2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008



# Adaptive Controller Design and Disturbance Attenuation for Sequentially Interconnected SISO Linear Systems under Noisy Output Measurements with Partly Measured Disturbances

Sheng Zeng Emmanuel Fernandez

*Abstract*— **In this paper, we present robust adaptive controller design for a special class of linear system, which is composed of two sequentially interconnected SISO linear subsystems, $\mathbf{S_1}$ and $\mathbf{S_2}$, under noisy output measurements and with partly measured disturbances. Based on worst-case analysis approach, we formulate the robust adaptive control problem as a nonlinear $H^\infty$-optimal control problem under imperfect state measurements, and then solve it using game-theoretic approach. The *cost-to-come* function analysis is carried out to derive the estimators and identifiers of $\mathbf{S_1}$ and $\mathbf{S_2}$, and integrator backstepping methodology is applied recursively to obtain the control law. The design paradigm is the same as [1] with the only difference being the treatment of the measured disturbance. The same result of [1] is achieved. In addition, the designed controller achieves the disturbance attenuation level *zero* or arbitrary positive disturbance attenuation level with respect to the measured disturbances. Moreover, for the measured disturbances that the controller can achieve disturbance attenuation level *zero* with respect to, the asymptotic tracking objective is achieved even if they are only uniformly bounded without being of finite energy.**

*Index Terms*— **Nonlinear $H^\infty$ control; cost-to-come function analysis; measured disturbances; adaptive control.**

## I. Introduction

Adaptive control has attracted a lot of research attention in control theory for many decades. In the certainty equivalence based adaptive controller design [2] the unknown parameters of the uncertainty system are substituted by their online estimates, which are generated through a variety of identifiers, as long as the estimates satisfy certain properties independent of the controller. This approach leads to structurally simple adaptive controllers and has been demonstrated its effectiveness for linear systems with or without stochastic disturbance inputs [3] when long term asymptotic performance is considered. Yet, the certainty equivalence approach is unsuccessful to generalize to systems with severe nonlinearities. Also, early designs based on this approach were shown to be nonrobust [4] when the system is subject to exogenous disturbance inputs and unmodeled dynamics. Then, the stability and the performance of the closed-loop system becomes an important issue. This has motivated the study of robust adaptive control in the 1980s and 1990s, and the study of nonlinear adaptive control in the 1990s.

The topic of adaptive control design for nonlinear systems was studied intensely in the last decade after the celebrated characterization of feedback linearizable or partially feedback linearizable systems [5]. A breakthrough is achieved when the integrator backstepping methodology [6] was introduced to design adaptive controllers for parametric strict-feedback and parametric pure-feedback nonlinear systems systematically. Since then, a lot of important contributions were motivated by this approach, and a complete list of references can be found in the book [7]. Moreover, this nonlinear design approach has been applied to linear systems to compare performance with the certainty equivalence approach. However, simple designs using this approach without taking into consideration the effect of exogenous disturbance inputs have also been shown to be nonrobust when the system is subject to exogenous disturbance inputs.

The robustness of closed-loop adaptive systems has been an important research topic in late 1980s and early 1990s. Various adaptive controllers were modified to render the closed-loop systems robust [8]. Despite their successes, they still fell short of directly addressing the disturbance attenuation property of the closed-loop system.

The objectives of robust adaptive control are to improve transient response, to accommodate unmodeled dynamics, and to reject exogenous disturbance inputs, which are the same as the objectives to motivate the study of the $H^\infty$- optimal control problem. $H^\infty$-optimal control was proposed as a solution to the robust control problem, where these objectives are achieved by studying only the disturbance attenuation property for the closed-loop system. The game-theoretic approach to $H^\infty$-optimal control developed for the linear quadratic problems, offers the most promising tool to generalize the results to nonlinear systems [9]. Worst-case analysis based adaptive control design was proposed in late 1990s to address the disturbance attenuation property directly, and it is motivated by the success of the game-theoretic approach to $H^\infty$-optimal control problems [10]. In this approach, the robust adaptive control problem is formulated as a nonlinear $H^\infty$ control problem under imperfect state measurements. By *cost-to-come function* analysis, it is converted into an $H^\infty$ control problem with full information measurements. This full information measurements problem is then solved using nonlinear design tools for a suboptimal solution. This design paradigm has been applied to worst-case parameter identification problems [11], which has led to new classes of parametrized identifiers for linear and nonlinear systems. It has also been applied to adaptive control problems [12], [13], [14], [15], which has led to new classes of parametrized robust adaptive controllers for linear and nonlinear systems. In [12], adaptive control for strict-feedback nonlinear systems was considered under noiseless output measurements, and more general class of nonlinear systems was studied in [14]. In [13], single-input and single output (SISO) linear systems were considered under noisy output measurements. SISO linear systems with partly measured disturbance was studied in[15], which leads to disturbance feedforward structure in the adaptive controller. In [1], adaptive control for a sequentially interconnected SISO linear system was considered, and a special class of unobservable systems was also studied using the proposed approach.

In this paper, we study the adaptive control design for sequentially interconnected SISO linear systems, $\mathbf{S}_1$ and $\mathbf{S}_2$(see Figure 1), under noisy output measurements and partly measured disturbance using the similar approaches as [13] and [1]. We assume that

The work in this paper was done when Sheng Zeng was with Department of Electrical and Computer Engineering, University of Cincinnati, Cincinnati, OH. He is now with Hewlett-Packard Company, 18110 SE 34th St., Vancouver, WA 98683. Email: sheng.zeng@hp.com.

Emmanuel Fernandez is with Department of Electrical and computer Engineering, 837 Rhodes Hall, Mail Loc. 0030, University of Cincinnati, Cincinnati, OH 45221-0030. Tel:513-556-4785; Fax: 513-556-7326. Email: emmanuel@ececs.uc.edu.

978-1-4244-2079-7/08/$25.00 ©2008 AACC. 

the linear systems satisfy the same assumption as [1], and the adaptive control design follows the same design method discussed above. The robust adaptive controller achieves asymptotic tracking of the reference trajectories when disturbance inputs are of finite energy. The closed-loop system is totally stable with respect to the disturbance inputs and the initial conditions. Furthermore, the closed-loop system admits a guaranteed disturbance attenuation level with respect to the exogenous disturbance inputs, where ultimate lower bound for the achievable attenuation performance level is equal to the noise intensity in the measurement channel of $\mathbf{S_1}$. The results are as same as those in [1]. In addition, the controller achieves arbitrary positive distance attenuation level with respect to the measured disturbances by proper scaling. Moreover, if the measured disturbances satisfy the assumption 2 for $\check{w}_{1,b}$ and $\check{w}_{2,b}$, the proposed controller achieves disturbance attenuation level *zero* with respect to the measured disturbances, which further leads to a stronger asymptotic tracking property, namely, the tracking error converges to zero when the unmeasured disturbances are $\mathcal{L}_2 \cap \mathcal{L}_\infty$, and the measured disturbances are $\mathcal{L}_\infty$ only.

The balance of the paper is organized as follows. In Section II, we present the formulation of the adaptive control problem and discuss the general solution methodology. In Section III, we first obtain parameter identifier and state estimator using the *cost-to-come function* analysis in Subsection III-A, then we derive the adaptive control law in Subsection III-B. We present the main results on the robustness of the system in Section IV, whose proofs are omitted due to page limitation. The paper ends with some concluding remarks and an acknowledgement in Section V.

The following notation will be used in this paper. For any vector $z \in \mathbb{R}^n$, and any $n \times n$-dimensional symmetric matrix $M$, where $n \in \mathbb{N}$, $|z|_M^2 = z' M z$. For any matrix $M$, the vector $\overrightarrow{M}$ is formed by stacking up its column vectors. For any symmetric matrix $M$, $\overleftarrow{M}$ denotes the vector formed by stacking up the column vector of the lower triangular part of $M$. For $n \times n$-dimensional symmetric matrices $M_1$ and $M_2$, where $n \in \mathbb{N}$, we write $M_1 > M_2$ if $M_1 - M_2$ is positive definite; we write $M_1 \geq M_2$ if $M_1 - M_2$ is positive semi-definite. For $n \in \mathbb{N}$, the set of $n \times n$-dimensional positive definite matrices is denoted by $\mathcal{S}_{+n}$. For $n \in \mathbb{N} \cup \{0\}$, $I_n$ denotes the $n \times n$-dimensional identity matrix. For any $n \in \mathbb{N}$ and $k \in \{1, \cdots, n\}$, $e_{n,k}$ denotes $\left[ \begin{array}{ccc} 0_{1\times(k-1)} & 1 & 0_{1\times(n-k)} \end{array} \right]'$.

## II. Problem Formulation

We consider the robust adaptive control problem for the system which is described by the block diagram in Figure 1.

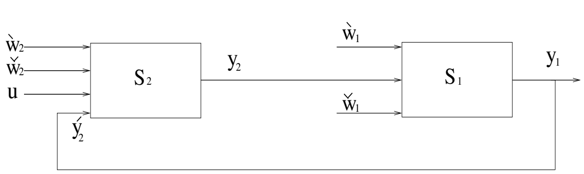


Fig. 1. Diagram of MIMO System.

We assume that the system dynamics for $\mathbf{S_i}$ is given by, $i = 1, 2$

$$\begin{aligned} \dot{\grave{x}}_i &= \grave{A}_i \grave{x}_i + \epsilon_{i,s} \grave{B}_1 y_2 + (1 - \epsilon_{i,s})(\grave{B}_2 u + \grave{A}_{2,y} \acute{y}_2) + \grave{D}_i \grave{w}_i \\ &\quad + \grave{\check{D}}_i \check{w}_i; \quad \grave{x}_i(0) = \grave{x}_{i,0} \qquad (1a) \\ y_i &= \grave{C}_i \grave{x}_i + \grave{E}_i \grave{w}_i \qquad (1b) \end{aligned}$$

where $\epsilon_{1,s} = 1$ and $\epsilon_{2,s} = 0$; $\grave{x}_i$ is the $n_i$-dimensional state vectors, $n_i \in \mathbb{N}$; $u$ is the scalar control input; $y_i$ is the scalar measurement output; $\grave{w}_i$ is $\grave{q}_i$-dimensional unmeasured disturbance input vector, $\grave{q}_i \in \mathbb{N}$; $\check{w}_i$ is $\check{q}_i$-dimensional measured disturbance input vector, $\check{q}_i \in \mathbb{N}$; the elements of $\check{w}_i$ are $\left[ \begin{array}{ccc} \check{w}_{i,1} & \cdots & \check{w}_{i,\check{q}_i} \end{array} \right]'$; $\acute{y}_2 = y_1$; the matrices $\grave{A}_i$, $\grave{A}_{i,y}$, $\grave{B}_i$, $\grave{C}_i$, $\grave{D}_i$, $\grave{\check{D}}_i$, and $\grave{E}_i$ are of the appropriate dimensions, generally unknown or partially unknown, $i = 1, 2$. For subsystem $\mathbf{S_1}$, the transfer function from $y_2$ to $y_1$ is $H_1(s) = \grave{C}_1 (sI_{n_1} - \grave{A}_1)^{-1} \grave{B}_1$, for subsystem $\mathbf{S_2}$, the transfer function from $u$ to $y_2$ is $H_2(s) = \grave{C}_2 (sI_{n_2} - \grave{A}_2)^{-1} \grave{B}_2$. All signals in the system are assumed to be continuous.

The subsystems $\mathbf{S_1}$ and $\mathbf{S_2}$ satisfy the following assumptions,

*Assumption 1:* For $i = 1, 2$, the pair $(\grave{A}_i, \grave{C}_i)$ is observable; the transfer function $H_i(s)$ is known to have relative degree $r_i \in \mathbb{N}$, and is strictly minimum phase. The uncontrollable part of $\mathbf{S_1}$ (with respect to $y_2$) is stable in the sense of Lyapunov; any uncontrollable mode corresponding to an eigenvalue of the matrix $\grave{A}_1$ on the $j\omega$-axis is uncontrollable from $\left[ \begin{array}{cc} \grave{w}_1' & \check{w}_1' \end{array} \right]'$. The uncontrollable part of $\mathbf{S_2}$ (with respect to $u$) is stable in the sense of Lyapunov; any uncontrollable mode corresponding to an eigenvalue of the matrix $\grave{A}_2$ on the $j\omega$-axis is uncontrollable from $\left[ \begin{array}{ccc} \grave{w}_2' & \acute{y}_2 & \check{w}_2' \end{array} \right]'$. ⋄

Based on Assumption 1, for $i = 1, 2$, there exists a state diffeomorphism: $x_i = \grave{T}_i \grave{x}_i$, and a disturbance transformation: $w_i = \grave{M}_i \grave{w}_i$, such that $\mathbf{S_i}$ can be transformed into the following state space representation,

$$\begin{aligned} \dot{x}_i &= A_i x_i + (y_i \bar{A}_{i,211} + (\epsilon_{i,s}(y_2 - u) + u)\bar{A}_{i,212} \\ &\quad + \sum_{j=1}^{\check{q}_i} \check{w}_{i,j} \bar{A}_{i,213j} + (1 - \epsilon_{i,s}) \acute{y}_2 \bar{A}_{2,214}) \theta_i + \check{D}_i \check{w}_i \\ &\quad + B_i(\epsilon_{i,s}(y_2 - u) + u) + (1 - \epsilon_{i,s}) A_{2,y} \acute{y}_2 + D_i w_i \quad (2a) \\ y_i &= C_i x_i + E_i w_i \qquad (2b) \end{aligned}$$

where $\theta_i$ is the $\sigma_i$-dimensional vector of unknown parameters for the subsystem $\mathbf{S}_i$, $\sigma_i \in \mathbb{N}$; the matrices $A_i$, $\bar{A}_{i,211}$, $\bar{A}_{i,212}$, $\bar{A}_{i,2131}$, $\cdots$, $\bar{A}_{i,213\check{q}_i}$, $\bar{A}_{2,214}$, $A_{2,y}$, $B_i$, $D_i$, $\check{D}_i$, $C_i$, and $E_i$ are known and have the following structures, $A_i = (a_{i,jk})_{n_i \times n_i}; a_{i,j\,(j+1)} = 1, a_{i,jk} = 0$, for $1 \leq j \leq r_i - 1$ and $j + 2 \leq k \leq n_i$; $\bar{A}_{i,212} = \left[ \begin{array}{ccc} \mathbf{0}_{\sigma_i \times (r_i - 1)} & \bar{A}'_{i,2120} & \bar{A}'_{i,212r_i} \end{array} \right]'$, $C_i = \left[ \begin{array}{cc} 1 & \mathbf{0}_{1 \times (n_i - 1)} \end{array} \right]$, $A_{i,2120}$ is a row vector, $B_i = \left[ \begin{array}{cccc} \mathbf{0}_{1 \times (r_i - 1)} & b_{i,p0} & \cdots & b_{i,p(n_i - r_i)} \end{array} \right]'$, $b_{i,pj}$ $j = 0, 1, \cdots, n_i - r_i$ are constants. We denote the elements of $x_1$ and $x_2$ by $\left[ \begin{array}{ccc} x_{1,1} & \cdots & x_{1,n_1} \end{array} \right]'$ and $\left[ \begin{array}{ccc} x_{2,1} & \cdots & x_{2,n_2} \end{array} \right]'$, with initial conditions $x_{1,0}$ and $x_{2,0}$, respectively.

*Assumption 2:* The measured disturbance $\check{w}_i$ can be partitioned as: $\check{w}_i = \left[ \begin{array}{cc} \check{w}'_{i,a} & \check{w}'_{i,b} \end{array} \right]'$ where $\check{w}_{i,a}$ is $\check{q}_{i,a}$ dimensional, $\check{q}_{i,a} \in \mathbb{N} \cup \{0\}$. The transfer function from each element of $\check{w}_{1,a}$ to $y_1$ has relative degree less than $r_1 + r_2$; and the transfer function from each element of $\check{w}_{2,a}$ to $y_2$ has relative degree less than $r_2$. ⋄

Based on Assumption 2, the matrix $\check{D}_i$ can be partitioned into $\left[ \begin{array}{cc} \check{D}_{i,a} & \check{D}_{i,b} \end{array} \right]$, where $\check{D}_{i,a}$ and $\check{D}_{i,b}$ have $n_i \times \check{q}_{i,a}$- and $n_i \times \check{q}_{i,b}$-dimensional, respectively; $\check{D}_{i,b} = \left[ \begin{array}{ccc} \mathbf{0}'_{(r_i - 1) \times \check{q}_{i,b}} & \check{D}'_{i,b0} & \check{D}'_{i,br_i} \end{array} \right]'$, and $\bar{A}_{i,213\,j} = \left[ \begin{array}{ccc} \mathbf{0}'_{(r_i - 1) \times \sigma_i} & \bar{A}'_{i,213\,j0} & \bar{A}'_{i,213\,jr_i} \end{array} \right]'$, $j = \check{q}_{i,a} + 1, \cdots, \check{q}_i$, where $\check{D}_{i,b0}$ and $\bar{A}_{i,213\,j0}$, $j = \check{q}_{i,a} + 1, \cdots, \check{q}_{i,a} + \check{q}_{i,b}$, are row vectors, $i = 1, 2$.

We call (2) the design model, and we make the following two assumptions.

*Assumption 3:* For $i = 1, 2$, the matrices $E_i$ are such that $E_i E_i' > 0$. Define $\zeta_i := (E_i E_i')^{-\frac{1}{2}}$ and $L_i := D_i E_i'$. ⋄

Due to the structures of $A_i$, $\bar{A}_{i,212}$ and $B_i$, the high frequency gain of the transfer function $H_i(s)$, $b_{i,0}$, is equal to $b_{i,p0} + \bar{A}_{i,212\,0} \theta_i$, $i = 1, 2$.

To guarantee the stabilizability of the identified system, we make the following assumption on the parameter vectors $\theta_1$ and $\theta_2$.

*Assumption 4:* The sign of $b_{i,0}$ is known; there exists a known smooth nonnegative radially-unbounded strictly convex function $P_i : \mathbb{R}^{\sigma_i} \to \mathbb{R}$, such that the true value $\theta_i \in \Theta_i := \{\bar{\theta}_i \in \mathbb{R}^{\sigma_i} \mid P_i(\bar{\theta}_i) \leq 1\}$; moreover, $\forall \bar{\theta}_i \in \Theta_i$, $\operatorname{sgn}(b_{i,0})(b_{i,p0} + \bar{A}_{i,212\,0}\bar{\theta}_i) > 0$, $i = 1, 2$. $\diamond$

Assumption 4 delineates *a priori* convex compact sets where the parameter vectors $\theta_1$ and $\theta_2$ lie in, respectively. This will guarantee the stability of the closed-loop system and the boundedness of the estimate of $\theta_1$ and $\theta_2$.

We make the following assumption about the reference signal, $y_d$.

*Assumption 5:* The reference trajectory, $y_d$, is $r_1+r_2$ times continuously differentiable. Define vector $Y_d := [y_d^{(0)}, \cdots, y_d^{(r_1+r_2)}]'$, where $y_d^{(0)} = y_d$, and $y_d^{(j)}$ is the $j$th order time derivative of $y_d$, $j = 1, \cdots, r_1 + r_2$; define $Y_{d0} := [y_d^{(0)}(0), \cdots, y_d^{(r_1+r_2-1)}(0)]' \in \mathbb{R}^{r_1+r_2}$. The signal $Y_d$ is available for feedback. $\diamond$

The uncertainty of subsystem $\mathbf{S_1}$ is $\grave{\omega}_1 := (x_{1,0}, \theta_1, \grave{w}_{1[0,\infty)}, \check{w}_{1[0,\infty)}, Y_{d0}, y_{d[0,\infty)}^{(r_1+r_2)}) \in \grave{\mathcal{W}}_1 := \mathbb{R}^{n_1} \times \Theta_1 \times \mathcal{C} \times \mathcal{C} \times \mathbb{R}^{r_1+r_2} \times \mathcal{C}$, which comprises the initial state $x_{1,0}$, the true value of the parameters $\theta_1$, the unmeasured disturbance waveform $\grave{w}_{1[0,\infty)}$, the measured disturbance waveform $\check{w}_{1[0,\infty)}$, the initial conditions of the reference trajectory $Y_{d0}$, and the waveform of the $(r_1 + r_2)$ th order derivative of the reference trajectory, $y_{d[0,\infty)}^{(r_1+r_2)}$. The uncertainty for subsystem $\mathbf{S_2}$ is $\grave{\omega}_2 := (x_{2,0}, \theta_2, \grave{w}_{2[0,\infty)}, \check{w}_{2[0,\infty)}) \in \grave{\mathcal{W}}_2 := \mathbb{R}^{n_2} \times \Theta_2 \times \mathcal{C} \times \mathcal{C}$, which comprises the initial state $x_{2,0}$, the true value of the parameters $\theta_2$, the unmeasured disturbance waveform $\grave{w}_{2[0,\infty)}$, and the measured disturbance waveform $\check{w}_{2[0,\infty)}$.

Our objective is to derive a control law, which is generated by the following mapping,

$$u(t) = \mu(y_{2[0,t]}, \acute{y}_{2[0,t]}, Y_{d[0,t]}, \check{w}_1, \check{w}_2) \tag{3}$$

where $\mu : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathbb{R}$, such that $x_{1,1}$ can asymptotically track the reference trajectory $y_d$, while rejecting the uncertainty $(\grave{\omega}_1, \grave{\omega}_2) \in \grave{\mathcal{W}}_1 \times \grave{\mathcal{W}}_2$, and keeping the closed-loop signals bounded. The control law $\mu$ must also satisfy that, $\forall (\grave{\omega}_1, \grave{\omega}_2) \in \grave{\mathcal{W}}_1 \times \grave{\mathcal{W}}_2$, there exists a solution $\grave{x}_{1[0,\infty)}$ and $\grave{x}_{2[0,\infty)}$ to the system (1), which yields a continuous control signal $u_{[0,\infty)}$. We denote the class of these admissible controllers by $\mathcal{M}_\mu$.

For design purposes, instead of attenuating the effect of $\begin{bmatrix} \grave{w}_1' & \check{w}_1' & \grave{w}_2' & \check{w}_2 \end{bmatrix}'$ we design the adaptive controller to attenuate the effect of $\begin{bmatrix} w_1' & \check{w}_1' & w_2' & \check{w}_2 \end{bmatrix}'$. This is done to allow our design paradigm to be carried out. This will result in a guaranteed attenuation level with respect to $\grave{\omega}_1$ and $\grave{\omega}_2$. To simplify the notation, we take the uncertainty $\omega_1 := (x_{1,0}, \theta_1, w_{1[0,\infty)}, \check{w}_{1[0,\infty)}, Y_{d0}, y_{d[0,\infty)}^{(r_1+r_2)}) \in \mathcal{W}_1 := \mathbb{R}^{n_1} \times \Theta_1 \times \mathcal{C} \times \mathcal{C} \times \mathbb{R}^{r_1+r_2} \times \mathcal{C}$, and $\omega_2 := (x_{2,0}, \theta_2, w_{2[0,\infty)}, \check{w}_{2[0,\infty)}) \in \mathcal{W}_2 := \mathbb{R}^{n_2} \times \Theta_2 \times \mathcal{C} \times \mathcal{C}$.

We state the control objective precisely as follows,

*Definition 1:* A controller $\mu \in \mathcal{M}_\mu$ is said to achieve *disturbance attenuation level* $\gamma$ with respect to $\begin{bmatrix} w_1' & \check{w}_{1,a}' & w_2' & \check{w}_{2,a}' \end{bmatrix}'$, and *disturbance attenuation level zero* with respect to $\begin{bmatrix} \check{w}_{1,b}' & \check{w}_{2,b}' \end{bmatrix}'$, if there exists functions $l_1(t, \theta_1, x_1, y_{1[0,t]}, y_{2[0,t]}, \check{w}_{1[0,t]}, \check{w}_{2[0,t]}, Y_{d[0,t]})$, $l_2(t, \theta_2, x_2, y_{1[0,t]}, y_{2[0,t]}, \check{w}_{1[0,t]}, \check{w}_{2[0,t]}, Y_{d[0,t]})$, and a known nonnegative constant $l_0(\check{x}_{1,0}, \check{x}_{2,0}, \check{\theta}_{1,0}, \check{\theta}_{2,0})$, such that

$$\sup_{\grave{w}_1 \in \grave{\mathcal{W}}_1, \grave{w}_2 \in \grave{\mathcal{W}}_2} J_{\gamma t_f} \leq 0; \quad \forall t_f \geq 0 \tag{4}$$

and $l_1 \geq 0$ and $l_2 \geq 0$ along the closed-loop trajectory, where

$$J_{\gamma t_f} := J_{1,\gamma t_f} + J_{2,\gamma t_f} \tag{5a}$$

$$J_{i\gamma t_f} := \int_0^{t_f} \Big( \epsilon_{i,s}(C_1 x_1 - y_d)^2 - \gamma^2 |w_i|^2 - \gamma^2 |\check{w}_{i,a}|^2 + l_i \Big) \, \mathrm{d}\tau - \gamma^2 \left| \begin{bmatrix} \theta_i' - \check{\theta}_{i,0}' & x_{i,0}' - \check{x}_{i,0}' \end{bmatrix}' \right|^2_{\bar{Q}_{i,0}} \tag{5b}$$

$\check{\theta}_{i,0} \in \Theta_i$ is the initial guess of $\theta_i$; $\check{x}_{i,0} \in \mathbb{R}^{n_i}$ is the initial guess of $x_{i,0}$; $\bar{Q}_{i,0} > 0$ is a $(n_i + \sigma_i) \times (n_i + \sigma_i)$-dimensional weighting matrix, quantifying the level of confidence in the estimate $\begin{bmatrix} \check{\theta}_{i,0}' & \check{x}_{i,0}' \end{bmatrix}'$; $\bar{Q}_{i,0}^{-1}$ admits the structure $\begin{bmatrix} Q_{i,0}^{-1} & Q_{i,0}^{-1}\Phi_{i,0}' \\ \Phi_{i,0} Q_{i,0}^{-1} & \Pi_{i,0} + \Phi_{i,0} Q_{i,0}^{-1} \Phi_{i,0}' \end{bmatrix}$, $Q_{i,0}$ and $\Pi_{i,0}$ are $\sigma_i \times \sigma_i$- and $n_i \times n_i$-dimensional positive definite matrices, respectively, $i = 1, 2$.

Clearly, when the inequality (4) is achieved, the squared $\mathcal{L}_2$ norm of the output tracking error $C_1 x_1 - y_d$ is bounded by $\gamma^2$ times the squared $\mathcal{L}_2$ norm of the transformed disturbance input $\begin{bmatrix} w_1' & \check{w}_{1,a}' & w_2' & \check{w}_{2,a}' \end{bmatrix}'$, plus some constant. When the $\mathcal{L}_2$ norm of $\grave{w}_1$, $\grave{w}_2$, $\check{w}_1$, and $\check{w}_2$ are finite, the squared $\mathcal{L}_2$ norm of $C_1 x_1 - y_d$ is also finite, which implies $\lim_{t \to \infty} (C_1 x_1(t) - y_d(t)) = 0$, under additional assumptions.

Let $\xi_i$ denote the expanded state vector $\xi_i = [\theta_i', x_i']'$, $i = 1, 2$, and note that $\dot{\theta}_i = 0$, we have the following expanded dynamics for system (2),

$$\begin{aligned}
\dot{\xi}_1 &= \begin{bmatrix} \mathbf{0}_{\sigma_1 \times \sigma_1} & \mathbf{0}_{\sigma_1 \times n_1} \\ y_1 \bar{A}_{1,211} + y_2 \bar{A}_{1,212} + \sum_{j=1}^{\check{q}_1} \check{w}_{1,j} \bar{A}_{1,213j} & A_1 \end{bmatrix} \xi_1 \\
&\quad + \begin{bmatrix} \mathbf{0}_{\sigma_1 \times 1} \\ B_1 \end{bmatrix} y_2 + \begin{bmatrix} \mathbf{0}_{\sigma_1 \times q_1} \\ D_1 \end{bmatrix} w_1 + \begin{bmatrix} \mathbf{0}_{\sigma_1 \times \check{q}_1} \\ \check{D}_1 \end{bmatrix} \check{w}_1 \\
&=: \bar{A}_1(y_1, y_2)\xi_1 + \bar{B}_1 y_2 + \bar{D}_1 w_1 + \bar{\check{D}}_1 \check{w}_1 \\
y_1 &= \begin{bmatrix} \mathbf{0}_{1 \times \sigma_1} & C_1 \end{bmatrix} \xi_1 + E_1 w_1 =: \bar{C}_1 \xi_1 + E_1 w_1 \\
\dot{\xi}_2 &= \begin{bmatrix} \mathbf{0}_{\sigma_2 \times \sigma_2} \\ y_2 \bar{A}_{2,211} + u \bar{A}_{2,212} + \sum_{j=1}^{\check{q}_2} \check{w}_{2,j} \bar{A}_{2,213j} + \acute{y}_2 \bar{A}_{2,214} \end{bmatrix. \\
&\qquad \left. \begin{matrix} \mathbf{0}_{\sigma_2 \times n_2} \\ A_2 \end{matrix} \right] \xi_2 + \begin{bmatrix} \mathbf{0}_{\sigma_2 \times 1} \\ B_2 \end{bmatrix} u + \begin{bmatrix} \mathbf{0}_{\sigma_2 \times 1} \\ A_{2,y} \end{bmatrix} \acute{y}_2 \\
&\quad + \begin{bmatrix} \mathbf{0}_{\sigma_2 \times q_2} \\ D_2 \end{bmatrix} w_2 + \begin{bmatrix} \mathbf{0}_{\sigma_2 \times \check{q}_2} \\ \check{D}_2 \end{bmatrix} \check{w}_2 \\
&=: \bar{A}_2(y_1, y_2, u)\xi_2 + \bar{B}_2 u + \bar{A}_{2,y} \acute{y}_2 + \bar{D}_2 w_2 + \bar{\check{D}}_2 \check{w}_2 \\
y_2 &= \begin{bmatrix} \mathbf{0}_{1 \times \sigma_2} & C_2 \end{bmatrix} \xi_2 + E_2 w_2 =: \bar{C}_2 \xi_2 + E_2 w_2
\end{aligned}$$

The worst-case optimization of the cost function (4) can be carried out in two steps as depicted in the following equations.

$$\sup_{\grave{\omega}_1 \in \grave{\mathcal{W}}_1, \, \grave{\omega}_2 \in \grave{\mathcal{W}}_2} J_{\gamma t_f} \leq \sup_{\omega_m \in \mathcal{W}_m} \Bigg( \sum_{i=1}^{2} \sup_{\omega_i \in \mathcal{W}_i | \omega_m \in \mathcal{W}_m} J_{i,\gamma t_f} \Bigg) \tag{6}$$

where $\omega_m$ is the measured signals of the system, and defined as

$$\begin{aligned} \omega_m &:= (y_{1[0,\infty)}, y_{2[0,\infty)}, \check{w}_{1[0,\infty)}, \check{w}_{2[0,\infty)}, Y_{d0}, y_{d[0,\infty)}^{(r_1+r_2)}) \\ &\in \mathcal{W}_m := \mathcal{C} \times \mathcal{C} \times \mathcal{C} \times \mathcal{C} \times \mathbb{R}^{r_1+r_2} \times \mathcal{C}. \end{aligned}$$

The inner supremum operators will be carried out first. We maximize over $\omega_i$ given that the measurement $\omega_m$ is available for estimator design, $i = 1, 2$. In this step, the control input, $u$, is a function only depended on $\omega_m$, then $u$ is an open-loop time function and available for the optimization. Using *cost-to-come* function analysis, we derive the dynamics of the estimators for subsystem $\mathbf{S}_1$ and $\mathbf{S}_2$ independently.

The outer supremum operator will be carried out second. In this step, we use a backstepping procedure to design the controller $\mu$.

This completes the formulation of the robust adaptive control problem.

## III. Adaptive Control Design

In this section, we present the adaptive control design, which involves estimation design and control design. First, we discuss estimation design.

### A. Estimation Design

The design paradigm is the same as [15] to derive the identifier of subsystem $\mathbf{S}_1$, and the only difference is that the actual control input to subsystem $\mathbf{S}_1$ is the output of subsystem $\mathbf{S}_2$, $y_2$. In this paper, we only summarize the estimation design for $\mathbf{S}_2$ briefly due to the paper limitation. In this step, the measurements waveform $\omega_m$ is assumed to be known. Since the control input, $u$, is a causal function of $\omega_m$, then it is known. Again, we will apply the *cost-to-come function* methodology to derive the estimator.

Set function $l_2$ in (5$b$) to $|\xi_2 - \hat{\xi}_2|^2_{\bar{Q}_2} + 2(\xi_2 - \check{\xi}_2)' l_{2,2} + \check{l}_2$, where $\hat{\xi}_2 = [\hat{\theta}_2', \hat{x}_2']'$ is the worst-case estimate for the expanded state $\xi_2$, $\check{\xi}_2$ is the estimate of $\xi_2$, $\bar{Q}_2$ is a matrix-valued weighting function, $l_{2,2}$ and $\check{l}_2$ are two design functions to be introduced later, the cost function of subsystem $\mathbf{S_2}$, (5$b$), is then of a linear quadratic structure. By *cost-to-come* function analysis, we obtain the dynamics of worst-case covariance matrix $\bar{\Sigma}_2$, and state estimator $\check{\xi}_2$. We partition $\bar{\Sigma}_2$ as $\bar{\Sigma}_2 = \begin{bmatrix} \Sigma_2 & \bar{\Sigma}_{2,12} \\ \bar{\Sigma}_{2,21} & \bar{\Sigma}_{2,22} \end{bmatrix}$ and introduce $\Phi_2 := \bar{\Sigma}_{2,21}\Sigma_2^{-1}$ and $\Pi_2 := \gamma^2(\bar{\Sigma}_{2,22} - \bar{\Sigma}_{2,21}\Sigma_2^{-1}\bar{\Sigma}_{2,12})$, then the weighting matrix $\bar{\Sigma}_2$ is positive definite if and only if $\Sigma_2$ and $\Pi_2$ are positive definite. To guarantee the boundedness of $\Sigma_2$, we choose weighing matrix $\bar{Q}_2$ as follows,

$$\begin{aligned}\bar{Q}_2 \quad = \quad & \begin{bmatrix} -\Phi_2' \\ I_{n_2} \end{bmatrix} \gamma^4 \Pi_2^{-1} \Delta_2 \Pi_2^{-1} \begin{bmatrix} -\Phi_2' \\ I_{n_2} \end{bmatrix}' \\ & + \begin{bmatrix} \epsilon_2 \Phi_2' C_2' \gamma^2 \zeta_2^2 C_2 \Phi_2 & \mathbf{0}_{\sigma_2 \times n_2} \\ \mathbf{0}_{n_2 \times \sigma_2} & \mathbf{0}_{n_2 \times n_2} \end{bmatrix} \end{aligned} \tag{7}$$

where $\Delta_2(t) = \gamma^{-2}\beta_{2,\Delta}\Pi_2(t) + \Delta_{2,1}$, with $\beta_{2,\Delta} \geq 0$ being a constant and $\Delta_{2,1}$ being an $n_2 \times n_2$- dimensional positive-definite matrix, and $\epsilon_2$ is a scalar function defined by $\epsilon_2 = K_{2,c}^{-1}\mathrm{Tr}(\Sigma_2^{-1})$ or $\epsilon_2 = 1$. $K_{2,c} \geq \gamma^2 \mathrm{Tr}(Q_{2,0})$ is a design constant, $Q_{2,0}$ is an $\sigma_2 \times \sigma_2$-dimensional positive-definite matrix, and given in (8a). Then the dynamics of $\Sigma_2$, $\Phi_2$, $\Pi_2$ are given by,

$$\dot{\Sigma}_2 = (\epsilon_2 - 1)\Sigma_2 \Phi_2' C_2' \gamma^2 \zeta_2^2 C_2 \Phi_2 \Sigma_2; \Sigma_2(0) = \gamma^{-2} Q_{2,0}^{-1} \tag{8a}$$

$$\begin{aligned}\dot{\Pi}_2 =& (A_2 - \zeta_2^2 L_2 C_2 + \beta_{2,\Delta}/2 I_{n_2})\Pi_2 + \Pi_2 (A_2 - \zeta_2^2 L_2 C_2 \\ & + \beta_{2,\Delta}/2 I_{n_2})' - \Pi_2 C_2' \zeta_2^2 C_2 \Pi_2 + D_2 D_2' \\ & - \zeta_2^2 L_2 L_2' + \gamma^2 \Delta_{2,1}\end{aligned} \tag{8b}$$

$$\begin{aligned}\dot{\Phi}_2 =& A_{2,f}\Phi_2 + y_2 \bar{A}_{2,211} + u \bar{A}_{2,212} + \sum_{j=1}^{\check{q}_2} \bar{A}_{2,213j} \check{w}_{2,j} \\ & + \acute{y}_2 \bar{A}_{2,214}; \qquad \Phi_1(0) = \Phi_{1,0}\end{aligned} \tag{8c}$$

where $A_{2,f} := A_2 - \zeta_2^2 L_2 C_2 - \Pi_2 C_2' C_2 \zeta_2^2$ is Hurwitz. By Lemma 1 in [1], we have the covariance matrix $\Sigma_2$ upper and lower bounded as follows, $K_{2,c}^{-1} I_{\sigma_2} \leq \Sigma_2(t) \leq \Sigma_2(0) = \gamma^{-2} Q_{2,0}^{-1}$, $\gamma^2 \mathrm{Tr}(Q_{2,0}) \leq \mathrm{Tr}(\Sigma_2(t))^{-1} \leq K_{2,c}$, whenever $\Sigma_2$ exists on $[0, t_f]$ and $\Phi_2$ is continuous on $[0, t_f]$. To avoid the calculation of $\Sigma_2^{-1}$ online, we define $s_{2,\Sigma} = \mathrm{Tr}(\Sigma_2^{-1})$.

To guarantee the estimates parameter to be bounded and the estimate of high frequency gain to be bounded away from zero without persistently exciting signals, we introduce the following soft projection design on the parameter estimate.

Define $\rho_2 := \inf\{P_2(\bar{\theta}_2) \mid \bar{\theta}_2 \in \mathbb{R}^{\sigma_2}, b_{2,p0} + \bar{A}_{2,212\,0}\bar{\theta}_2 = 0\}$. By Assumption 4 and [15] we have $1 < \rho_2 \leq \infty$. Fix any $\rho_{2,o} \in (1, \rho_2)$, we define the open set $\Theta_{2,o} := \{\bar{\theta}_2 \mid P_2(\bar{\theta}) < \rho_{2,o}\}$. Our control design will guarantee that the estimate $\check{\theta}_2$ lies in $\Theta_{2,o}$, which immediately implies $|b_{2,p0} + \bar{A}_{2,212\,0}\check{\theta}_2| > c_{2,0} > 0$, for some $c_{2,0} > 0$. Moreover, the convexity of $P_2$ implies the following inequality: $\frac{\partial P_2}{\partial \theta_2}(\check{\theta}_2)\,(\theta_2 - \check{\theta}_2) < 0 \quad \forall \check{\theta}_2 \in \mathbb{R}^{\sigma_2} \backslash \Theta_2$. To incorporate the modifier to the estimates dynamics, we introduce $l_{2,2} = [-(P_{2,r}(\check{\theta}_2)' \ \mathbf{0}_{1 \times n_2}]'$, where

$$P_{2,r}(\check{\theta}_2) := \begin{cases} \frac{\exp\left(\frac{1}{1 - P_2(\check{\theta}_2)}\right)}{\left(\rho_{2,o} - P_2(\check{\theta}_2)\right)^3} \left(\frac{\partial P_2}{\partial \theta_2}(\check{\theta}_2)\right)' & \forall \theta_2 \in \Theta_{2,o} \backslash \Theta_2 \\ 0_{\sigma_2 \times 1} & \forall \theta_2 \in \Theta_2 \end{cases}$$

and the dynamics of $\check{\xi}_2$ is then given as follows,

$$\begin{aligned}\dot{\check{\xi}}_2 =& -\bar{\Sigma}_2 \left[ \ (P_{2,r}(\check{\theta}_2))' \quad \mathbf{0}_{1 \times n_2} \ \right]' + \bar{A}_2 \check{\xi}_2 + \bar{B}_2 u + \bar{A}_{2,y} \acute{y}_2 \\ & + \bar{\bar{D}}_2 \check{w}_2 - \bar{\Sigma}_2 \bar{Q}_2 (\hat{\xi}_2 - \check{\xi}_2) + \zeta_2^2 \left(\gamma^2 \bar{\Sigma}_2 \bar{C}_2' + \bar{L}_2\right)(y_2 - \bar{C}_2 \check{\xi}_2)\end{aligned}$$

where $\check{\xi}_2 = [\check{\theta}_2' \ \check{x}_2]'$ with initial condition $[\check{\theta}_{2,0}' \ \check{x}_{2,0}']'$, and $\bar{L}_2$ is defined as $\bar{L}_2 = [\mathbf{0}_{1 \times \sigma_2} \ L_2']'$. This completes the estimation design of $\mathbf{S_2}$.

Associated with the identifier and estimator of subsystem $\mathbf{S}_i$, $i = 1, 2$, we introduce the value function $W_i : \mathbb{R}^{n_i + \sigma_i} \times \mathbb{R}^{n_i + \sigma_i} \times \mathcal{S}_{+(n_i + \sigma_i)} \to \mathbb{R}$ as $W_i(\xi_i, \check{\xi}_i, \bar{\Sigma}_i) = |\theta_i - \check{\theta}_i|^2_{\Sigma_i^{-1}} + \gamma^2 |x_i - \check{x}_i - \Phi_i(\theta_i - \check{\theta}_i)|^2_{\Pi_i^{-1}}$, whose time derivative is as follows

$$\begin{aligned}\dot{W}_i =& -\epsilon_{i,s}|x_{1,1} - y_d|^2 - \gamma^4 |x_i - \hat{x}_i - \Phi_i(\theta_i - \hat{\theta}_i)|^2_{\Pi_i^{-1}\Delta_i \Pi_i^{-1}} \\ & - \epsilon_i \gamma^2 \zeta_i^2 |\theta_i - \hat{\theta}_i|^2_{\Phi_i' C_i' C_i \Phi_i} + \epsilon_{i,s}|C_i \check{x}_i - y_d|^2 \\ & + |\xi_{i,c}|^2_{\bar{Q}_i} - \gamma^2 \zeta_i^2 |y_i - C_i \check{x}_i|^2 + \gamma^2 |w_i|^2 - \gamma^2 |w_i - w_{i,*}|^2 \\ & + 2(\theta_i - \check{\theta}_i)' P_{i,r}(\check{\theta}_i) + \epsilon_i \epsilon_{i,s} |\theta_i - \hat{\theta}_i|^2_{\Phi_i' C_i' C_i \Phi_i}\end{aligned} \tag{9}$$

where $w_{i,*}$ is the worst-case disturbance, given by $w_{i,*} : \mathbb{R} \times \mathbb{R}^{n_i + \sigma_i} \times \mathbb{R}^{n_i + \sigma_i} \times \mathcal{S}_{+(n_i + \sigma_i)} \longrightarrow \mathbb{R}$

$$\begin{aligned}w_{i,*}(\xi_i, \check{\xi}_i, \bar{\Sigma}_i, w_i) = \ & \zeta_i^2 E_i' (y_i - \bar{C}_i \xi_i) + \gamma^{-2}(I_{q_i} - \zeta_i^2 E_i' E_i) \\ & \bar{D}_i' \bar{\Sigma}_i^{-1} (\xi_i - \check{\xi}_i); \quad i = 1, 2\end{aligned}$$

We note that (9) holds when $\Sigma_i > 0$ and $\theta_i \in \Theta_{i,0}$, and the last term in $\dot{W}_i$ is nonpositive, zero on the set $\Theta_i$ and approaches $-\infty$ as $\check{\theta}_i$ approaches the boundary of the set $\Theta_{i,o}$, which guarantees the boundedness of $\check{\theta}_i$, $i = 1, 2$.

Then (5) can be equivalently written as, $i = 1, 2$:

$$\begin{aligned}J_{i,\gamma t_f} =& \int_0^{t_f} \Big( \epsilon_{i,s}|C_1 \check{x}_1 - y_d|^2 + |\xi_{i,c}|^2_{\bar{Q}_i} + \check{l}_i - \gamma^2 \zeta_i^2 |y_i - C_i \check{x}_i|^2 \\ & - \gamma^2 |w_i - w_{i,*}|^2 - \gamma^2 |\check{w}_{i,a}|^2 \Big) \mathrm{d}\tau - l_{i,0} - |\xi_{i,e}(t_f)|^2_{(\bar{\Sigma}_i(t_f))^{-1}}\end{aligned}$$

where $\xi_{i,e} = \xi_i - \check{\xi}_i$. This completes the identification design step.

### B. Control Design

In this section, we describe the controller design for the uncertain system under consideration. Note that, we ignored some terms in the cost function (5) in the identification step, since they are constant when $y_1$, $y_2$, $\check{w}_1$, $\check{w}_2$ and $\acute{y}_2$ are given. In the control design step, we will include such terms. Then, based on the cost function (5), the controller design is to guarantee that the following supremum is less than or equal to zero for all measurement waveforms,

$$\sup_{\check{w}_1\in\check{\mathcal{W}}_1,\check{w}_2\in\check{\mathcal{W}}_2} J_{\gamma t_f} \leq \sup_{\omega_m\in\mathcal{W}_m}\Bigg\{\int_0^{t_f}\Big(|C_1\check{x}_1-y_d|^2+\sum_{i=1}^{2}\Big(\check{l}_i +|\xi_{i,c}|^2_{\bar{Q}_i}-\gamma^2\zeta_i^2|y_i-C_i\check{x}_i|^2-\gamma^2|\check{w}_{i,a}|^2\Big)\Big)\,\mathrm{d}\tau \quad (10)$$

where function $\check{l}_1$ is part of the weighting function $l_1$, and $\check{l}_2$ is part of the weighting function $l_2$ to be designed, which are constants in the identifier design step and are therefore neglected.

By equation (10), we observe that the cost function is expressed in term of the states of the estimator we derived, whose dynamics are driven by the measurement $y_1$, $y_2$, $\check{w}_1$, $\check{w}_2$, $\acute{y}_2$, the reference trajectory $y_d$, the input $u$, and the worst-case estimate for the expanded state vector $\hat{\xi}_1$ and $\hat{\xi}_2$, which are signals we can either measure or construct. This is then a nonlinear $H^\infty$-optimal control problem under full information measurements. Since $\acute{y}_2 = y_1$ in the adaptive system under consideration, we can equivalently deal with the transformed variables, $v = \left[\begin{array}{cc} v_1' & v_2' \end{array}\right]'$, instead of considering $y_1$, $y_2$, $\check{w}_1$, $\check{w}_2$, and $\acute{y}_2$ as the maximizing variable, and we will attenuate disturbance $v_a = \left[\begin{array}{cc} v_{1,a}' & v_{2,a}' \end{array}\right]'$, and cancel the disturbance $\check{w}_{1,b}$ and $\check{w}_{2,b}$, where $v_i = \left[\begin{array}{ccc} \zeta_i\,(y_i-C_i\check{x}_i) & \check{w}_{i,a}' & \check{w}_{i,b}' \end{array}\right]'$, $v_{i,a} = \left[\begin{array}{cc} \zeta_i\,(y_i-C_i\check{x}_i) & \check{w}_{i,a}' \end{array}\right]'$, $i=1,2$.

For $i = 1,2$, we introduce the matrix $M_{i,f} := \left[\begin{array}{cccc} A_{i,f}^{n_i-1}p_{i,n_i} & \cdots & A_{i,f}p_{i,n_i} & p_{i,n_i} \end{array}\right]$, where $p_{i,n_i}$ is a $n_i$-dimensional vector such that the pair $(A_{i,f},p_{i,n_i})$ is controllable. We note that $\acute{y}_2 = y_1$, then the following $3n_1+4n_2+\check{q}_1+\check{q}_2$-dimensional prefiltering system for $y_1$, $y_2$, $u$, $\check{w}_1$, $\check{w}_2$, and $\acute{y}_2$ generates the $\Phi_1$ and $\Phi_2$ online: $\dot{\eta}_i = A_{i,f}\eta_i + p_{i,n_i}y_i$; $\dot{\eta}_{\check{w}_i,j} = A_{i,f}\eta_{\check{w}_i,j}+p_{i,n_i}\check{w}_{i,j}$; $\eta_{\check{w}_i,j}(0) = \eta_{\check{w}_i,j0}$, $j=1,\cdots,\check{q}_i$; $\dot{\lambda}_i = A_{i,f}\lambda_i + \epsilon_{i,J}p_{1,n_1}y_2 + (1-\epsilon_{i,J})p_{2,n_2}u$; $\lambda_i(0) = \lambda_{i,0}$ $\dot{\eta}_{2,y} = A_{2,f}\eta_{2,y}+p_{2,n_2}\acute{y}_2$; $\eta_{2,y}(0) = \eta_{2,y0}$, $\dot{\lambda}_{i,o} = A_{i,f}\lambda_{i,o}$; $\lambda_{i,o}(0) = p_{i,n_i}$.

$$\begin{aligned}\Phi_i \;=\; & (1-\epsilon_{i,J})\left[\begin{array}{cccc} A_{2,f}^{n_2-1}\eta_{2,y} & \cdots & A_{2,f}\eta_{2,y} & \eta_{2,y}\end{array}\right]M_{i,f}^{-1} \\ & \cdot\bar{A}_{2,214} + \left[\begin{array}{cccc} A_{i,f}^{n_i-1}\lambda_i & \cdots & A_{i,f}\lambda_i & \lambda_i\end{array}\right]M_{i,f}^{-1}\bar{A}_{i,212} \\ & + \left[\begin{array}{cccc} A_{i,f}^{n_i-1}\lambda_{i,o} & \cdots & A_{i,f}\lambda_{i,o} & \lambda_{i,o}\end{array}\right]M_{i,f}^{-1}\Phi_{i,o0} \\ & + \sum_{j=1}^{\check{q}_i}\left[\begin{array}{cccc} A_{i,f}^{n_i-1}\eta_{\check{w}_i,j} & \cdots & A_{i,f}\eta_{\check{w}_i,j} & \eta_{\check{w}_i,j}\end{array}\right]M_{i,f}^{-1} \\ & \cdot\bar{A}_{i,213j} + \left[\begin{array}{cccc} A_{i,f}^{n_i-1}\eta_i & \cdots & A_{i,f}\eta_i & \eta_i\end{array}\right]M_{i,f}^{-1}\bar{A}_{i,211}\end{aligned}$$

where $\eta_{i,0}$, $\lambda_{i,0}$, $\eta_{2,y\,0}$ and $\Phi_{i,o0}$ are the initial conditions such that the above equation holds at $t=0$.

The variables to be designed at this stage include $\check{x}_{2,1}$, $u$, $\xi_{1,c}$, and $\xi_{2,c}$. Note that the structures of $A_1$ and $A_2$ in the dynamics is in strict-feedback form, we will use the backstepping methodology, see [7], to design the control input $u$, which will guarantee the global boundedness of the closed-loop system states and the asymptotic convergence of the tracking error. Since there are the nonnegative definite weighting on $\xi_{1,c}$ and $\xi_{2,c}$ in the cost function (10), we can not use integrator backstepping to design feedback law for $\xi_{1,c}$ and $\xi_{2,c}$. Hence, we set $\xi_{1,c} = \xi_{2,c} = 0$ in the backstepping procedure. After the completion of the backstepping procedure, we will then optimize the choice of $\xi_{1,c}$ and $\xi_{2,c}$ based on the value function obtained. Note that $\Sigma_1$, $\Pi_1$, $s_{1,\Sigma}$, $\check{\theta}_1$, $\Sigma_2$, $\Pi_2$, $s_{2,\Sigma}$, and $\check{\theta}_2$ are always bounded by the design in Section III-A. Since $\Phi_1$ is driven by control $y_2$, and $\Phi_2$ is explicitly driven by $u$, they can not be stabilized in conjunction with $\check{x}_1$ and $\check{x}_2$ in the backstepping design. We will assume they are bounded and prove later they are indeed so under the derived control law.

In view of $y_2 = \zeta_2^{-1}e'_{\check{q}_a+2,\check{q}_{1,a}+2}v_a + \check{x}_{2,1}$, we will treat $\check{x}_{2,1}$ as the virtual control input of subsystem $\mathbf{S_1}$, where $\check{q}_a = \check{q}_{1,a}+\check{q}_{2,a}$ to carry out the backstepping design for subsystem $\mathbf{S_1}$, and then derive the robust adaptive controller, $\mu$ , for closed-loop system. We will show later that the control law $u := \mu$ guarantees the boundedness of the closed-loop system states and the asymptotic convergence of tracking error. For detailed equations of the backstepping design, see the full version of the paper.

For the closed-loop adaptive nonlinear system, we have the following value function, $U = W_1 + W_2 + V_{2,r_2}$, where $V_{2,r_2}$ is the value function defined in the backstepping procedure, and its time derivative is given by

$$\begin{aligned}\dot{U} = & -|x_{1,1}-y_d|^2 - \sum_{j=1}^{2}\Big(\gamma^4|x_j-\hat{x}_j-\Phi_j(\theta_j-\hat{\theta}_j)|^2_{\Pi_j^{-1}\Delta_j\Pi_j^{-1}} \\ & +\epsilon_j\,(\gamma^2\zeta_j^2-1)|\theta_j-\hat{\theta}_j|^2_{\Phi_j'C_j'C_j\Phi_j} - 2\,(\theta_j-\check{\theta}_j)'P_{j,r}(\check{\theta}_j) + |\tilde{\eta}_j|^2_{Y_j} \\ & +\sum_{k=1}^{r_j}\beta_{j,k}z_{j,k}^2 - \gamma^2|w_j|^2 + \gamma^2|w_j-w_{j,opt}|^2 - \gamma^2|\check{w}_j|^2 \\ & +\gamma^2|\check{w}_j-\check{w}_{j,opt}|^2\Big) + \frac{1}{4}\left|\varsigma_{1,(r_1+r_2)}\right|^2_{\bar{Q}_1} - \epsilon_2|\theta_2-\hat{\theta}_2|_{\Phi_2'C_2'C_2\Phi_2} \\ & +\frac{1}{4}\left|\varsigma_{2,r_2}\right|^2_{\bar{Q}_2} - \left|\xi_{1,c}+\frac{1}{2}\varsigma_{1,(r_1+r_2)}\right|^2_{\bar{Q}_1} - \left|\xi_{2,c}+\frac{1}{2}\varsigma_{2,r_2}\right|^2_{\bar{Q}_2}\end{aligned}$$

where $z_{i,j} := \check{x}_{i,j} - \alpha_{i,j}$, $\alpha_{i,j}$ is the virtual control law, $i = 1,2$, $j = 1,\cdots,r_i$; $\varsigma_{1,r_1+r_2}$ and $\varsigma_{2,r_2}$ are functions obtained after backsteping design; $w_{1,opt}$ and $w_{2,opt}$ are the worst case disturbance with respect to the value function $U$, $\nu_{2,r_2}$ is a function obtained after step $r_1+r_2+1$. All variables above are well defined in our full version paper.

Then the optimal choice for the variable $\xi_{i,c}$ and $\hat{\xi}_i$, $i=1,2$, are: $\xi_{1,c*} = -\frac{1}{2}\varsigma_{1,r_1+r_2} \iff \hat{\xi}_{1,*} = \check{\xi}_1 - \frac{1}{2}\varsigma_{1,r_1+r_2}$; $\xi_{2,c*} = -\frac{1}{2}\varsigma_{2,r_2} \iff \hat{\xi}_{2,*} = \check{\xi}_2 - \frac{1}{2}\varsigma_{2,r_2}$, which yields that the closed-loop system is dissipative with storage function $U$ and supply rate: $-|x_{1,1}-y_d|^2+\gamma^2|w_1|^2+\gamma^2|w_2|^2+\gamma^2|\check{w}_{1,a}|^2+\gamma^2|\check{w}_{2,a}|^2$. This optimal choice for $\hat{\xi}_i$, $i=1,2$, results in the first proposed adaptive control law.

The optimal choice of $\xi_{i,c*}$ is generally very complicated. We could simply choose $\xi_{i,c} = 0$, i.e., $\hat{\xi}_i = \check{\xi}_i$. Since it will result in a simplified identifier structure, this suboptimal choice of $\hat{\xi}_i$ may be preferable over the optimal one. This suboptimal choice for $\hat{\xi}_i$ results in the second proposed adaptive control law, $i=1,2$.

This completes the adaptive controller design step. We will discuss the robustness and tracking properties of the proposed adaptive control laws in the next section.

## IV. Main Result

In this Section, we present the main result by stating two theorems.

For the first adaptive control law, with the optimal choice of $\xi_{i,c*}$, the closed-loop system dynamics with initial condition $X_0$ are

$$\dot{X} \;=\; F + G(X)\left[\begin{array}{cc} w_1' & w_2' \end{array}\right] + G_{\check{w}}(X)\cdot\left[\begin{array}{cc} \check{w}_1' & \check{w}_2' \end{array}\right]$$

where $F$, $G$ and $G_M$ are smooth mapping of $\mathcal{D}\times\mathbb{R}$, $\mathcal{D}$ and $\mathcal{D}$, respectively; and the initial condition $X_0 \in \mathcal{D}_0 := \{X_0 \in \mathcal{D} \mid \theta_i\in\Theta_i, \check{\theta}_{i,0}\in\Theta_i, \Sigma_i(0) = \gamma^{-2}Q_{i,0}^{-1} > 0, \mathrm{Tr}\left((\Sigma_i(0))^{-1}\right) \leq K_{i,c}, s_{i,\Sigma}(0) = \gamma^2\mathrm{Tr}(Q_{i,0}); i = 1,2\}$. And $U$ satisfies an Hamilton-Jacobi-Isaacs equation, $\forall X\in\mathcal{D}, \forall y_d^{(r_1+r_2)}\in\mathbb{R}$.

$$\begin{aligned}&\frac{\partial U}{\partial X}(X)F(X,y_d^{(r_1+r_2)}) + \frac{1}{4\gamma^2}\frac{\partial U}{\partial X}(X)\left[\begin{array}{cc} G(X) & G_{\check{w}}(X)\end{array}\right] \\ &\quad\left[\begin{array}{cc} G(X)' & G_{\check{w}}(X)'\end{array}\right]'\left(\frac{\partial U}{\partial X}(X)\right)' + Q(X,y_d^{(r_1+r_2)}) = 0\end{aligned}$$

where $Q : \mathcal{D} \times \mathbb{R} \to \mathbb{R}$ is smooth and given by

$$Q(X, y_d^{(r_1+r_2)}) = |x_{1,1} - y_d|^2 + \sum_{j=1}^{2} \Big( -2\,(\theta_j - \check{\theta}_j)' P_{j,r}(\check{\theta}_j)$$
$$\epsilon_j\,(\gamma^2 \zeta_j^2 - 1)|\theta_j - \hat{\theta}_j|^2_{\Phi_j' C_j' C_j \Phi_j} + \frac{1}{4}\left|\varsigma_{j,r_j}\right|^2_{\bar{Q}_j} + |\tilde{\eta}_j|^2_{Y_j}$$
$$+ \sum_{k=1}^{r_j} \beta_{j,k} z_{j,k}^2 \Big) + \epsilon_2 |\theta_2 - \hat{\theta}_2|_{\Phi_2' C_2' C_2 \Phi_2}$$

The closed-loop adaptive system possesses a strong stability property, which will be stated precisely in the following theorem.

*Theorem 1:* Consider the robust adaptive control problem formulated in Section II with Assumptions $1-5$ holding. The robust adaptive controller $\mu$ with the optimal or suboptimal choice of $\xi_{i,c}$, achieves the following strong robustness properties for the closed-loop system.

1) Given $c_w \geq 0$, and $c_d \geq 0$, there exists a constant $c_c \geq 0$ and compact sets $\Theta_{1,c} \subset \Theta_{1,o}$, and $\Theta_{2,c} \subset \Theta_{2,o}$ such that for any uncertainty $(x_{1,0}, \theta_1, \grave{w}_{1,[0,\infty)}, \check{w}_{1,[0,\infty)}, Y_{d0}, y_{d[0,\infty)}^{(r_1+r_2)}) \in \grave{\mathcal{W}}_1$ and $(x_{2,0}, \theta_2, \grave{w}_{2,[0,\infty)}, \check{w}_{1,[0,\infty)}) \in \grave{\mathcal{W}}_2$ with $|x_{i,0}| \leq c_w$; $|\grave{w}_i(t)| \leq c_w$; $|\check{w}_i(t)| \leq c_w$; $|Y_d(t)| \leq c_d$; $\forall t \in [0,\infty), i = 1,2$. All closed-loop state variables $x_1$, $\check{x}_1$, $\check{\theta}_1$, $\Sigma_1$, $s_{1,\Sigma}$, $\eta_1$, $\eta_{1,d}$, $\Phi_{1,u}$, $x_2$, $\check{x}_2$, $\check{\theta}_2$, $\Sigma_2$, $s_{2,\Sigma}$, $\eta_2$, $\eta_{2,d}$, $\Phi_{2,u}$ are bounded as follows, $\forall t \in [0,\infty)$, $|x_i(t)| \leq c_c$; $|\check{x}_i(t)| \leq c_c$; $\check{\theta}_i(t) \in \Theta_{i,c}$; $|\eta_i(t)| \leq c_c$; $|\eta_{i,d}(t)| \leq c_c$; $|\Phi_{i,u}(t)| \leq c_c$, $K_{i,c}^{-1} I \leq \Sigma_i(t) \leq \gamma^{-2} Q_{i,0}^{-1}; \gamma^2 \mathrm{Tr}(Q_{i,0}) \leq s_{i,\Sigma}(t) \leq K_{i,c}; i = 1,2$. The inputs are also bounded $|u(t)| \leq c_u$, and $\hat{\xi}_1 \leq c_u$, $\hat{\xi}_2 \leq c_u$, $\forall t \in [0,\infty)$, for some constant $c_u \geq 0$. Furthermore, there exists constant $c_\lambda \geq 0$ such that $|\lambda_{i,0}(t)| \leq c_\lambda$, $|\lambda_i(t)| \leq c_\lambda$, $|\eta_{\check{w}_{i,j}}(t)| \leq c_\lambda$, $i = 1,2$, $j = 1,\cdots,\check{q}_i$, and $|\eta_{2,y}(t)| \leq c_\lambda$, $\forall t \geq 0$.
2) For any uncertainty $(x_{1,0}, \theta_1, \grave{w}_{1,[0,\infty)}, \check{w}_{1,[0,\infty)}, Y_{d0}, y_{d[0,\infty)}^{(r_1+r_2)}) \in \grave{\mathcal{W}}_1$, and $(x_{2,0}, \theta_2, \grave{w}_{2,[0,\infty)}, \check{w}_{2,[0,\infty)}) \in \grave{\mathcal{W}}_2$ the controller $\mu \in \mathcal{M}$ achieves disturbance attenuation level $\gamma$ with respect to $w_1$ and $w_2$, arbitrary disturbance attenuation level $\check{\gamma}$ with respect to $\check{w}_{1,a}$ and $\check{w}_{2,a}$, and disturbance attenuation level zero with respect to $\check{w}_{1,b}$ and $\check{w}_{2,b}$.
3) For any uncertainty $(x_{1,0}, \theta_1, \grave{w}_{1,[0,\infty)}, \check{w}_{1,[0,\infty)}, Y_{d0}, y_{d[0,\infty)}^{(r_1+r_2)}) \in \grave{\mathcal{W}}_1$, and $(x_{2,0}, \theta_2, \grave{w}_{2,[0,\infty)}, \check{w}_{2,[0,\infty)}) \in \grave{\mathcal{W}}_2$ with $\grave{w}_{1,[0,\infty)} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, $\grave{w}_{2,[0,\infty)} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, $\check{w}_{1,a[0,\infty)} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, $\check{w}_{2,a[0,\infty)} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, $\check{w}_{1,b[0,\infty)} \in \mathcal{L}_\infty$, $\check{w}_{2,b[0,\infty)} \in \mathcal{L}_\infty$, and $Y_{d[0,\infty)} \in \mathcal{L}_\infty$, the noiseless output of the system, $x_{1,1}$, asymptotically tracks the reference trajectory, $y_d$, *i.e.*, $\lim_{t\to\infty}(x_{1,1}(t) - y_d(t)) = 0$.

Based on Theorem 1, we note that the above controller $\mu$ can achieve disturbance attenuation level $\gamma$ with respect to $w_1$ and $w_2$. We have the following theorem about the ultimate lower bound on the achievable performance level $\gamma$ for the adaptive system.

*Theorem 2:* Consider the robust adaptive control problem formulated in Section III, under the assumptions of Theorem 1, the ultimate lower bound on the achievable performance level is only relevant to the Subsystem $\mathbf{S_1}$, i.e., $\gamma \geq \zeta_1^{-1}$ or $\gamma > \zeta_1^{-1}$.

## V. Conclusions

In this paper, we present the robust adaptive control for the same linear system as [1] except that we assume part of the disturbance are measured. We assume that the subsystem $\mathbf{S}_1$ and $\mathbf{S}_2$ satisfy the same assumptions as [1], and we use the same design method as [1] to derive the controller, where the measures of transient response, disturbance attenuation, and asymptotic tracking are incorporated into a single game-theoretic cost function, and then *cost-to-come function* analysis is applied to obtain the finite dimensional estimators of $\mathbf{S_1}$ and $\mathbf{S_2}$ independently. The integrator backstepping methodology is finally applied to obtain the controller. The controller achieves the same result as [1], namely the total stability of the closed-loop system, the desired disturbance attenuation level, and asymptotic tracking of the reference trajectory when the disturbance is of finite energy and uniformly bounded. In addition, the proposed controller may achieve arbitrary positive disturbance attenuation level with respect to the measured disturbances by proper scaling. The contribution of the measurements of part of the disturbance inputs is that we can design an adaptive controller with disturbance feedforward structure with respect to $\check{w}_{1,b}$ and $\check{w}_{2,b}$ to eliminate their effect on the squared $\mathcal{L}_2$ norm of the tracking error. Moreover, the asymptotic tracking is achieved even if the measured disturbances are only uniformly bounded without requiring them to be of finite energy.


## Acknowledgement

The authors are grateful to Dr. Zigang Pan for his substantial contributions and comments to this paper.


## References


[1] S. Zeng and E. Fernandez, "Adaptive controller design and disturbance attenuation for sequentially interconnected siso linear systems under noisy output measurements," in *Proceedings of the 2007 American Control Conference*, New York, NY, 9–13, July 2007, pp. 1925–1930.

[2] G. C. Goodwin and K. S. Sin, *Adaptive Filtering, Prediction and Control*. Englewood Cliffs: Prentice-Hall, 1984.

[3] P. R. Kumar, "A survey of some results in stochastic adaptive control," *SIAM Journal on Control and Optimization*, vol. 23, no. 3, pp. 329–380, 1985.

[4] C. E. Rohrs, L. Valavani, M. Athans, and G. Stein, "Robustness of continuous-time adaptive control algorithms in the presence of unmodeled dynamics," *IEEE Transactions on Automatic Control*, vol. 30, pp. 881–889, 1985.

[5] A. Isidori, *Nonlinear Control Systems*, 3rd ed. London: Springer-Verlag, 1995.

[6] I. Kanellakopoulos, P. V. Kokotović, and A. S. Morse, "Systematic design of adaptive controllers for feedback linearizable systems," *IEEE Transactions on Automatic Control*, vol. 36, pp. 1241–1253, 1991.

[7] M. Krstić, I. Kanellakopoulos, and P. V. Kokotović, *Nonlinear and Adaptive Control Design*. New York, NY: Wiley, 1995.

[8] P. A. Ioannou and J. Sun, *Robust Adaptive Control*. Upper Saddle River, NJ: Prentice Hall, 1996.

[9] G. Didinsky, "Design of minimax controllers for nonlinear systems using cost-to-come methods," Ph.D. dissertation, University of Illinois, Urbana, IL, August 1994.

[10] T. Başar and P. Bernhard, *$H^\infty$-Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach*, 2nd ed. Boston, MA: Birkhäuser, 1995.

[11] Z. Pan and T. Başar, "Parameter identification for uncertain linear systems with partial state measurements under an $H^\infty$ criterion," *IEEE Transactions on Automatic Control*, vol. 41, pp. 1295–1311, September 1996.

[12] I. E. Tezcan and T. Başar, "Disturbance attenuating adaptive controllers for parametric strict feedback nonlinear systems with output measurements," *Journal of Dynamic Systems, Measurement and Control, Transactions of the ASME*, vol. 121, no. 1, pp. 48–57, March 1999.

[13] Z. Pan and T. Başar, "Adaptive controller design and disturbance attenuation for SISO linear systems with noisy output measurements," University of Illinois at Urbana-Champaign, Urbana, IL, CSL report, July 2000.

[14] G. Arslan and T. Başar, "Disturbance attenuating controller design for strict-feedback systems with structurally unknown dynamics," *Automatica*, vol. 37, no. 8, pp. 1175–1188, August 2001.

[15] S. Zeng and Z. Pan, "Adaptive controller design and disturbance attenuation for SISO linear systems with noisy output measurements and partly measured disturbances," in *Proceedings of the 2004 American Control Conference*, Boston, MA, June–July 2004, pp. 4523–4528.