Vol. 42, No. 2-4, 1995.

[3] P.S. Sheeba and D. Ghose, Optimal resource partitioning in a military conflict based on Lanchester attrition models, *Proceedings of the IEEE Conference on Decision and Control, and the European Control Conference,* Seville, Spain, December 2005, pp. 5859-5864.

[4] P.S. Sheeba and D. Ghose, Optimal resource partitioning in conflicts based on Lanchester $(n, 1)$ attrition model, *Proceedings of the American Control Conference,* Minneapolis, Minnesota, June 2006, pp. 638-643.

[5] J.G. Taylor, Optimal commitment of forces in some Lanchester-type combat models, *Operations Research,* Vol. 27, No.1, January 1979, pp. 96-114.

[6] D.M. Roberts and D.M. Conolly: An extension of the Lanchester square law to inhomogeneous forces with an application to force allocation methodlogy, *Journal of the Operational Research Society,* Vol. 43, No. 8, August 1992, pp. 741-752.

[7] G.T. Kaup, D.J. Kaup and N.M. Finkelstein, The Lanchester (n,1) problem, *Journal of the Operational Research Society,* Vol. 56, No. 12, December 2005, pp. 1399-1407.

[8] R.K. Colegrave and J.M. Hyde, The Lanchester square-law model extended to a (2,2) conflict, *IMA Journal of Applied Mathematics,* Vol. 51, No. 2, 1993, pp. 95-109.