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First-State Contractive Model Predictive Control of Nonholonomic Mobile Robots

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Abstract— **In this paper, a first-state contractive (FSC) model predictive control (MPC) algorithm is developed for the trajectory tracking and point stabilization problems of nonholonomic mobile robots. Different from other stabilizing MPC methods, which address stability by adding terminal state penalties in the performance index and imposing constraints on the terminal state at the end of the prediction horizon, the proposed MPC algorithm guarantees its stability by adding a contractive constraint on the first state at the beginning of the prediction horizon. The resulting MPC scheme is denoted as *first-state contractive* MPC (FSC-MPC). In the absence of disturbances, it can be shown that the proposed algorithm is exponentially stable. Simulation results are provided to verify the effectiveness of the method. Moreover, it is shown that the FSC-MPC algorithm has simultaneous tracking and point stabilization capability.**

I. Introduction

Unmanned ground vehicles (UGV) can provide a promising and efficient alternative to existing techniques in a wide range of applications. Due to the advancement in electronics and computing, small UGVs with satisfactory sensing and computational capabilities now can be built within a reasonable budget. The challenge here lies in designing control algorithms to handle complex environments and a wide variety of requirements.

A nonholonomic model (*e.g.*, unicycle) is commonly adopted to describe vehicle's kinematics in UGV motion coordination. Therefore, fundamental control problems, trajectory tracking and point stabilization of nonholonomic mobile robots, are inevitably encountered. During the past decades, these problems have received a lot of attention and numerous control algorithms can be found in the existing literature. A detailed summary of developments in control of nonholonomic systems can be found in [1].

The trajectory tracking problem focuses on stabilizing robots to a time-varying trajectory. Nonlinear feedback controllers are mostly found in the literature. Early results include [2], [3], in which local asymptotic controllers are developed. Other techniques, such as sliding mode [4] and output feedback linearization [5] have been widely used. However, according to the authors in [6], the nonlinear internal dynamics of the closed-loop system under output feedback linearization controllers exhibit unstable properties when robots track a trajectory moving backward. So far, input constraints are usually ignored in the existing approaches.

The point stabilization problem, which considers stabilizing robots to a final goal posture, is more challenging. According to Brockett's theorem [7], a smooth time-invariant feedback control law doesn't exist. Many researchers have contributed a lot of algorithms to overcome these difficulties and most of the existing approaches can be classified into two categories: (i) discontinuous feedback laws, and (ii) time-varying algorithms. To mention a few, see [3], [8] for discontinuous feedback controllers and [9], [10] for time-varying controllers. Other techniques, such as dynamic feedback linearization [11] are also found in the literature. An interesting result is reported in [12] that with a special choice of state-space variables, smooth feedback controllers can be designed. Some drawbacks of the existing approaches are reported in [8], [13], such as the slow convergence of time-varying control laws and the complex design of discontinuous controllers. In addition, most of the existing approaches do not consider input constraints.

Only a few controllers [14], [15], which can handle the tracking and stabilization problems in the same control structure, are found in the literature. Common approaches are switching controllers between tracking and point stabilization.

Recently, model predictive control (MPC) or receding horizon control (RHC) has gained more and more attention in the control community. The inherent ability of MPC to handle constrained systems makes it a promising technique for the control of nonholonomic mobile robots. MPC controllers are reported in [15] for trajectory tracking and [16] for point stabilization. However, because of the stability condition, the MPC controller in [15] cannot track a trajectory moving backward.

In this paper, we proposed a novel MPC approach for the control of nonholonomic mobile robots. From the literature, most stabilizing MPC methods address stability by adding terminal state penalties in the performance index and/or imposing constraints on the terminal state at the end of the prediction horizon. However, the proposed MPC algorithm guarantees its stability by adding a contractive constraint on the first state at the beginning of the prediction horizon. More specifically, the contributions of this paper are threefold: (i) the exponential stability of our MPC controller is guaranteed by adding a first-state contractive constraint. This means that the convergence is faster and no terminal region calculation is required; (ii) tracking a trajectory moving backward is no longer a problem under our MPC controller and (iii),

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the proposed MPC controller has simultaneous tracking and point stabilization capability, in contrast to most of the existing controllers in the literature.

The rest of the paper is organized as follows. Section II introduces the robot kinematic model and the trajectory tracking and point stabilization problems of a nonholonomic mobile robot. A *first-state contractive* MPC algorithm is proposed in Section III. Stability results of the proposed algorithm are found in Section IV. In Section V, simulation results are provided to show the effectiveness of the method. Finally, concluding remarks and future work are given in Section VI.

II. PRELIMINARIES

This paper deals with the problem of designing control laws for the motion control of nonholonomic mobile robots. In this section, a brief introduction of the kinematic model used for the mobile robots and the two fundamental classes of problems, trajectory tracking and point stabilization are given.

A. Kinematic Model

Consider the planar motion of mobile robots under the nonholonomic constraint of pure rolling and non-slipping, the kinematic model is given as follows (see Figure 1)

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}, \qquad (1)$$

where $(x, y) \in \mathbb{R}^2$ denotes the position of the robot in a Cartesian coordinate frame, $\theta \in (-\pi, \pi]$ represents the orientation of the robot with respect to the positive X axis, and $v \in \mathcal{V} \subseteq \mathbb{R}$ and $\omega \in \mathcal{W} \subseteq \mathbb{R}$ are the control inputs representing linear and angular velocities, respectively.

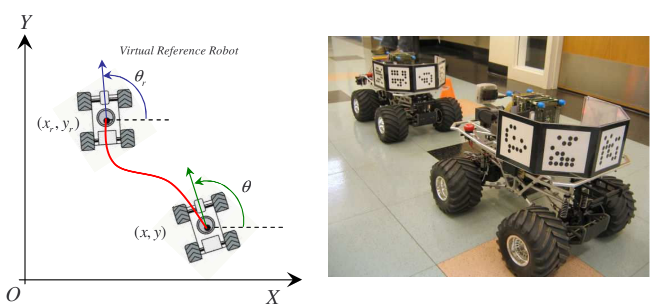

Fig. 1. Nonholonomic mobile robots.

Although there is no consideration of motor dynamics and other mechanical effects, this simplified model is sufficient to describe the nonholonomic mobile robots' motion.

Since system (1) falls in the form of driftless systems

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \omega, \qquad (2)$$

and the accessibility rank condition is globally satisfied [17]

$$\text{rank}\{g_1, g_2, [g_1, g_2]\} = \text{rank} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ \sin\theta & 0 & -\cos\theta \\ 0 & 1 & 0 \end{bmatrix} = 3, \qquad (3)$$

where $g_1 = [\cos\theta \;\; \sin\theta \;\; 0]^T$, $g_2 = [0 \;\; 0 \;\; 1]^T$ and $[g_1, g_2]$ is the Lie bracket of g_1 and g_2, system (1) is controllable. Note that, for nonlinear systems, the existence of continuous time-invariant state feedback control laws cannot be implied from the controllability.

B. Trajectory Tracking

Let a triplet $\mathbf{z}_c = [x \;\; y \;\; \theta]^T$ describe the position and the orientation of a mobile robot. The reference trajectories can be described by a virtual reference robot with a state vector $\mathbf{z}_r = [x_r \;\; y_r \;\; \theta_r]^T$, an input vector $\mathbf{u}_r = [v_r \;\; \omega_r]^T$ and the kinematic model (see Figure 1)

$$\dot{\mathbf{z}}_r = \begin{bmatrix} \dot{x}_r \\ \dot{y}_r \\ \dot{\theta}_r \end{bmatrix} = \begin{bmatrix} \cos\theta_r & 0 \\ \sin\theta_r & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u}_r. \qquad (4)$$

Then the trajectory tracking problem can be defined [3].

***Definition** 2.1:* The trajectory tracking problem, under the assumption that the virtual reference robot is not at rest ($v_r = \omega_r = 0$) when $t \to +\infty$, is to find a feedback control law $\mathbf{u} = [v \;\; \omega]^T$, such that

$$\lim_{t \to \infty} (\mathbf{z}_r - \mathbf{z}_c) = 0,$$

with any initial robot posture $\mathbf{z}_c(0)$.

By transforming the reference state $\mathbf{z}_r$ in a local coordinate system attached to the tracking robot, an error state $\mathbf{z}_e$ can be defined [2]

$$\mathbf{z}_e := \begin{bmatrix} x_e \\ y_e \\ \theta_e \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} (\mathbf{z}_r - \mathbf{z}_c). \qquad (5)$$

Taking the derivative of (5) and rearranging with (1), (4), the error model becomes

$$\begin{aligned} \dot{x}_e &= \omega y_e - v + v_r \cos\theta_e, \\ \dot{y}_e &= -\omega x_e + v_r \sin\theta_e, \\ \dot{\theta}_e &= \omega_r - \omega. \end{aligned} \qquad (6)$$

Let us define $\mathbf{u}_e$,

$$\mathbf{u}_e := \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -v + v_r \cos\theta_e \\ \omega_r - \omega \end{bmatrix}, \qquad (7)$$

and then linearize system (6) about the the equilibrium point ($\mathbf{z}_e = 0$, $\mathbf{u}_e = 0$). We obtain

$$\dot{\mathbf{z}}_e = \begin{bmatrix} 0 & \omega_r & 0 \\ -\omega_r & 0 & v_r \\ 0 & 0 & 0 \end{bmatrix} \mathbf{z}_e + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u}_e. \qquad (8)$$

The controllability of system (8) can be easily checked. However, when the virtual reference robot stops ($v_r = \omega_r = 0$), the controllable property is lost.

C. Point Stabilization

For the point stabilization problem, one can have the following definition.

***Definition** 2.2:* Given an arbitrary constant reference position and orientation $\mathbf{z}_d = [x_d \;\; y_d \;\; \theta_d]^T$, the point stabilization problem is to find a feedback control law $\mathbf{u} = [v \;\; \omega]^T$, such that

$$\lim_{t\to\infty} (\mathbf{z}_d - \mathbf{z}_c) = 0,$$

with any initial robot posture $z_c(0)$.

Without loss of generality, we use $\mathbf{z}_d = [0 \;\; 0 \;\; 0]^T$ as the constant reference posture (since by coordinate transforming, any arbitrary posture can be transformed to $[0 \;\; 0 \;\; 0]^T$). Then the problem becomes to find a feedback control law which drives the system (1) back to the origin aligning with the X axis.

It is a well-known result that a smooth time-invariant feedback control law does not exist for the point stabilization problem [7]. However, with the analysis in Section II-A, system (1) is still controllable.

Consider a Cartesian to polar coordinate transformation [12] (see Figure 2), a polar state $\mathbf{z}_q = [l \;\; \phi \;\; \alpha]^T$ can be defined

$$\begin{aligned} l &= \sqrt{x^2 + y^2} \\ \phi &= \text{arctan2}(-y, -x) \\ \alpha &= \phi - \theta \end{aligned} \tag{9}$$

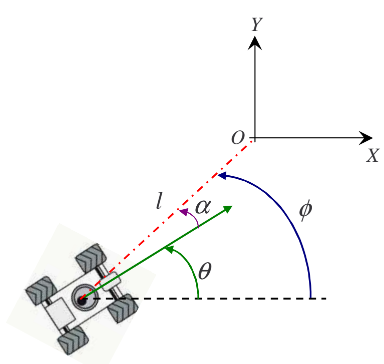

Fig. 2. Coordinate transformation.

Then the kinematic model (1) becomes

$$\begin{aligned} \dot{l} &= -v\cos\alpha \\ \dot{\phi} &= \frac{v\sin\alpha}{l} \\ \dot{\alpha} &= -\omega + \frac{v\sin\alpha}{l} \end{aligned} \tag{10}$$

Note, when $l = 0$, which means that robot reaches the origin, the new kinematic model is not defined.

III. First-State Contractive MPC

Without considering disturbances and model uncertainties, systems like (6) and (10) can be generally expressed and converted into the following nonlinear set of difference equations

$$\mathbf{z}(k+1) = \mathbf{f}(\mathbf{z}(k), \mathbf{u}(k)), \quad \mathbf{z}(0) = \mathbf{z}_0, \tag{11}$$

with a state vector $\mathbf{z}(k) \in \mathcal{Z}$ and an input vector $\mathbf{u}(k) \in \mathcal{U}$, $k \in \mathbb{Z}^*$. $\mathcal{Z} \subset \mathbb{R}^m$ is the state constraints which contains the origin in its interior. $\mathcal{U} \subset \mathbb{R}^n$ is the input constraints which is a compact subset of $\mathbb{R}^n$ containing the origin in its interior. Usually, we have $\mathcal{U} = \{\mathbf{u} \in \mathbb{R}^n : \mathbf{u}_{\min} \le \mathbf{u} \le \mathbf{u}_{\max}\}$. $\mathbf{u}_{\min}$ and $\mathbf{u}_{\max}$ are known constants in $\mathbb{R}^n$. Function $\mathbf{f} : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ is assumed to be continuous.

The control goal is to find $\mathbf{u}(k)$ which drives the system (11) toward the equilibrium ($\mathbf{z}(k) = 0$ and $\mathbf{u}(k) = 0$).

To obtain the current control $\mathbf{u}(k)$ at time t_k, where k is a nonnegative integer ($k \in \mathbb{Z}^*$), a finite-horizon optimal control problem

$$\begin{aligned} \min_{\mathbf{u}} \quad & J_H(\mathbf{z}, k, \mathbf{u}), \\ \text{subject to:} \quad & \mathbf{z}(k+1) = \mathbf{f}(\mathbf{z}(k), \mathbf{u}(k)), \\ & \mathbf{z}(k) \in \mathcal{Z}, \\ & \mathbf{u}(k) \in \mathcal{U}, \end{aligned} \tag{12}$$

must be solved online for an MPC algorithm. The performance index $J_H(\mathbf{z}, k, \mathbf{u})$ is defined as

$$J_H(\mathbf{z}, k, \mathbf{u}) := \sum_{i=1}^{H} L(\mathbf{z}(k+i; k), \mathbf{u}(k+i-1; k)), \tag{13}$$

where $H \in \mathbb{N}$ is the horizon length (for simplicity, the prediction horizon equals the control horizon in this paper). The incremental cost is defined as

$$L(\mathbf{z}, \mathbf{u}) := \|\mathbf{z}\|_Q^2 + \|\mathbf{u}\|_R^2, \tag{14}$$

where $\|\mathbf{z}\|_Q$ and $\|\mathbf{u}\|_R$ denote the weighted 2-norm, which are defined as $\|\mathbf{z}\|_Q^2 := \mathbf{z}^T Q \mathbf{z}$ and $\|\mathbf{u}\|_R^2 := \mathbf{u}^T R \mathbf{u}$. Q and R are positive definite symmetric matrices of appropriate dimensions.

Since a finite horizon is used, the controller found in (12) is not guaranteed to be stable. Many researchers have contributed to the stability of nonlinear MPC with some important methods (see [18] for a detailed discussion).

To achieve stability, the core idea behind the methods mentioned above is to add terminal state penalties in the performance index and impose constraints on the terminal state at the end of the prediction horizon. Therefore, those methods can be denoted as *terminal-state constrained* MPC (TSC-MPC). However, in the implementation of most MPC schemes, only the first control of the control sequence yield by optimization is applied to the plant at each sampling instance. All the other controls are discarded. Only the first state at the beginning of the prediction horizon is directly affected by this implementation.

Motivated by this observation and the *contractive* MPC scheme developed in [19], a new MPC algorithm is proposed here. To be specific, we obtain the current control $\mathbf{u}(k)$

at time t_k by solving the following finite-horizon optimal control problem online

$$\begin{aligned} \min_{\mathbf{u}} \quad & J_H(\mathbf{z}, k, \mathbf{u}), \\ \text{subject to:} \quad & \mathbf{z}(k+1) = \mathbf{f}(\mathbf{z}(k), \mathbf{u}(k)), \\ & \mathbf{z}(k) \in \mathcal{Z}, \\ & \mathbf{u}(k) \in \mathcal{U}, \\ & \|\mathbf{z}(k+1)\|_{\hat{P}} \leq \rho \|\mathbf{z}(k)\|_{\hat{P}}, \end{aligned} \tag{15}$$

where $J_H(\mathbf{z}, k, \mathbf{u})$ is defined in (13) and (14). $\hat{P}$ is a positive definite symmetric matrix and $\rho \in (0,\ 1)$.

Note, the last inequality constraint in (15) can be called *first-state contractive* constraint. This means the first state at the beginning of the prediction horizon, $\mathbf{z}(k+1)$, is contracted in norm with respect to the current state, $\mathbf{z}(k)$. Therefore, the proposed MPC algorithm can be denoted as *first-state contractive* MPC (FSC-MPC). The user adjustable parameter ρ is called *contractive parameter*, which addresses the contraction rate.

The FSC-MPC controller can be implemented as follows.

FSC-MPC Algorithm

Data: prediction horizon $H \in \mathbb{N}$; sampling time $\delta_T \in \mathbb{R}^+$; weights Q, R, $\hat{P} > 0$; constraints $\mathbf{u}_{\min}$, $\mathbf{u}_{\max} \in \mathbb{R}^m$; contractive parameter $\rho \in (0,\ 1)$; $k \in \mathbb{Z}^*$.

Step 0: set $k = 0$; set initial control prediction $\hat{\mathbf{u}}(i;k) = 0$, $i \in [1, \ldots, H-1]$.

Step 1: measure the states $\mathbf{z}(k)$ at time t_k; with control prediction $\hat{\mathbf{u}}(i;k)$, solve the optimal control problem (15) and obtain a control sequence $\mathbf{u}^*(i;k)$.

Step 2: apply the first control $\mathbf{u}^*(1;k)$ in the control sequence $\mathbf{u}^*(i;k)$ to system (11) for the time interval $[t_k,\ t_{k+1}]$, where $t_{k+1} = t_k + \delta_T$.

Step 3: update the control prediction as follows

$$\hat{\mathbf{u}}(i;k) = \begin{cases} \mathbf{u}^*(i+1;k) & i \in [1, \ldots, H-2] \\ \mathbf{u}^*(i;k) & i = H-1 \end{cases};$$

set $k = k+1$; go back to **Step 1**.

Note, in **Step 1**, an assumption is that for all $k \in \mathbb{Z}^*$, a feasible solution of the optimal control problem (15), satisfying all the constraints, always exists. However, global optimal solution is not strictly required here. Any feasible or local optimal solutions is acceptable. This approach might compromise the performance, but the stability property of the algorithm will not be affected.

IV. Stability Results

In this section, the stability of FSC-MPC algorithm will be proven. Before we give the main results, let's make the following assumptions.

***Assumption** 4.1:* There exists a constant $\beta \in (0,\ \infty)$ such that for all $\mathbf{z}(k) \in B_\beta := \{\mathbf{z} \in \mathcal{Z} | \|\mathbf{z}\|_{\hat{P}} \leq \beta\}$, a *contractive parameter* $\rho \in (0,\ 1)$ can be found so that at time t_k, a feasible solution of the optimal control problem (15), satisfying all the constraints, always exists for all $k \in \mathbb{Z}^*$.

***Assumption** 4.2:* For all $t \in [t_k,\ t_{k+1}]$, $k \in \mathbb{Z}^*$, there exists a constant $\kappa \in (0,\ \infty)$, such that the transient state, $\mathbf{z}(t)$, satisfies $\|\mathbf{z}(t)\|_{\hat{P}} \leq \kappa \|\mathbf{z}(k)\|_{\hat{P}}$.

Note, Assumption 4.2 means that systems with finite escape time are not under consideration.

***Theorem** 4.3:* Suppose that the optimal control problem is feasible at time t_0 and Assumptions 4.1 and 4.2 are satisfied. The FSC-MPC algorithm described in Section III for system (11) is exponentially stable in the sense that the state trajectory of the closed-loop system satisfies the following inequality

$$\|\mathbf{z}(t)\|_{\hat{P}} \leq \kappa \|\mathbf{z}(0)\|_{\hat{P}} e^{-\frac{(1-\rho)}{\delta_T}(t-t_0)}, \tag{16}$$

where δ_T is the sampling time.

Proof: Since the optimal control problem is feasible at time t_0, from Assumption 4.1, the optimal control problem is feasible at time t_k, $k \in \mathbb{Z}^*$. Therefore, we have

$$\|\mathbf{z}(k)\|_{\hat{P}} \leq \rho \|\mathbf{z}(k-1)\|_{\hat{P}} \leq \cdots \leq \rho^k \|\mathbf{z}(0)\|_{\hat{P}}. \tag{17}$$

Now with Assumption 4.2 and (17), $\mathbf{z}(t)$ satisfies the following inequality

$$\|\mathbf{z}(t)\|_{\hat{P}} \leq \kappa \rho^k \|\mathbf{z}(0)\|_{\hat{P}}, \tag{18}$$

where $t \in [t_k,\ t_{k+1}]$, for all $k \in \mathbb{Z}^*$.

Since $\rho \in (0,\ 1)$, we have $e^{(\rho-1)} - \rho \geq 0$, which means $e^{(\rho-1)k} \geq \rho^k \geq 0$, for all $k \in \mathbb{Z}^*$. Inequality (18) can be rewritten as follows

$$\|\mathbf{z}(t)\|_{\hat{P}} \leq \kappa \|\mathbf{z}(0)\|_{\hat{P}} e^{-(1-\rho)k}. \tag{19}$$

Since $k = (t_k - t_0)/\delta_T$ and $(t - t_0)/\delta_T \leq (t_k - t_0)/\delta_T = k$, for all $t \in [t_0,\ t_k]$, we have

$$e^{-(1-\rho)k} \leq e^{-\frac{(1-\rho)}{\delta_T}(t-t_0)} \tag{20}$$

Therefore, from inequalities (19) and (20), we conclude

$$\|\mathbf{z}(t)\|_{\hat{P}} \leq \kappa \|\mathbf{z}(0)\|_{\hat{P}} e^{-\frac{(1-\rho)}{\delta_T}(t-t_0)}.$$

According to [20], the closed-loop system is exponentially stable, so does the FSC-MPC algorithm. ■

V. Simulation Results

The effectiveness of the FSC-MPC algorithm presented in Section III is investigated by numerical simulations. In the figures, each robot is depicted by an arrow within a circle (dotted circle for virtual reference robot). The orientation of the robot is shown by the orientation of the arrow.

A. Trajectory Tracking

In this section, the simulation results of our FSC-MPC controller, Kanayama's controller proposed in [2] and Samson's controller proposed in [3] are compared. Specifically, the controllers proposed in [2] and [3] are

$$\begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} v_r \cos\theta_e + K_x x_e \\ \omega_r + v_r(K_y y_e + K_\theta \sin\theta_e) \end{bmatrix}, \tag{21}$$

$$\begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} v_r \cos\theta_e + K_1 x_e \\ \omega_r + K_2 v_r \frac{\sin\theta_e}{\theta_e} + K_3 \theta_e \end{bmatrix}, \tag{22}$$

respectively.

The reference trajectory starts from posture $\mathbf{z}_r(0) = [0\ \ 0\ \ 0]^T$ with constant control inputs $[v_r\ \ \omega_r]^T = [1\ \ 0]^T$. In addition, we assume that two perturbations occur at time 10 s and 20 s which change the orientation of the virtual reference robot from 0 rad to $\pi/2$ rad and from $\pi/2$ rad back to 0 rad, respectively. Total simulation time is 30 s.

The initial condition of tracking robot is $\mathbf{z}_c(0) = [0\ \ 3\ \ 0]^T$. Controller parameters are selected as follows. For Kanayama's controller, $K_x = 1$, $K_y = 4$ and $K_\theta = 4$. For Samson's controller, $K_1 = 1$, $K_2 = 4$ and $K_3 = 4$. Sampling time for these two controllers is $\delta_T = 0.1$ s. For FSC-MPC controller, $H = 6$, $\rho = 0.95$, $Q = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\hat{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Sampling time is $\delta_T = 0.5$ s. Control input constraints are

$$-4(\text{m/s}) \leq v \leq 4(\text{m/s}), \quad -0.8(\text{rad/s}) \leq \omega \leq 0.8(\text{rad/s}).$$

The system responses of the three controllers are shown in Figure 3. All the controllers can drive the tracking robot back to the reference trajectory. Since constraints are put on the control inputs, our FSC-MPC controller is outperformed by Kanayama's and Samson's controllers.

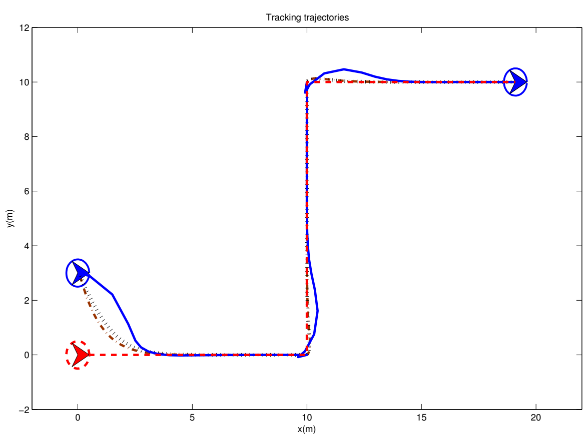

Fig. 3. Tracking trajectories. Dashed: Reference. Solid: FSC-MPC. Dotted: Samson. Dash-dot: Kanayama.

B. Point Stabilization

In this section, the simulation results of our FSC-MPC controller and Aicardi's controller proposed in [12] are compared. Specifically, the controllers proposed in [12] is

$$\begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} K_1 e \cos\alpha \\ k_2\alpha + K_1 \frac{\cos\alpha \sin\alpha}{\alpha}(\alpha + K_3\phi) \end{bmatrix} \tag{23}$$

Three initial robot postures are used in the simulation. They are

$$\mathbf{z}_c(0) = \left\{ \begin{bmatrix} 1 \\ 0 \\ \pi/2 \end{bmatrix}, \begin{bmatrix} -0.5 \\ 0.867 \\ \pi/2 \end{bmatrix}, \begin{bmatrix} -0.5 \\ -0.867 \\ \pi/2 \end{bmatrix} \right\}.$$

The final posture is $\mathbf{z}_d = [0\ \ 0\ \ 0]^T$. Controller parameters are selected as follows. For Aicardi's controller, $K_1 = 3$, $K_2 = 6$ and $K_3 = 1$. Sampling time is $\delta_T = 0.05$ s and the simulation lasts 4 s. For FSC-MPC controller, controller parameters, sampling time and control input constraints are the same as those in Section V-A. Total simulation time is 10 s.

The trajectories generated by Aicardi's controller and our FSC-MPC controller from different initial postures are shown in Figure 4. The FSC-MPC controller successfully stabilizes the robot at the desired final posture.

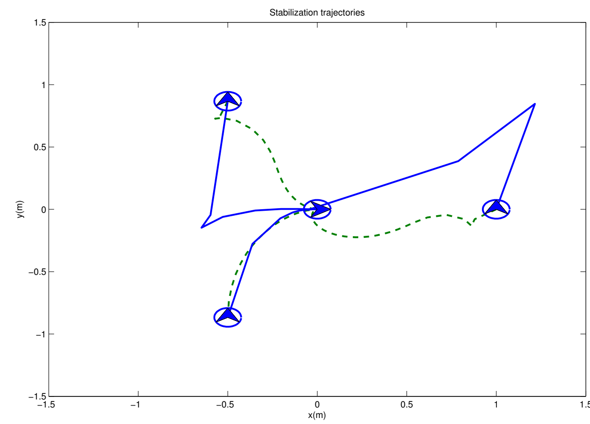

Fig. 4. Stabilization trajectories. Solid: FSC-MPC. Dashed: Aicardi

Specifically, we use the integral of norm squared actual control inputs ($\sum_1^k \|\mathbf{u}\|^2\delta_T$) as a metric to evaluate the control energy. The control energy expended by each controller from different initial postures are shown in Table I. The FSC-MPC requires much less control energy in comparison with the Aicardi's controller.

Initial Posture	Aicardi	FSC-MPC
$[1\ \ 0\ \ \pi/2]^T$	1772.7122	10.4999
$[-0.5\ \ 0.867\ \ \pi/2]^T$	586.2641	6.1507
$[-0.5\ \ -0.867\ \ \pi/2]^T$	69.7075	3.7015

TABLE I

THE INTEGRAL OF NORM SQUARED ACTUAL CONTROL INPUTS FOR STABILIZATION.

C. Simultaneous Tracking and Stabilization

A simulation is illustrated in this section which shows that our FSC-MPC controller has the ability of simultaneous tracking and stabilization. Usually, simultaneous tracking and stabilization is not considered under a single controller approach. Most of the existing controllers for trajectory tracking of nonholonomic mobile robots will fail when the virtual reference robot stops or moves backward.

The virtual reference robot starts moving backward from posture $\mathbf{z}_r(0) = [0\ \ 0\ \ \pi/2]^T$ with constant control inputs $[v_r\ \ \omega_r]^T = [-1\ \ 0.1]^T$. Then, it stops at time $t = 5$ s. The initial condition of the tracking robot is $\mathbf{z}_c(0) = [10\ 10\ \pi/2]^T$. We compare our FSC-MPC controller with

Samson's controller (22). Controller parameters, sampling time, simulation time and control input constraints are the same as those in Section V-A.

The results are shown in Figures 5The FSC-MPC controller successfully stabilizes the tracking robot to the final posture where the reference robot stops. Meanwhile, Samson's controller experiences some extreme maneuvers and only stops the tracking robot to a neighboring position.

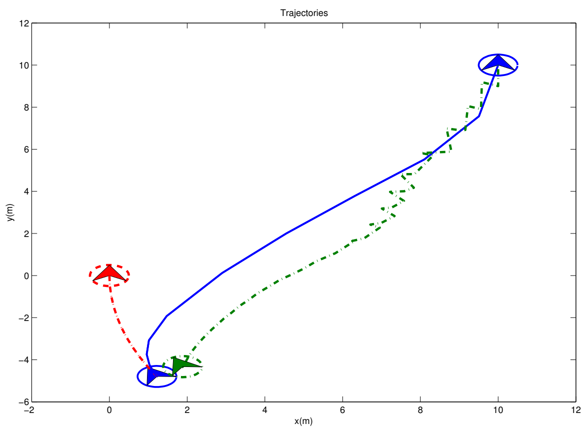

Fig. 5. Trajectories of simultaneous tracking and stabilization. Dashed: Reference. Solid: FSC-MPC. Dash-Dot: Samson.

VI. Conclusions

In this paper, a *first-state contractive* model predictive control (FSC-MPC) algorithm is developed for the trajectory tracking and point stabilization problems of nonholonomic mobile robots. Stability of the proposed MPC scheme is guaranteed by adding a first-state contractive constraint. Simulation results show that the proposed FSC-MPC controller can generate satisfactory system responses while requires much less control energy in comparison with other well-known controllers. In addition, the proposed FSC-MPC algorithm has the ability of simultaneous tracking and stabilization, in contrast to controllers available in the literature.

For all simulations, an initial feasible solution is required for the proposed FSC-MPC controller. Like most of the MPC schemes, a trial-and-error approach is used. The choice of the contractive parameter is critical for the initial feasible solution. A value close to 1 is preferred. However, a small value will give faster convergence rate when the system approaches the equilibrium point. As part of our future work, we are investigating adaptive or time-varying schemes of the contractive parameter, and experimental verifications of the FSC-MPC on the MARHES [21] testbed.

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