

Satellite Attitude Control Using Statistical Game Theory

Jong-Ha Lee, Ronald W. Diersing and Chang-Hee Won

Abstract— We consider the application of multi-objective statistical game theory to a remote sensing satellite attitude control. Statistical game theory is a generalization of mixed H_2/H_∞ control, where we have two objective functions and we optimize the higher order cumulants of these objective functions. We use previously developed satellite attitude model with thrusters, gravity torquers and a reaction wheel cluster. Then we control the satellite attitude using the statistical game (Minimal Cost Variance/ H_∞) control, H_∞ control, and mixed H_2/H_∞ control. Throughout the simulations, statistical game control has an extra degree of freedom to improve the performance and reduce the overshoot and undershoot compared to either H_∞ control and H_2/H_∞ control. The simulations show that the performance of Minimal Cost Variance/ H_∞ is 8.8% and 5.4% faster than H_2/H_∞ and H_∞ control, respectively. Moreover, the control actions of MCV/ H_∞ is reduced by 67% compared to H_∞ control and 55% compared to H_2/H_∞ control. So, we achieve both performance improvement while saving the control energy. In the case of the stability margin, MCV/ H_∞ control has the highest stability margin, H_∞ control has the lowest, and H_2/H_∞ control an intermediate value between those two.

I. INTRODUCTION

In the stochastic optimal control problem, the system is given as a stochastic differential equation and an optimal controller is determined to minimize the expected values of a cost function. Historically, stochastic optimal control was based upon the mean of a cost function. However, the mean or the first cumulant is only one of the cumulants that describe the distribution of a random variable. Other statistical quantities, such as the variance or the skewness can be considered to minimize the whole distribution of the cost function. This is the main idea of statistical control. In statistical control, a performance index is minimized that is based upon other statistical quantities such as the variance and skewness of the cost function, not just the mean [1]. Recently, we have applied the method to structural control and showed promising results [2].

Multi-objective control is a control method in which the control must concern itself with not only one performance index, as in traditional optimal control, but several. The most prevalent multi-objective control is the mixed H_2/H_∞ control, which the control wishes to minimize an H_2 norm while keeping the H_∞ norm constrained. This approach was started in [3], and the Nash game approaches to this problem

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are proposed in [4]. In this game, two players; a control and a disturbance are considered. They both wish to minimize their respective performance indices when the other player has played their equilibrium solution [5]. In the stochastic version of this problem, the players then wish to minimize the mean of their cost functions.

Minimal Cost Variance (MCV) control minimizes the variance, instead of the mean, of the cost function. The work was started in [6]. The full state feedback MCV control results for a nonlinear system and nonquadratic cost function are given in [7], and a linear system and quadratic cost function case is developed in [8]. The work of Diersing and Sain [9] examined to combine cost cumulants and MCV control to minimize a linear combination of the first two cumulants the mean and the variance while satisfying H_∞ constraint. This is called MCV/ H_∞ control and it shows that these control methods have been applied successfully to vibration control problems, such as the control structures excited by seismic disturbance [9].

In this paper, multi-objective statistical control is used for a Low Earth Orbit (LEO) satellite. For this, we consider MCV/ H_∞ control with actual commercial satellite parameters. The work of [11] compared mixed H_2/H_∞ control with H_2 control, and H_∞ control method to stabilize attitude control. This comparison was performed for a one player system. Here, we consider a stochastic version of the system with quadratic cost function. When the system has a stochastic white noise present in addition to a bounded power disturbance, the players' performance indices can be cast in terms of the mean value of the two quadratic costs. In a Nash game approach, a control and a disturbance is taken as two players. The equilibrium solution is found by solving coupled Riccati equations.

We present multi-objective statistical control method with equilibrium solutions in Section II. In Section IV, Korea Multi Purpose Satellite (KOMPSAT) system model with three magnetic torquers and a four reaction wheel cluster as actuators is provided. In section V, the performance of two control, H_∞ and H_2/H_∞ is compared to that of MCV/ H_∞ control with real parameters of a LEO satellite. Finally, Section VI concludes this paper.

II. MULTI-OBJECTIVE STATISTICAL CONTROL

The system is described as a linear system

$$dx(t) = [Ax(t) + Bu(t) + Dv(t)] dt + Ed\xi(t) \quad (1)$$

where x is the state, u is the control, v is the disturbance, and ξ is a Brownian motion with variance W . The regulated

outputs of the system are

$$z_1(t) = H_1(t)x(t) + G_1(t)u(t) \quad (2)$$

$$z_2(t) = H_2(t)x(t) + G_2(t)u(t) \quad (3)$$

We assume that $H_i' H_i = Q_i$, and $G_i' G_i = R_i$ for $i=1,2$, where Q_1 , Q_2 , and R_2 are assumed positive semi-definite and R_1 is assumed positive definite. This stochastic differential equation has two cost functions. The first cost function, J_1 , is to be associated with the control u and the second, J_2 is for the disturbance v . The players' cost functions will be assumed to be quadratic;

$$J_1 = \int_{t_0}^{t_f} (z_1'(t)z_1(t)) dt \quad (4)$$

$$= \int_{t_0}^{t_f} (x'(t)Q_1x(t) + u'(t)R_1u(t)) dt$$

$$J_2 = \int_{t_0}^{t_f} (\delta^2 v'(t)v_1x(t) - z_2'(t)z_2(t)) dt \quad (5)$$

$$= \int_{t_0}^{t_f} (\delta^2 v'(t)v(t) - x'(t)Q_2x(t) - u'(t)R_2u(t)) dt$$

With the system and costs defined, we can discuss the control method. The game to be considered here is one in which the first player, the control u , wishes to minimize a performance index consisting of a linear combination of cumulants given by $\varphi_1 = E_{tx}\{J_1\} + \gamma Var_{tx}\{J_1\}$, where γ is some positive constant and Var_{tx} is the variance using the conditional expectation. On the other hand, the second player, the disturbance v , wishes to minimize the mean of its cost function. That is the disturbance has $\varphi_2 = E_{tx}\{J_2\}$ as its performance index. Because both players will be assumed to have feedback information available to them, \mathcal{U}_M will be the information pattern for the control and \mathcal{W}_F will be the information pattern for the disturbance. We call this MCV/ H_∞ control. We assume that the costs are quadratic. That is $\mathcal{M}(t, x) = x'\mathcal{M}(t)x + m(t)$, $\mathcal{V}(t, x) = x'\mathcal{V}(t)x + v(t)$, and $\mathcal{P}(t, x) = x'\mathcal{P}(t)x + p(t)$ where \mathcal{M} , \mathcal{V} , \mathcal{P} are matrix functions of time and m , v , p are scalar functions of time. The MCV/ H_∞ equilibrium solution for the control is determined as

$$\begin{aligned} \min_{\mu \in \mathcal{U}_M} \{ & x' \dot{\mathcal{M}}x + \dot{m} + 2(Ax + B\mu + D\nu^*)' \mathcal{M}x \\ & + tr(EWE'(\mathcal{M}) + \gamma\mathcal{V}) + x'Q_1x + \mu'R_1\mu \\ & + \gamma[x'\dot{\mathcal{V}}x + \dot{v} + 2(Ax + B\mu + D\nu^*)' \mathcal{V}x \\ & + 4MEWE' \mathcal{M}] \} = 0 \end{aligned} \quad (6)$$

and minimizing this gives

$$u^* = \mu^*(t, x) = -R^{-1}B'[\mathcal{M}(t) + \gamma\mathcal{V}(t)]x(t) \quad (7)$$

Similarly, for the disturbance

$$\begin{aligned} \min_{\nu \in \mathcal{W}_F} \{ & x' \dot{\mathcal{P}}x + \dot{p} + 2(Ax + B\mu^* + D\nu)' \mathcal{P}x \\ & + tr(EWE'(\mathcal{P})) + \gamma^2 v'v - x'Q_2x \\ & - \mu'R_2\mu^* \} = 0 \end{aligned} \quad (8)$$

which by minimization yields

$$v^* = \nu^*(t, x) = -\frac{1}{\delta^2} D' \mathcal{P}(t)x(t) \quad (9)$$

Using this equilibrium solution (μ^*, ν^*) , we can determine three Riccati equations by substitution. \mathcal{M} , \mathcal{V} and \mathcal{P} are solutions of these three Riccati equations. There is the mean of the control's cost function

$$\begin{aligned} & \dot{\mathcal{M}} + A'\mathcal{M} + \mathcal{M}A + Q_1 - MBR_1^{-1}B'\mathcal{M} \\ & - \frac{1}{\delta^2} \mathcal{P}DD'\mathcal{M} - \frac{1}{\delta^2} \mathcal{M}DD'\mathcal{P} \\ & + \gamma^2 \mathcal{V}BR_1^{-1}B'\mathcal{V} = 0 \end{aligned} \quad (10)$$

where $\mathcal{M}(t_f) = Q_f^1$. Next, we derive an expression for the variance.

$$\begin{aligned} & \dot{\mathcal{V}} + A'\mathcal{V} + \mathcal{V}A - \gamma MBR_1^{-1}B'\mathcal{V} - \gamma \mathcal{V}BR_1^{-1}B'\mathcal{M} \\ & - \frac{1}{\delta^2} \mathcal{P}DD'\mathcal{V} - \frac{1}{\delta^2} \mathcal{V}DD'\mathcal{P} - 2\gamma \mathcal{V}BR_1^{-1}B'\mathcal{V} \\ & + 4MEWE' \mathcal{M} = 0 \end{aligned} \quad (11)$$

with $\mathcal{V}(t_f) = 0$. Finally an expression for the mean of the disturbance's cost is given by

$$\begin{aligned} & \dot{\mathcal{P}} + A'\mathcal{P} + \mathcal{P}A - (\mathcal{M} + \gamma\mathcal{V})BR_1^{-1}B'\mathcal{P} \\ & - \mathcal{P}BR_1^{-1}B'(\mathcal{M} + \gamma\mathcal{V}) - \frac{1}{\delta^2} \mathcal{P}DD'\mathcal{P} \\ & - Q_2 - MBR_1^{-1}R_2R_1^{-1}B'\mathcal{M} \\ & - \gamma MBR_1^{-1}R_2R_1^{-1}B'\mathcal{V} - \gamma \mathcal{V}BR_1^{-1}R_2R_1^{-1}B'\mathcal{M} \\ & - \gamma^2 \mathcal{V}BR_1^{-1}R_2R_1^{-1}B'\mathcal{V} = 0 \end{aligned} \quad (12)$$

with $\mathcal{P}(t_f) = Q_f^2$. Notice that when the system is linear and the costs are quadratic in the stochastic game, we know the equilibrium solution (μ^*, ν^*) when these Riccati equations are satisfied [2]. Also as γ goes to zero, the equilibrium solution becomes the solution of H_2/H_∞ control problem. This suggests that MCV/ H_∞ control is a cumulant generalization of H_2/H_∞ control [9].

III. MCV/ H_∞ CONTROL AS A GENERALIZATION OF H_2/H_∞ CONTROL

In the H_2/H_∞ control technique, one wishes to minimize the H_2 norm of the system while constraining the H_∞ norm to some value, in our case we will call this value δ . The Nash game approach to solving the H_2/H_∞ problem was undertaken by [4] and [10]. The H_2/H_∞ control problem can be approached by finding a Nash equilibrium solution to a two person game in which one player is the control and the other player is the disturbance. For the stochastic case, the control wishes to minimize the mean of its cost function J_1 and the disturbance wishes to minimize the mean of J_2 , the disturbance's cost function as given in (5). The control wishing to minimize the mean of J_2 corresponds with the H_2 norm, while the mean of J_2 corresponds with a constraint on the H_∞ norm. The stochastic differential equation governing the system is still (1). For $E\{J_2\} \geq 0$, we have

$$E \left\{ \int_{t_0}^{t_f} (\delta^2 v'(t)v(t) - z_2'(t)z_2(t)) dt \right\} \geq 0. \quad (13)$$

With some manipulation this becomes

$$E \left\{ \int_{t_0}^{t_f} \delta^2 v'(t)v(t)dt \right\} \geq E \left\{ \int_{t_0}^{t_f} z_2'(t)z(t)dt \right\}$$

and by interchanging the integration and expectation, we obtain

$$\int_{t_0}^{t_f} \delta^2 E \{v'(t)v(t)\} dt \geq \int_{t_0}^{t_f} E \{z_2'(t)z(t)\} dt.$$

This then becomes

$$\int_{t_0}^{t_f} E \left\{ \|z_2(t)\|^2 \right\} dt \leq \delta^2 \int_{t_0}^{t_f} E \left\{ \|v(t)\|^2 \right\} dt \quad (14)$$

where $\|z(t)\|^2 = z'(t)z(t)$. However, the two-norm for the stochastic case and signal w , as given in [10], is defined as

$$\|w\|_{2,[t_0,t_f]}^2 = \int_{t_0}^{t_f} E \left\{ \|w(t)\|^2 \right\} dt. \quad (15)$$

Using this and substituting into (14), we have

$$\|z_2(t)\|_{2,[t_0,t_f]} \leq \delta \|v(t)\|_{2,[t_0,t_f]}. \quad (16)$$

The H_∞ norm of a system T_{z_2v} is defined as

$$\|T_{z_2v}\|_{\infty,[t_0,t_f]} = \sup_v \frac{\|z_2\|_{2,[t_0,t_f]}}{\|v\|_{2,[t_0,t_f]}}. \quad (17)$$

What we can see is that by minimizing the mean of J_2 with respect to the disturbance v , we in effect constrain the H_∞ norm of the system.

Using the Nash game approach to the H_2/H_∞ problem gives the following solutions from [4],[10].

$$\begin{aligned} u^* &= \mu^*(t, x) = -R^{-1}(t)B(t)\mathcal{M}(t)x(t) \\ v^* &= \nu^*(t, x) = -\frac{1}{\delta^2}D'\mathcal{P}(t)x(t) \end{aligned} \quad (18)$$

with

$$\begin{aligned} \dot{\mathcal{M}} + A'\mathcal{M} + \mathcal{M}A + Q_1 - \mathcal{M}BR_1^{-1}B'\mathcal{M} \\ - \frac{1}{\delta^2}\mathcal{P}DD'\mathcal{M} - \frac{1}{\delta^2}\mathcal{M}DD'\mathcal{P} = 0 \end{aligned} \quad (19)$$

and

$$\begin{aligned} \dot{\mathcal{P}} + A'\mathcal{P} + \mathcal{P}A - \mathcal{M}BR_1^{-1}B'\mathcal{P} \\ - \mathcal{P}BR_1^{-1}B'\mathcal{M} - \frac{1}{\delta^2}\mathcal{P}DD'\mathcal{P} \\ - Q_2 - \mathcal{M}BR_1^{-1}R_2R_1^{-1}B'\mathcal{M}. \end{aligned} \quad (20)$$

In MCV control, we minimize the variance of J_1 while holding the mean of J_1 to a constraint. This can be seen as a generalization of H_2 control. Meanwhile, we also consider some uncertainty to the system that can be better viewed through the H_∞ technique. In the same vein, we can view MCV/H_∞ as a generalization of H_2/H_∞ . Therefore we generalize the Nash game approach to the MCV/H_∞ case, where the control considers the variance of its cost J_1 and the disturbance still considers the mean of J_2 . The disturbance's performance index of the mean of its cost corresponds to a constraint on the H_∞ norm, as has been shown.

Notice that these Riccati equations (20) are very similar to those in the MCV/H_∞ case. In fact, as the parameter

γ goes toward zero, the MCV/H_∞ control Riccati equations become these from the H_2/H_∞ problem. This is a generalization of the H_2/H_∞ problem. As the parameter γ goes toward zero, the weight of the variance on the performance index becomes less and less, which puts the weight on the mean constraint. When it is in fact zero, the problem is simply the H_2/H_∞ control problem. So through the use of cumulants and Nash game theory, we can have a generalization of H_2/H_∞ . Furthermore, we can allow the control to use higher order cumulants as in the MCV control problem, but also design for some uncertainties that can be represented through the H_∞ norm of the system.

IV. SYSTEM MODEL

We consider the satellite's attitude control using four reaction wheels and three thrusters. The actual system description and its parameters of a LEO satellite KOMPSAT is given in [11]. The KOMPSAT is a sun-synchronous remote sensing satellite with the inclination of 98.13 degrees, the altitude of 685 km, and the total weight of 509 kg. The general nonlinear satellite attitude dynamic model is given by

$$\begin{aligned} I_g \dot{\underline{\omega}} = -\underline{\omega} \times (I_t \underline{\omega} + L' I_w \underline{\Omega}) - L' \underline{\tau}_w \\ + \underline{\tau}_{thruster} + \underline{\tau}_{gravity} + \underline{v} \end{aligned} \quad (21)$$

where \underline{v} represents the disturbances due to the magnetic field, solar radiation pressure, and atmospheric drag. The states are given as $\underline{x} = [\phi, \theta, \psi, \omega_x, \delta\omega_y, \omega_z, \Omega_1, \Omega_2, \Omega_3, \Omega_4]'$ $\equiv [\underline{\phi}_c, \underline{\omega}, \underline{\Omega}]'$ where $\omega_x, \omega_y, \omega_z$ are the angular velocities in Body Fixed Coordinate (BFC) system, Ω_i are the wheel speeds, ϕ is the roll Euler angle, θ is the pitch Euler angle, and ψ is the yaw Euler angle. We assume that roll, pitch, and yaw angles are available using the conical earth sensor, the fine sun sensor, and the gyros. In addition, the wheel speeds are also available from the measurement of the reaction wheel tachometer. Also the following are defined:

- 1) n : Orbital rate
- 2) I_t : Total amount of inertia for satellite body (3×3)
- 3) I_w : Wheel moment of inertia matrix (4×4)
- 4) $I_g = I_t - L' I_w L$: Total moment inertial minus moment of inertia of the wheels (3×3)
- 5) L : Wheel orientation matrix (4×3)

$$L = \begin{bmatrix} \cos \alpha \sin \beta & \sin \alpha \sin \beta & \cos \beta \\ -\sin \alpha \sin \beta & \cos \alpha \sin \beta & \cos \beta \\ -\cos \alpha \sin \beta & -\sin \alpha \sin \beta & \cos \beta \\ \sin \alpha \sin \beta & -\cos \alpha \sin \beta & \cos \beta \end{bmatrix} \quad (22)$$

To have zero initial conditions we let $\delta\omega_y = \omega_y + n$, and find the general linear equation form of (21) for any Torque Equilibrium Attitude (TEA) of the states. For the simplicity sake, we assume the case of the attitude hold mode where the TEA values are fixed at the following values: $\underline{x}_0 = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0]'$. Assuming the gravity gradient torque for a point mass, we have the following linearized equations.

$$\begin{aligned} \dot{\underline{\omega}} \equiv I_g^{-1} n N_1 \underline{\omega} + I_g^{-1} n N_2 \underline{\Omega} + I_g^{-1} 3n^2 N_3 \underline{\phi}_c \\ - I_g^{-1} L' \tau_{wheel} + I_g^{-1} \tau_{thruster} + I_g^{-1} \underline{v}, \end{aligned} \quad (23)$$

$$\dot{\underline{\Omega}} = I_w^{-1} \underline{\tau}_w - L \dot{\underline{\omega}}, \quad (24)$$

and

$$\begin{aligned} \dot{\underline{\phi}}_c = & [0, 0, -n]' \phi + [n, 0, 0]' \psi + [1, 0, 0]' \omega_x \\ & + [0, 1, 0]' (\omega_y + n) + [0, 0, 1]' \omega_z. \end{aligned} \quad (25)$$

Because $\delta\omega_y = \omega_y + n$, we have

$$\begin{aligned} \dot{\phi} &= n\psi + \omega_x \\ \dot{\theta} &= \omega_y + n = \delta\omega_y \\ \dot{\psi} &= -n\phi + \omega_z \end{aligned} \quad (26)$$

Using (23), (24), and (26), we can obtain a linearized differential equation of (1) with

$$A = \begin{bmatrix} X_1 & I_3 & 0_{3 \times 4} \\ 3I_g^{-1}n^2N_3 & I_g^{-1}nN_1 & I_g^{-1}nN_2 \\ -3LI_g^{-1}n^2N_3 & -LI_g^{-1}nN_1 & -LI_g^{-1}nN_2 \end{bmatrix} \quad (27)$$

where

$$X_1 = \begin{bmatrix} 0 & 0 & n \\ 0 & 0 & 0 \\ -n & 0 & 0 \end{bmatrix} \quad (28)$$

$$N_1 = \begin{bmatrix} 0 & 0 & I_t(3,3) - I_t(2,2) \\ 0 & 0 & 0 \\ I_t(2,2) - I_t(1,1) & 0 & 0 \end{bmatrix} \quad (29)$$

$$N_2 = \begin{bmatrix} I_w(1,1)L_t(3,1) & I_w(2,2)L_t(3,2) \\ 0 & 0 \\ -I_w(1,1)L_t(1,1) & -I_w(2,2)L_t(1,2) \\ I_w(3,3)L_t(3,3) & I_w(4,4)L_t(3,4) \\ 0 & 0 \\ -I_w(3,3)L_t(1,3) & -I_w(4,4)L_t(1,4) \end{bmatrix} \quad (30)$$

$$N_3 = \begin{bmatrix} I_t(3,3) - I_t(2,2) & 0 & 0 \\ 0 & I_t(3,3) - I_t(1,1) & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (31)$$

$L_t = L'$ and $X(i, j)$ is the element of a matrix X in the i th row and j th column. The control is given by the torque due to reaction wheels and the thrusters, $\underline{u} = [\underline{\tau}_w, \underline{\tau}_{thruster}]'$. The control's input matrix is given as

$$B = \begin{bmatrix} 0_{3 \times 4} & 0_{3 \times 3} \\ -I_g^{-1}L' & I_g^{-1} \\ I_w^{-1} + LI_g^{-1}L' & -LI_g^{-1} \end{bmatrix} \quad (32)$$

and the disturbance's input matrix is

$$D = \begin{bmatrix} 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} \\ I_g^{-1} & I_g^{-1} & I_g^{-1} \\ -LI_g^{-1} & -LI_g^{-1} & -LI_g^{-1} \end{bmatrix} \quad (33)$$

and E is a vector of ones.

Control Method	ω_x	Time (sec)
H_∞ ($\delta = 4$)	$5.0e-5$	128.38
H_2/H_∞ ($\delta = 4$)	$5.0e-5$	123.77
MCV/ H_∞ ($\delta = 4$ and $\gamma = 1.0e-7$)	$5.0e-5$	117.12

TABLE I

ANGULAR VELOCITY ω_x AND TIME OF THREE CONTROL METHODS

V. SIMULATION RESULTS

In this section, we investigate MCV/ H_∞ control performance and compare the angular velocity and the control action as well as the stability margin of H_∞ and H_2/H_∞ control. All simulations in this paper are based on the following parameters; orbital rate $n = 1.06e-3$ rad/s, total moment of inertia for the spacecraft body,

$$I_t = \begin{bmatrix} 217.38 & 0 & 0 \\ 0 & 95.57 & 0 \\ 0 & 0 & 154.57 \end{bmatrix}$$

and the moment of inertia matrix for the reaction wheels,

$$I_w = 0.0077 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We let the state weighting matrix $Q_1 = Q_2 = Q = I_{10 \times 10}$ and the control weighting matrix $R_1 = R_2 = R = I_{7 \times 7}$. The initial condition is given in the simulation as $x_0 = [1deg, 1deg, 1deg, 0, -n, 0, 0, 0, 0, 0]$.

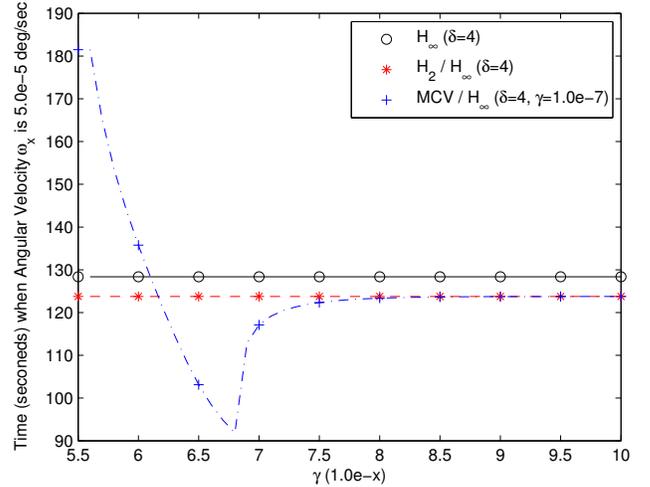


Fig. 1. Time when Angular Velocity ω_x is $5.0e-5$ deg/sec Versus γ

Before we start, we have to determine the parameter δ of H_∞ , H_2/H_∞ , and MCV/ H_∞ control and the parameter γ of MCV/ H_∞ control. For fixed δ , we can always find γ which will improve the performance of the system. In the first simulation, we fixed $\delta = 4$ and vary γ for the performance comparison. We define the settling time as the

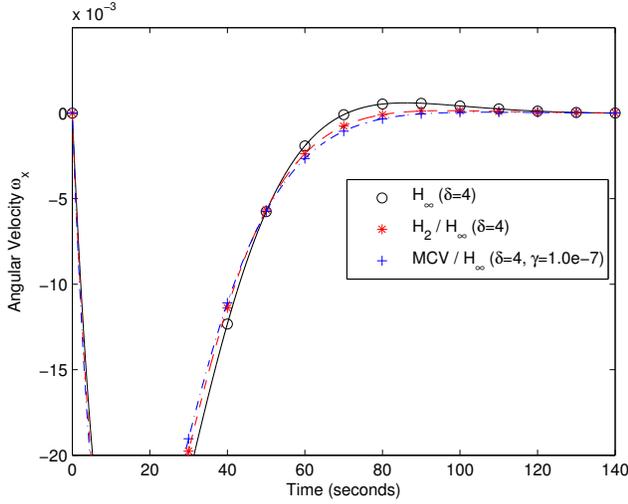


Fig. 2. Angular Velocity ω_x Versus Time

Control Method	Overshoot	Undershoot
H_∞ ($\delta = 4$)	$6.30e-4$	$-3.12e-2$
H_2/H_∞ ($\delta = 4$)	$1.40e-4$	$-3.09e-2$
MCV/H_∞ ($\delta = 4$ and $\gamma = 1.0e-7$)	$5.42e-5$	$-3.01e-2$

TABLE II

OVERSHOOT AND UNDERSHOOT ANGULAR VELOCITY ω_x OF THREE CONTROL METHODS

time required of angular velocity ω_x to reach $5.0e-5$ deg/sec. Figure 1 shows the time when the angular velocity ω_x is $5.0e-5$ deg/sec versus δ for H_∞ , H_2/H_∞ , and MCV/H_∞ control. Notice that if δ is 4, three Riccati equations have unique solutions \mathcal{M} , \mathcal{V} , \mathcal{P} from $\gamma = 1.0e-5.6$. Since the settling time of MCV/H_∞ control is faster than that of H_∞ and H_2/H_∞ control from $\gamma = 1.0e-6.2$ to $1.0e-8$, we use $\gamma = 1.0e-7$ for the following simulation. Also one can notice that if γ of MCV/H_∞ control approaches 0, then we have H_2/H_∞ control. From this, we can verify that MCV/H_∞ control is a generalization of H_2/H_∞ control.

Figure 2 shows the angular velocity ω_x versus time graph for H_∞ , H_2/H_∞ , and MCV/H_∞ control. Table I shows the angular velocity ω_x and the time when ω_x is $5.0e-5$ deg/sec. It shows that the time required to reach $5.0e-5$ deg/sec of ω_x in the MCV/H_∞ control is 117.12 sec which is 6.65 sec quicker than that of H_2/H_∞ control. Table II shows the largest overshoot and undershoot values of Figure 2. MCV/H_∞ control has the smallest overshoot

Control Method	Time	$ \tau_{wheel} $
H_∞ ($\delta = 4$)	128.38s	$1.9e-10$
H_2/H_∞ ($\delta = 4$)	123.77s	$1.4e-10$
MCV/H_∞ ($\delta = 4$ and $\gamma = 1.0e-7$)	117.12s	$6.3e-11$

TABLE III

CONTROL ACTION DUE TO REACTION WHEELS FOR THREE CONTROL METHODS

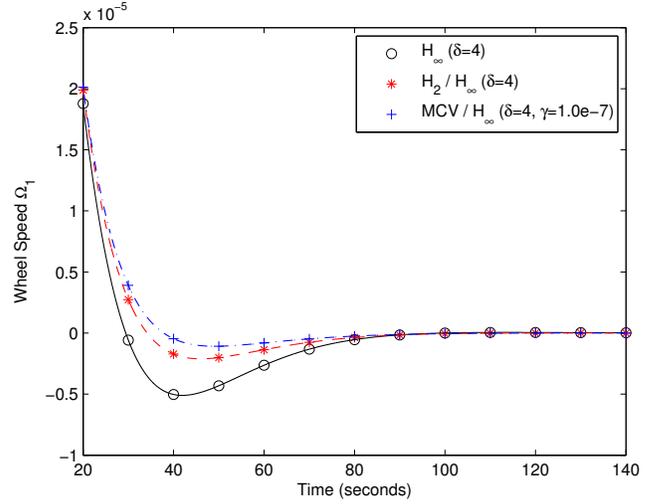


Fig. 3. Wheel Speed Ω_1 Versus Time

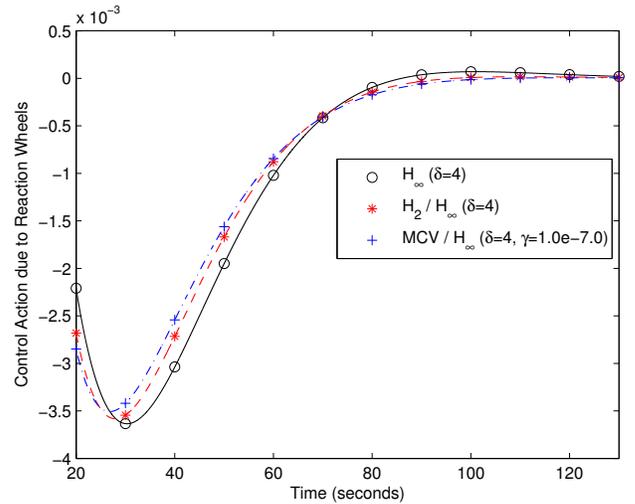


Fig. 4. Control Action due to Reaction Wheels Versus Time

and H_∞ control has the highest overshoot. H_2/H_∞ control performance is between the other two control methods.

The wheel speed Ω_1 is also simulated in Figure 3. In this plot, one can see that MCV/H_∞ control has the smallest undershoot at $-1.1e-6$, and has the fastest settling time. H_∞ control has the largest undershoot at $-5.1e-6$, and has the slowest settling time. The undershoot and the settling time of H_2/H_∞ control is in between MCV/H_∞ and H_∞ control. We note that the other three wheel speeds Ω_2 , Ω_3 , and Ω_4 achieve the similar performance as Ω_1 .

We have two control actions given by reaction wheel torques τ_{wheel} and thruster torques $\tau_{thruster}$. Figure 4 plots the magnitude of τ_{wheel} versus time and Table III shows the amount of τ_{wheel} for the specific times. We choose the times to match the times of Table I. At those times, we found that the magnitude of τ_{wheel} value of MCV/H_∞ control is

Control Method	Time	$\mathcal{I}_{thruster}$
H_∞ ($\delta = 4$)	128.38s	$2.4e-5$
H_2/H_∞ ($\delta = 4$)	123.77s	$2.3e-5$
MCV/ H_∞ ($\delta = 4$ and $\gamma = 1.0e-7$)	117.12s	$7.8e-6$

TABLE IV
CONTROL ACTION DUE TO THRUSTERS FOR THREE CONTROL METHODS

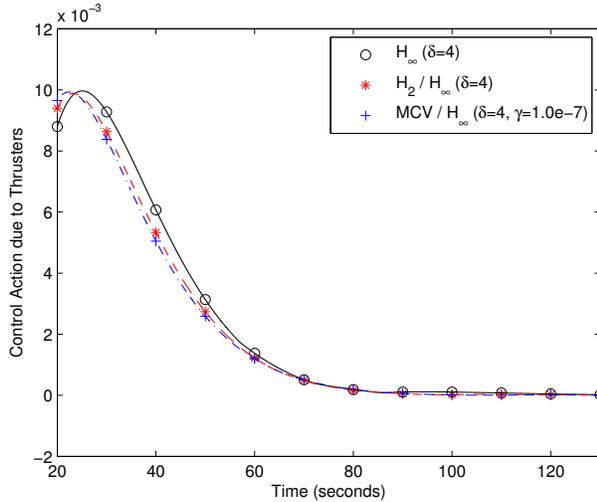


Fig. 5. Control Action due to Thrusters Versus Time

Control Method	Max. Real Part
H_∞ ($\delta = 4$)	$-4.56e-2$
H_2/H_∞ ($\delta = 4$)	$-5.44e-2$
MCV/ H_∞ ($\delta = 4$ and $\gamma = 1.0e-7$)	$-5.67e-2$

TABLE V
ROBUST STABILITY COMPARISON

the smallest and H_∞ control is the largest. We also show the control action due to thrusters $\mathcal{I}_{thrusters}$ in Figure 5 and Table IV. From these results, we find that MCV/ H_∞ control requires the smallest control effort, and H_∞ control requires the largest. Again, H_2/H_∞ is in between two controls.

Finally, we provide the stability results. Table V shows the maximum real part of the closed loop system poles for H_∞ , H_2/H_∞ , and MCV/ H_∞ control. H_∞ control has the largest closed loop eigenvalue, MCV/ H_∞ control has the smallest eigenvalue, and H_2/H_∞ is in between. From the stability point of view, MCV/ H_∞ control gives the highest stability margin, H_∞ control gives the smallest, and H_2/H_∞ control is in between.

VI. CONCLUSIONS

In this paper, multi-objective statistical control method for satellite attitude control is studied. In doing this, we consider a stochastic version of a system and a system with two players, a control and a disturbance. A Nash game approach was taken to solve the case when two players wish to optimize their own performance indices, namely

two different linear combinations of cost cumulants. The performance comparison of MCV/ H_∞ control with two different control methods H_∞ and H_2/H_∞ is provided. We compared these three control methods for $\delta = 4$ and $\gamma = 1.0e-7$. By comparing those performances, MCV/ H_∞ control has the fastest settling time in reaching $5.0e-5$ deg/sec angular velocity ω_x , and requires the least control effort, and H_∞ control has the slowest settling time and requires the most control effort. H_2/H_∞ control showed the intermediate performance between MCV/ H_∞ and H_∞ control. MCV/ H_∞ control showed the smallest overshoot and undershoot in time response because we optimized with respect to the variations. Finally, from the stability point of view, MCV/ H_∞ control has the largest stability margin, H_∞ control has the smallest, and again H_2/H_∞ control is in between. We conclude that MCV/ H_∞ control is a cumulant generalization of H_2/H_∞ control and the performance of MCV/ H_∞ control is better than H_∞ and H_2/H_∞ control for satellite's attitude control.

REFERENCES

- [1] K. D. Pham, M. K. Sain, and S. R. Liberty, "Cost Cumulant Control: State-Feedback, Finite-Horizon Paradigm with Application to Seismic Protection", *Special Issue of Journal of Optimization Theory and Applications*, Vol. 116, No. 3, pp 685-710, December 2002.
- [2] R. W. Diersing, " H_∞ , Cumulants, and Control", *Ph.D. Dissertation, Department of Electrical Engineering, University of Notre Dame*, August 2006.
- [3] D. S. Bernstein, W. M. Hassas, "LQG Control with an H_∞ Performance Bound: A Riccati Equation Approach", *IEEE Transactions on Automatic Control*, Vol. 34, No. 3, pp. 293-305, 1989.
- [4] D. J. N. Limebeer, B. D. O. Anderson, and D. Hendel, "A Nash Game Approach to Mixed H_2/H_∞ control", *IEEE Transactions on Automatic Control*, Vol. 39, No. 1, pp. 69-82, January 1994.
- [5] T. Basar, G. J. Olsder, *Dynamic Noncooperative Game Theory*, 2ed., SIAM, Philadelphia, 1999.
- [6] M. K. Sain, "Relative Costs of Mean and Variance Control for a Class of Linear Noisy Systems", *Proceedings 3rd Annual Allerton Conference on Circuit and System Theory*, pp.121-129, 1965.
- [7] C.-H. Won, "Nonlinear n-th Cost Cumulant Control and Hamilton-Jacobi-Bellman Equations for Markov Diffusion Process", *Proceedings of 44th IEEE Conference on Decision and Control*, Seville, Spain, pp. 4524-2529, 2005.
- [8] M. K. Sain, C.-H. Won, B. F. Spencer, Jr., and S. R. Liberty, "Cumulants and Risk-Sensitive Control: A Cost Mean and Variance Theory with Application to Seismic Protection of Structures", *Advances in Dynamic Games and Applications, Annals of the International Society of Dynamic Games*, Volume 5, pp. 427-459, Jerzy A Filar, Vladimir Gaitsgory, and Koichi Mizukami, Editors, Boston, Birkhauser, 2000.
- [9] R. W. Diersing and M. K. Sain, "A Cost Cumulant Control Problem: The Nash Game Solution, *Proceedings: American Control Conference*, Portland, Oregon, June 8, 2005.
- [10] X. Chen, K. Zhou, "Multiobjective H_2/H_∞ Control Design," *SIAM Journal of Control and Optimization*, vol. 40, no. 2, pp. 628-660, 2001.
- [11] C.-H. Won, "Comparative Study of Various Control Methods for Attitude Control of a LEO Satellite," *Aerospace Science and Technology*, No. 5, pp. 323-333, 1999.