2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Nonlinear Output Feedback $\mathcal{H}_\infty$ Control for Polynomial Nonlinear Systems

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***Abstract*— In this paper, we propose a computational scheme of solving the output feedback $\mathcal{H}_\infty$ control problem for a class of nonlinear systems with polynomial vector field. The output feedback control design problem will be decomposed into a state feedback and an output estimation problems. Resorting to higher order Lyapunov functions, two Hamilton-Jacobian-Isaacs (HJI) inequalities are first formulated as semi-definite optimization conditions. Sum-of-squares (SOS) programming techniques are then applied to obtain computationally tractable solutions, from which a nonlinear control law will be constructed. The closed-loop system is asymptotically stabilizable by the nonlinear output feedback control and achieves good $\mathcal{H}_\infty$ performance under the exogenous disturbances.**

I. INTRODUCTION

The analysis and control of nonlinear systems are among the most challenging problems in systems and control theory. In the past decade, a theoretical framework for exploring $\mathcal{H}_\infty$ control of nonlinear systems has been proposed in [17], [4], [18]. Interpreting nonlinear $\mathcal{H}_\infty$ control in terms of dissipativity and differential game [1], the solution to this problem has been related to an appropriate Hamilton-Jacobi-Isaacs (HJI) equation. For hyperbolic nonlinear systems whose linearized plant are stabilizable, the solution of HJI equation was characterized [17], [18] by an invariant manifold of Hamiltonian vector fields using differential geometric theory. Later on, the result has been generalized to non-hyperbolic nonlinear systems via output feedback control [4]. It was further shown that the solution to output feedback control problem is determined by a pair of coupled HJI equations. Parallel to linear $\mathcal{H}_\infty$ control theory, a separation principle was also established under a detectability hypothesis [3]. However, how to solve nonlinear $\mathcal{H}_\infty$ control problem in a numerically efficient way to make it practically useful to physical world remains an unsolved issue.

It is well known that the HJI partial differential equation (PDE) reduces to Riccati algebraic equations for linear systems, which can be solved easily by efficient numerical algorithms. In the nonlinear context, however, there is no systematic numerical algorithm currently available for the solution of this PDE. Therefore, the key of nonlinear $\mathcal{H}_\infty$ control theory is the solvability of HJI equations. To this end, various approaches have been proposed to solve HJI equation numerically. In [5], [16], Taylor series expansion of the storage function $V(x)$ were considered to solve HJI equation term by term in an iterative fashion, provided that the linearized model of nonlinear systems has a solution. However, the solutions from these approaches do not have a closed form and they may not converge to an analytic solution. On the other hand, a convex parametrization of nonlinear $\mathcal{H}_\infty$ control problem was derived in [8] based on a pair of positive definite matrix functions $P(x), Q(x)$. Unfortunately, it is difficult, if not impossible, to specify the form of $P(x)$ such that $\partial V(x)/\partial x = 2x^T P(x)$ except for the trivial case when $P(x)$ is a constant matrix. Therefore, the proposed formulation does not naturally lead to computationally tractable solution algorithms for nonlinear $\mathcal{H}_\infty$ control. In [14], the $\mathcal{L}_2$ gain analysis problem for polynomial nonlinear systems was formulated as a convex state-dependent linear matrix inequality (LMI), which can be recast as a SOS optimization problem. This approach was shown promising to overcome the numerical difficulty in solving HJI inequality and provides an analytic solution at the same time. Reference [19] proposed an iterative method based on SOS programming [12], [2] to solve a special state feedback $\mathcal{H}_\infty$ control problem. As a powerful and promising technique, SOS programming has also been applied to solve nonlinear analysis [9], [15] and stabilization [11], [13] problems. The main advantages of SOS decomposition are the resulting computational tractability and the algorithmic character of the solution procedure [10]. This could help to provide coherent methodology of synthesizing Lyapunov functions for nonlinear systems. In addition, the importance of SOS technique also lies in its ability to provide tractable relaxations for many difficult optimization problems, such as nonlinear $\mathcal{H}_\infty$ control.

In this paper, we will focus on materializing $\mathcal{H}_\infty$ theory into an algorithmic procedure for a class of nonlinear systems with its vector field in polynomial form. It turns out that the nonlinear $\mathcal{H}_\infty$ control problem can be solved by establishing several convex optimization conditions based on the idea of SOS decomposition. Moreover, the resulting output feedback controller will be constructed to achieve closed-loop stability as well as $\mathcal{L}_2$ gain performance. Specifically, we will use polynomial type Lyapunov functions to convert original HJI inequalities into matrix inequalities for polynomial nonlinear systems. It has been shown that the parametric representation of Lyapunov functions in higher order form will provide an effective way to convert the difficult HJI inequality to a state-dependent LMI. Consequently, the testing of nonnegativity of generalized Gram matrix from the resulting LMI is solvable

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using SOS programming techniques with polynomial-time complexity. The proposed approach is applicable to both parabolic and non-parabolic types of polynomial nonlinear systems.

The notation used in this paper is fairly standard. We denote nonnegative integer set as $\mathbf{Z}_+$. $\mathbf{R}$ stands for the set of real numbers and $\mathbf{R}_+$ for the non-negative real numbers. $\mathbf{R}^{m\times n}$ is the set of real $m \times n$ matrices. For two integers $k_1, k_2,\ k_1 \le k_2$, $\mathbf{I}[k_1, k_2] = \{k_1, k_1+1, \cdots, k_2\}$. We use $\mathbf{S}^{n\times n}$ to denote real, symmetric $n \times n$ matrices, and $\mathbf{S}_+^{n\times n}$ for positive definite matrices. For an $M \in \mathbf{S}^{n\times n}$, $M > 0$ ($M \ge 0$) indicates that M is a positive definite (positive semi-definite) matrix and $M < 0$ ($M \le 0$) denotes a negative definite (negative semi-definite) matrix. A block diagonal matrix with matrices $X_1, X_2, \cdots, X_p$ on its main diagonal is denoted by $\operatorname{diag}\{X_1, X_2, \cdots, X_p\}$. $\frac{\partial V}{\partial x} = \begin{bmatrix}\frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} & \ldots & \frac{\partial V}{\partial x_n}\end{bmatrix}$ is the derivative of V with respect to x. $\|x\|_2$ is the $\mathcal{L}_2$ norm of x. A multivariate polynomial $p(x)$ is a sum of squares (SOS) if there exist polynomials $p_1(x), \ldots, p_\ell(x)$ such that $p(x) = \sum_{i=1}^{\ell} p_i^2(x)$.

II. SOS-Based Nonlinear $\mathcal{H}_\infty$ Analysis

Based on the dissipation system theory [17], the $\mathcal{H}_\infty$ analysis for a nonlinear system

$$\begin{cases} \dot{x} = A(x)x + B_1(x)w \\ z = C_1(x)x \end{cases} \tag{1}$$

with desired stability and $\mathcal{L}_2$ gain can be determined by a HJI inequality

$$\frac{\partial U}{\partial x}A(x)x + \frac{1}{4\gamma^2}\frac{\partial U}{\partial x}B_1(x)B_1^T(x)\frac{\partial U^T}{\partial x} + x^T C_1^T(x)C_1(x)x < 0.$$

Nevertheless, there does not exist computationally efficient algorithm to solve the above HJI inequality. Restricting to polynomial vector field, i.e., $A(x), B_1(x), C_1(x)$ as polynomial functions, one can derive the well-known bounded real lemma for polynomial nonlinear systems parallel to its linear counterpart as follows.

Lemma 1: The stability and achievable $\mathcal{L}_2$ gain properties for the polynomial nonlinear system (1) can be established by finding a positive definite Lyapunov function $U(x)$ such that following condition is satisfied

$$\begin{bmatrix} \frac{\partial U}{\partial x}A(x)x + x^T C_1^T(x)C_1(x)x & \frac{\partial U}{\partial x}B_1(x) \\ B_1^T(x)\frac{\partial U^T}{\partial x} & -4\gamma^2 I \end{bmatrix} < 0. \tag{2}$$

This can be easily shown by applying Schur complement to condition (2). Moreover, by specifying $U(x)$ in a polynomial function form, condition (2) becomes an LMI with state-dependent polynomial entries. With recent advances in SOS programming, it is advantageous to recast the resulting state-dependent LMI as a semidefinite programming problem, and solve it using computationally efficient SOS tools [10], [12]. The feasibility of SOS-based solution approach has been demonstrated in [14].

III. $\mathcal{H}_\infty$ Control of Polynomial Nonlinear Systems using Output Feedback

Consider a polynomial nonlinear system that is affine in exogenous disturbance and control input

$$\begin{cases} \dot{x} = A(x)x + B_1(x)w + B_2(x)u \\ z = C_1(x)x + D_{12}(x)u \\ y = C_2(x)x + D_{21}(x)w \end{cases} \tag{3}$$

where the system state $x \in \mathbf{R}^n$, control input $u \in \mathbf{R}^{n_u}$, exogenous disturbance $w \in \mathbf{R}^{n_w}$, controlled output $z \in \mathbf{R}^{n_z}$, and measured output $y \in \mathbf{R}^{n_y}$. It is assumed that all of the state space entries are polynomial functions of the state x with compatible dimensions. Moreover, the following standard assumptions will be made:

1) $(A(x)x, B_2(x))$ is reachable from zero, and $(C_2(x)x, A(x)x)$ is zero-state detectable for all $x \in \mathbf{R}^n$,
2) $D_{12}^T(x)\begin{bmatrix}C_1(x) & D_{12}(x)\end{bmatrix} = \begin{bmatrix}0 & I\end{bmatrix}$,
3) $\begin{bmatrix}B_1(x) \\ D_{21}(x)\end{bmatrix} D_{21}^T(x) = \begin{bmatrix}0 \\ I\end{bmatrix}$.

Our objective is to design an output feedback controller such that the nonlinear closed-loop system is asymptotically stable and its $\mathcal{L}_2$ gain from z to w is less than γ, i.e.

$$\|z\|_2 \le \gamma\|w\|_2 \text{ when } x(0) = 0.$$

This problem is called nonlinear output feedback $\mathcal{H}_\infty$ control problem.

A. Nonlinear Output Feedback $\mathcal{H}_\infty$ Synthesis Condition

It was shown in [4], [7] that the nonlinear output feedback $\mathcal{H}_\infty$ control problem is (locally) solvable if there exist two positive definite matrix functions $U(x), V(x)$ and a scalar γ such that

$$\mathcal{H}_{SF}(U,\gamma,x) < 0 \tag{4}$$

$$\mathcal{H}_{OE}(V,\gamma,x) - \mathcal{H}_{SF}(U,\gamma,x) < 0 \tag{5}$$

$$\left.\frac{\partial^2}{\partial x^2}(\mathcal{H}_{OE} - \mathcal{H}_{SF})\right|_{x=0} < 0 \tag{6}$$

$$W(x) := V(x) - U(x) \ge 0, \tag{7}$$

where

$$\mathcal{H}_{SF}(U,\gamma,x) := \frac{\partial U}{\partial x}A(x)x + x^T C_1^T(x)C_1(x)x + \frac{1}{4}\frac{\partial U}{\partial x}\left[\frac{1}{\gamma^2}B_1(x)B_1^T(x) - B_2(x)B_2^T(x)\right]\frac{\partial U^T}{\partial x}$$

$$\mathcal{H}_{OE}(V,\gamma,x) := \frac{\partial V}{\partial x}A(x)x + \frac{1}{4\gamma^2}\frac{\partial V}{\partial x}B_1(x)B_1^T(x)\frac{\partial V^T}{\partial x} + x^T\left[C_1^T(x)C_1(x) - C_2^T(x)C_2(x)\right]x.$$

Moreover, one of the output feedback controllers is given by

$$\begin{cases} \dot{x}_c = A(x_c)x_c + B_1(x_c)F_1(x_c) + B_2(x_c)F_0(x_c) \\ \qquad + L_0(x_c)\left[C_2(x_c)x_c - y\right] \\ u \ = F_0(x_c) \end{cases} \tag{8}$$

where the matrix functions $F_0(x), F_1(x)$ and $L_0(x)$ are defined as

$$F_0(x) = -\frac{1}{2}B_2^T(x)\frac{\partial U^T}{\partial x} \quad F_1(x) = \frac{1}{2\gamma^2}B_1^T(x)\frac{\partial U^T}{\partial x}$$
$$\frac{\partial W(x)}{\partial x}L_0(x) = -2x^T C_2^T(x).$$

Note that solvability conditions (4)-(5) are given in the form of HJI inequalities, which are partial differential inequalities and extremely difficult to solve. In the special case when both $U(x), V(x)$ are quadratic functions of state x, the solvability conditions (4)-(7) degenerate to a set of state-dependent LMIs. However, for general polynomial Lyapunov functions, the bilinear term $-\frac{\partial U}{\partial x}B_2(x)B_2^T(x)\frac{\partial U^T}{\partial x}$ in condition (4) will render a non-convex problem. In the following subsections, we will show how to obtain computationally tractable solutions for both conditions using SOS programming tools and develop a systematic design procedure for nonlinear output feedback $\mathcal{H}_\infty$ control problem with both of $U(x)$ and $V(x)$ as polynomial forms of x, i.e.

$$U(x) = \frac{1}{2}M^{[n_p]}(x)^T P M^{[n_p]}(x)$$
$$V(x) = \frac{1}{2}M^{[n_q]}(x)^T Q M^{[n_q]}(x).$$

with $n_p \geq n, n_q \geq n$.

B. Iterative Algorithm to Solve State Feedback $\mathcal{H}_\infty$ Condition

As mentioned above, the main obstruction in solving nonlinear H_∞ control problem is due to non-convexity of the state feedback condition (4). To overcome this difficulty, we will use an iterative algorithm to solve this state feedback condition similar to the one in [19].

Suppose there is a quadratic Lyapunov function $U_0(x) = \frac{1}{2}x^T P_0 x > 0$ and an initial state feedback controller $u_0 = K_0(x)x$ for state feedback $\mathcal{H}_\infty$ control, then the closed-loop plant becomes

$$\begin{cases} \dot{x} = [A(x) + B_2(x)K_0(x)]x + B_1(x)w \\ z = [C_1(x) + D_{12}(x)K_0(x)]x \end{cases}$$

Due to the special form of quadratic Lyapunov function, it sufficient to reformulate condition (4) as a SOS optimization problem:

$$-\mathcal{S}_{SF0} \quad \text{is SOS}, \tag{9}$$

where

$$\mathcal{S}_{SF0} := \begin{bmatrix} \left\{ \begin{matrix} \frac{1}{2}[A(x)R_0 + R_0A^T(x)] \\ +\frac{1}{4}[\frac{1}{\gamma^2}B_1(x)B_1^T(x) - B_2(x)B_2^T(x)] \end{matrix} \right\} & R_0C_1^T(x) \\ C_1(x)R_0 & -I \end{bmatrix}$$

and $R_0 = P_0^{-1} > 0$. Consequently, condition (9) can be solved using SOS programming techniques to obtain $U_0(x)$ and a closed-loop $\mathcal{H}_\infty$ performance γ_0. Moreover, a feasible state feedback control renders the closed-loop $\mathcal{L}_2$ gain less than γ_0 will be

$$u_0(x) = -\frac{1}{2}B_2^T(x)\frac{\partial U_0^T}{\partial x}. \tag{10}$$

On the other hand, applying lemma 1 to the closed loop system, an equivalent SOS condition for state feedback $\mathcal{H}_\infty$ control will be

$$-\mathcal{S}_{SFi} \quad \text{is SOS}, \tag{11}$$

where

$$\mathcal{S}_{SFi} := \begin{bmatrix} \left\{ \begin{matrix} \frac{\partial U_i}{\partial x}[A(x)x + B_2(x)u_{i-1}(x)] \\ +x^T C_1^T(x)C_1(x)x \end{matrix} \right\} & \frac{\partial U_i}{\partial x}B_1(x) & u_{i-1}^T(x) \\ B_1^T(x)\frac{\partial U_i^T}{\partial x} & -4\gamma_i^2 & 0 \\ u_{i-1}(x) & 0 & -I \end{bmatrix}.$$

For any fixed $u_{i-1}(x)$, condition (11) is convex about variables $U_i(x)$ and γ_i and can be used to solve for a $U_i(x)$ and γ_i through SOS programming. Using a polynomial representation of Lyapunov function $U_i(x)$ to solve condition (11), it is possible to achieve a better closed-loop performance $\gamma_i < \gamma_{i-1}$. It is clear that (11) is always feasible (at least $U_i(x) = U_{i-1}(x)$ works).

Then computing a new controller u_i from eqn. (10), the next round of iteration could be started by solving the SOS optimization (11) repeatedly.

In summary, an iterative algorithm to solve the nonlinear state feedback H_∞ condition (4) will be:

1) Initialization: Using SOS programming to obtain an initial quadratic Lyapunov function $U_0(x) = \frac{1}{2}x^T P_0 x$ and an initial polynomial state feedback controller u_0 by solving conditions (9) and (10) sequentially.
2) Iteration: Compute u_i using eqn. (10), and synthesize a polynomial Lyapunov function $U_i(x) = \frac{1}{2}M^{[n_{pi}]}(x)^T P_i M^{[n_{pi}]}(x) > 0, n_{pi} \geq n$ with its associated $\mathcal{L}_2$ gain γ_i by solving SOS condition (11).
3) If $|\gamma_i - \gamma_{i-1}| < \epsilon$ or the iteration number i is sufficiently large, let $\gamma = \gamma_i$ and STOP. Otherwise, set $i = i + 1$ and go to step 2.

Although we start from a quadratic Lyapunov function, the final Lyapunov function will be a complicated polynomial form of x. During the iteration, $u_i(x)$ could be specified as any type of polynomials of x. The resulting state feedback controller is given by

$$u_i(x) = -\frac{1}{2}B_2^T(x)\frac{\partial U_i^T}{\partial x}.$$

Finally, it is noted that the performance level γ_i will improve gradually as i increases and finally it converges to a sub-optimal solution.

Remark 1: Note that the synthesis conditions (9) and (11) are linear matrix inequalities with polynomial entries. SOSTOOL [12] provides an efficient way to obtain the tractable solutions by reformulating these conditions into SOS optimization problems. Generally speaking, a global nonlinear controller will be obtained. Nevertheless, it is often too restrictive to synthesize a global stabilizing controller. Moreover, in a restricted region, local controllers will often perform better than global controllers. In such situations, it is suggested to add state region constraints into the original condition $\mathcal{H}_{SF}(U,\gamma,x)<0$ and search for a local solution. Suppose that there are r state region constraints denoted by functions $R_j(x)<0, j\in\mathbf{I}[1,r]$. Then the modified state feedback $\mathcal{H}_\infty$ control condition will be

$$\mathcal{H}_{SF}(U,\gamma,x)-\sum_{j=1}^{r}\lambda_j(x)R_j(x)<0,$$

in which $\lambda_j(x)>0, j\in\mathbf{I}[1,r]$ are SOS multipliers. Consequently, the SOS conditions in the iterative algorithm for local state feedback control will be modified to

$$\begin{aligned}&-\mathcal{S}_{SFi}+\begin{bmatrix}\sum_{j=1}^{r}\lambda_j(x)R_j(x) & \underline{\mathbf{0}}\\ \underline{\mathbf{0}} & \underline{\mathbf{0}}\end{bmatrix}\quad\text{is SOS}\\ &\lambda_j(x)>0,\qquad j\in\mathbf{I}[1,r],\end{aligned}$$

where $\underline{\mathbf{0}}$'s are zero matrices with proper dimensions. The local region is specified by the intersection of constraints $R_j(x)<0, j\in\mathbf{I}[1,r]$.

C. Design Procedure for Nonlinear Output Feedback H_∞ Control

Based on the discussions in subsections III-A and III-B, we propose a control design procedure to solve the nonlinear output feedback H_∞ synthesis problem for polynomial nonlinear systems.

1) Using the iterative algorithm in subsection III-B, the HJI inequality (4) will be solved to obtain a polynomial Lyapunov function $U(x)=\frac{1}{2}M^{[n_p]}(x)^TPM^{[n_p]}(x), n_p\geq n$ and a performance level $\gamma>0$.
2) Calculate $F_0(x), F_1(x)$ by

$$F_0(x)=-\frac{1}{2}B_2^T(x)\frac{\partial U^T}{\partial x}\quad F_1(x)=\frac{1}{2\gamma^2}B_1^T(x)\frac{\partial U^T}{\partial x}.$$

3) Reformulate conditions (5)-(7) as another SOS optimization problem:

$$\begin{aligned}&\min\ \hat{\gamma} \qquad\qquad (12)\\ \text{s.t.}\quad &-\mathcal{S}_{OE}\quad\text{is SOS}\\ &-\frac{\partial^2}{\partial x^2}(\mathcal{H}_{OE}-\mathcal{H}_{SF})\Big|_{x=0}\quad\text{is SOS}\\ &V(x)-U(x)\quad\text{is SOS.}\end{aligned}$$

where

$$\mathcal{S}_{OE}:=\begin{bmatrix}\left\{\begin{array}{c}\frac{\partial V}{\partial x}A(x)x+x^TC_1^T(x)C_1(x)x\\ -x^TC_2^T(x)C_2(x)x\\ -\mathcal{H}_{SF}(U,\gamma,x)\end{array}\right\} & \frac{1}{2}\frac{\partial V}{\partial x}B_1(x)\\ \frac{1}{2}B_1^T(x)\frac{\partial V^T}{\partial x} & -\hat{\gamma}^2I\end{bmatrix}.$$

Solve condition (12) using SOS programming to obtain $V(x)=\frac{1}{2}M^{[n_q]}(x)^TQM^{[n_q]}(x), n_q\geq n$ and $\hat{\gamma}\leq\gamma$.
4) Solve $L_0(x)$ from the equation

$$\frac{\partial W(x)}{\partial x}L_0(x)=-2x^TC_2^T(x) \qquad (13)$$

as $L_0(x)=-2\mathcal{M}^{-1}(x)C_2^T(x)$, where $\mathcal{M}(x)$ is a non-singular polynomial function satisfying $\frac{\partial W}{\partial x}=x^T\mathcal{M}$.
5) Finally, construct the output feedback controller in the form of (8). The closed-loop $\mathcal{L}_2$ gain will be bounded by γ.

In the proposed control design procedure, we first decompose the nonlinear output feedback $\mathcal{H}_\infty$ control problem into nonlinear state feedback and output estimation problems. Both problems are then solved efficiently using SOS programming techniques with polynomial complexity. The Lyapunov function for the closed-loop system is given by $W(x)=V(x)-U(x)$, which could be any general form polynomials with higher order. Note that $L_0(x)$ is not unique for a given $W(x)$. Since $W(x)$ has at least order 2, it is clear that $\frac{\partial W(x)}{\partial x}$ could be rewritten as $x^T\mathcal{M}(x)$ for a non-singular polynomial matrix $\mathcal{M}(x)$ of dimension $n\times n$. Therefore, one solution of $L_0(x)$ satisfying (14) will be

$$L_0(x)=-2\mathcal{M}^{-1}(x)C_2^T(x).$$

As mentioned in Remark 1, the SOS optimization problem (12) could also be solved by adding state region constraints. This will help improve the solvability of condition (12) and lead to local solution of nonlinear output feedback $\mathcal{H}_\infty$ control.

IV. EXAMPLE

In this section, we will apply the proposed nonlinear output feedback $\mathcal{H}_\infty$ control design procedure in section III-C to a nonlinear mass-spring-damper system [6] shown in Fig. 1. The dynamic equation of the nonlinear system is given by

$$m\ddot{x}+g(x,\dot{x})+f(x)=\phi(x)u,$$

where m is the mass, u is control force, $f(x)$ represents nonlinear spring term, $g(x,\dot{x})$ is the damper term, $\phi(x)$ is nonlinearity associated with input channel. The parameters are chosen as $m=1, c_1=1, c_2=0.01, c_3=0.1, c_4=0.13$, $g(x,\dot{x})=c_1\dot{x}, f(x)=c_2x+c_3x^3, \phi(x)=1+c_4x^2$.

Let $x_1=x$ (displacement), $x_2=\dot{x}$ (velocity), the state space model of the system will be

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -\frac{c_2}{m}-\frac{c_3}{m}x_1^2 & -\frac{c_1}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ \frac{1}{m}+\frac{c_4}{m}x_1^2\end{bmatrix}u.$$

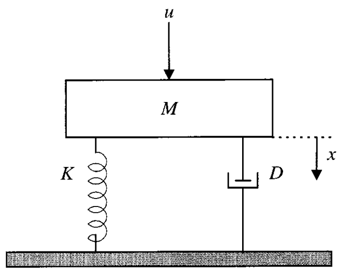

Fig. 1. Nonlinear mass-spring-damper system.

There are two disturbances acting on the system. One is environmental disturbance such as air force affecting the velocity of the mass. The other one is measuring noise acting on the output channel. Incorporating disturbances, the second-order polynomial nonlinear system will be written as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -0.01-0.1x_1^2 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0.8 & 0\end{bmatrix}\begin{bmatrix}w_1\\ w_2\end{bmatrix} + \begin{bmatrix}0\\ 1+0.13x_1^2\end{bmatrix}u$$

$$\begin{bmatrix}z_1\\ z_2\end{bmatrix} = \begin{bmatrix}0.6 & 0.3\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$y = \begin{bmatrix}1.61 & 1.38\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}w_1\\ w_2\end{bmatrix}.$$

For local control design, we introduce the state region constraint of the system as $R(x) = x_2^2 - 3 < 0$, i.e., the operating range of the velocity is restricted to the interval $[-\sqrt{3}, \sqrt{3}]$. Through the iterative algorithm and SOSTOOL [12], we solved optimization problem (4) in three iterations and obtained

$$\begin{aligned}
\gamma_0 &= 1.55\\
U_0(x) &= 1.5328x_1^2 + 1.2990x_1x_2 + 1.6206x_2^2\\
u_0(x) &= -1.0082x_1 + 1.3242x_2 - 0.1311x_1^3 - 0.1722x_1^2x_2\\
\gamma_1 &= 1.36\\
U_1(x) &= 0.4737x_1^4 + 2.2018x_1^2 + 2.1198x_1x_2 + 1.2474x_2^2\\
u_1(x) &= -1.0599x_1 - 1.2474x_2 - 0.1378x_1^3 - 0.1622x_1^2x_2\\
\gamma_2 &= 1.11\\
U_2(x) &= 0.43009x_1^4 + 2.0583x_1^2 + 2.0492x_1x_2 + 1.1894x_2^2\\
u_2(x) &= -1.0246x_1 - 1.1894x_2 - 0.1332x_1^3 - 0.1546x_1^2x_2\\
\gamma &= \gamma_3 = 1.02\\
U_3(x) &= 0.38271x_1^4 + 1.8763x_1^2 + 1.9283x_1x_2 + 1.1307x_2^2
\end{aligned}$$

Then we calculated $F_0(x), F_1(x)$ as

$$F_0(x) = -0.9642x_1 - 1.1307x_2 - 0.1253x_1^3 - 0.1470x_1^2x_2$$

$$F_1(x) = \begin{bmatrix}0.7483x_1 + 0.8776x_2\\ 0\end{bmatrix}.$$

Also solving the minimization problem (12) by SOS programming, we had

$$\begin{aligned}
\hat{\gamma} &= 0.93\\
V(x) &= 1.0483x_1^4 + 0.81589x_2x_1^3 + 1.048x_1^2x_2^2\\
&\quad + 2.8056x_1^2 + 2.0818x_1x_2 + 1.9599x_2^2
\end{aligned}$$

Consequently, we solved the matrix function $L_0(x)$ as

$$L_0(x) = -2\mathcal{M}^{-1}(x)C_2^T(x) = \begin{bmatrix}\frac{115.71-2.2191x_1^2-3.1802x_1x_2}{-11.0621-23.3645x_1^2-28.1979x_1x_2+0.2961x_2x_1^3+0.4243x_2^2x_1^2}\\ \frac{28.993+6.1972x_1^2+2.9451x_1x_2}{-11.0621-23.3645x1^2-28.19796x_1x_2+0.2961x_2x_1^3+0.4243x_2^2x_1^2}\end{bmatrix}.$$

Finally, it is straightforward to construct the output feedback controller by substituting $F_0(x), F_1(x), L_0(x)$ into the controller formula (8). The closed-loop $\mathcal{H}_\infty$ norm is bounded by $\gamma = 1.02$.

To demonstrate the design nonlinear control law, we first let disturbance $w = 0$ and verify the stability of the closed-loop system. The initial point is chosen as $x(0) = [2.5\ \ 1.5]^T$, which is inside the specified state region $R(x)$. The phase portraits of the open-loop and the closed-loop nonlinear systems are plotted in the first subplot of Fig. 2. In Comparison, it is observed that the closed-loop system is stabilized and the state trajectory converges to the origin within $10sec$. On the other hand, although $x = 0$ is also an equilibrium point for the open-loop system, it takes much longer time ($> 100sec$) to converge. The second subplot of Fig. 2 provides the control input profile for the nonlinear output feedback controller.

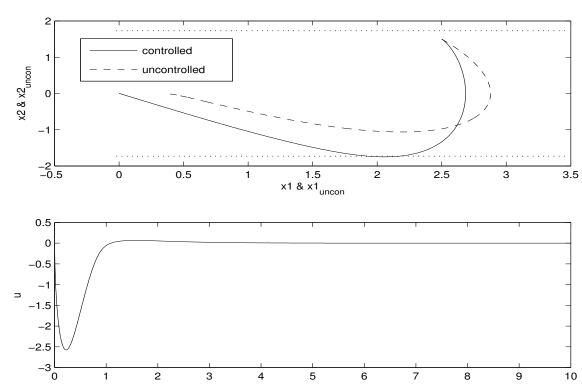

Fig. 2. State convergence and control input under an initial condition.

Secondly, we specify a piecewise constant disturbance $w(t)$ to the system as

$$w_1(t) = \begin{cases} 1.5, & 0 \le t < 2sec\\ 0, & 2sec \le t \le 10sec\end{cases}$$

$$w_2(t) = 1, \quad 0 \le t \le 10sec$$

In the first plot of Fig. 3, we computed the truncated norms $||z||_{2,T}/||w||_{2,T}$ for the energy amplification from disturbance w to output z over finite time interval $[0, T]$. As can be seen, the truncated norm is indeed less than γ. The second plot in Fig. 3 provides the state trajectory under the given disturbance.

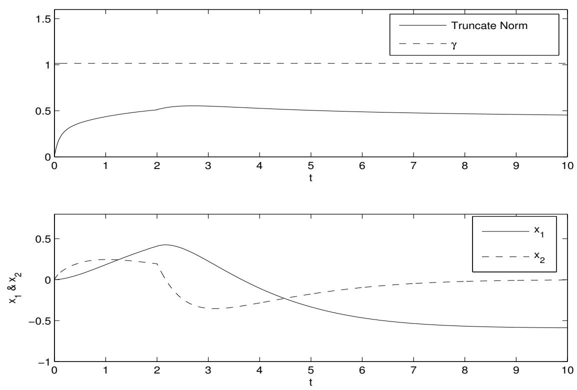

Fig. 3. Closed-loop disturbance attenuation for a disturbance input.

V. CONCLUSIONS

We have proposed a numerically efficient approach to solve challenging nonlinear output feedback $\mathcal{H}_\infty$ control problem. For polynomial nonlinear systems, it is advantageous to convert the HJIs associated with nonlinear $\mathcal{H}_\infty$ control to polynomial matrix inequalities, and solve the resulting matrix inequalities using SOS programming techniques. The proposed approach extends from trivial quadratic Lyapunov function case to higher-order Lyapunov functions, therefore it helps improving controlled performance and expanding stability region of nonlinear systems.

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