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Closed Form Solutions of the Sylvester and the Lyapunov Equations - Closed Form Gramians

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***Abstract*— In this paper, closed form expressions of linear MIMO system responses are used to express the solution of the Sylvester equation in closed form. The solution makes use of matrix polynomial formulations of e^{tA}. The solutions to the input and the output Lyapunov equations, i.e. the input and the output Gramians, are then presented as a special case. From these the Hankel singular values can be computed. The final expressions are presented in a form that emphasizes efficient computational implementation and the resulting time complexity.**

I. Introduction

There exists an extensive literature within the fields of ordinary differential equations, difference equations, matrix theory and Laplace transforms on closed formed expressions. The majority of such results, however, predates the computer era, and is not presented in a form that has onus on efficient algorithmic implementations. This fact, somewhat surprisingly, is still reflected in modern textbooks, e.g., in control theory, in the area of signals and systems as well as mathematics. In these textbooks, the corresponding types of results are presented in a restrictive setting, with little or no attention to how they could be implemented in general algorithms. Computer algorithms that have been developed over recent decades, e.g., within control theory and mathematics, on the other hand, are often based on general approaches to numerical solutions of ordinary differential equations and linear equations that do not make specific use of the structure that lies in the closed form expressions.

Naturally, much attention has been given to numerical methods during the past decades with the rapid development of fast computers. Those generally provide approximate solutions which are often applicable to large systems, see e.g., [1] regarding the computation of matrix exponentials and [2] and [3] regarding the solutions of Lyapunov equations. Despite the effectiveness and advantages of such numerical methods, closed form time domain solutions nevertheless provide direct, easy and accurate computation for small to midsize systems. Further, closed form solutions open a window of opportunities definitely worth exploring, e.g. in the control area for the design of controllers and model reduction, both in their own right for small to midsize systems and by combining them with numerical methods for large systems.

Closed form continuous time transfer function expressions (SISO case) were derived in [4] and extended to the case of repeated eigenvalues in [5]. The closed form lends itself well to computation and analysis of transfer function responses and opens up many new interesting applications, e.g., solving for optimal zero locations by minimizing transient responses [4]; tracking a given reference step response in [6], and addressing the model reduction problem by $\mathcal{L}_2/\mathcal{H}_2$ minimization in [7]. The closed form expressions were further used in the direct computation of coefficients for PID and generalized PID controllers in [8] and [9].

It is of interest to extend the results obtained for SISO systems to the MIMO case. Clearly, the results obtained for SISO systems can be used directly for MIMO systems in the transfer function matrix form. In the case of MIMO systems in the state space form, the computation of the matrix exponential e^{tA} becomes of interest. Many different approaches have been proposed to compute the matrix exponential based, e.g. on eigenvector expansions of the matrix A, rational approximations to the exponential function and exact polynomial representations making use of the Cayley Hamilton theorem, see e.g. [10], [11] and references therein. It should be noted that for large A matrices, the computation of the matrix exponential itself is not computationally attractive and may be plagued by roundoff error[11]. However, in the case of MIMO responses, the central computational task is to calculate the vector $e^{tA}b$ for a given vector b and a given matrix A. For this task, the computation can be arranged into a recursive procedure that lends itself to efficient implementation. These procedures can be derived in many different ways, making e.g. use of properties of confluent Vandermonde matrices and their inverses, interpolation polynomials and inverse Laplace transforms. In [12], we emphasized the connection with the Laplace transforms in computing e^{tA}, highlighting the potential benefits of the procedure by applying it to the task of calculating Gramians and solving the standard Lyapunov equation. This approach can e.g. be contrasted with that used in Matlab's `lyap` which transforms the corresponding system matrices to the Schur form, computes the solution of the resulting triangular system and transforms the solution back[13].

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While some of the basic ideas presented in [12] are certainly not new, cf e.g., [14] and [15], care has been taken to formulate them in a framework that can be readily implemented in a computational environment like Matlab, in an efficient manner. It should further be emphasized that the main motivation behind this work is to provide another tool in the linear systems toolbox, to be used along with methods that have already been developed, e.g. based on numerical approaches, indeed these may support each other in further development.

In this paper, we extend the work in [12] to solving the Sylvester equation in its most general form, and then apply the resulting closed form to the input and the output Lyapunov equations, resulting in closed form Gramians from which Hankel singular values can be computed. Finally, numerical examples illustrating these approaches are presented.

II. Mathematical Prerequisites

Consider the general state space representation of MIMO systems in the minimal form given by

$$\begin{array}{rcl} \dot{x} & = & Ax + Bu \\ y & = & Cx \end{array} \tag{1}$$

where A is an $n \times n$ matrix, B is $n \times p$ and C is $r \times n$. Assume the matrix A has the characteristic equation

$$\det(sI-A) = \sum_{i=0}^{n} a_i s^i = (s-\lambda_1)^{d_1}(s-\lambda_2)^{d_2}\cdots(s-\lambda_\nu)^{d_\nu}, \tag{2}$$

where $a_n = 1$, the rest of the $a_i's$ are real numbers and λ_i, $i = 1, \ldots, \nu$ are the eigenvalues of A. The corresponding Jordan matrix (assuming controllability and observability) is given by

$$J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_\nu \end{bmatrix} \tag{3}$$

with the diagonal blocks

$$J_i = \begin{bmatrix} \lambda_i & 1 & \cdots & \cdots & 0 \\ 0 & \lambda_i & 1 & & \vdots \\ 0 & 0 & \lambda_i & 1 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 & \lambda_i \end{bmatrix}, \tag{4}$$

each a $d_i \times d_i$ matrix.

Now, consider a basic rational function with a unity numerator:

$$\begin{aligned} F_b(s) &= \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} \\ &= \frac{1}{(s-\lambda_1)^{d_1}(s-\lambda_2)^{d_2}\cdots(s-\lambda_\nu)^{d_\nu}} \\ &= \sum_{i=1}^{\nu}\sum_{j=1}^{d_i} \frac{\kappa_{ij}}{(s-\lambda_i)^j}, \end{aligned} \tag{5}$$

where κ_{ij} are the basic partial fraction expansion coefficients which are easily computed recursively as in [5], i.e.,

$$\kappa_{ij} = \begin{cases} \prod_{q=1,q\neq i}^{\nu} \frac{1}{(\lambda_i-\lambda_q)^{d_q}}, \ j = d_i \\ \\ \sum_{q=1}^{d_i-j} \frac{\kappa_{i(j+q)}(-1)^q}{d_i-j} \times \\ \sum_{p=1,p\neq i}^{\nu} \frac{d_p}{(\lambda_i-\lambda_p)^q}, \ j = d_i-1, \ldots, 1. \end{cases} \tag{6}$$

The term basic response refers here to the response of a transfer function containing only poles and a unity numerator, i.e., the basic impulse response $y_b(t)$ is the solution of

$$y_b^{(n)}(t) + a_{n-1}y_b^{(n-1)}(t) + \ldots + a_0 y_b(t) = \delta(t), \qquad t > 0. \tag{7}$$

The basic response is then given by

$$y_b(t) = \sum_{i=1}^{\nu}\sum_{j=1}^{d_i} \kappa_{ij} \frac{t^{(d_i-j)}}{(d_i-j)!} e^{\lambda_i t} = \kappa^T \mathcal{E}(t), \qquad t > 0, \tag{8}$$

where

$$\kappa = \begin{bmatrix} \kappa_{11} & \cdots & \kappa_{1d_1} & \cdots & \kappa_{\nu 1} & \cdots & \kappa_{\nu d_\nu} \end{bmatrix}^T \tag{9}$$

and where $\mathcal{E}(t)$ is an $n \times 1$ vector containing the linearly independent basis functions

$$\mathcal{E}(t) = \begin{bmatrix} \mathcal{E}_1(t) \\ \mathcal{E}_2(t) \\ \vdots \\ \mathcal{E}_\nu(t) \end{bmatrix} \tag{10}$$

with

$$\mathcal{E}_i(t) = \begin{bmatrix} e^{\lambda_i t} \\ \frac{d}{d\lambda_i} e^{\lambda_i t} \\ \vdots \\ \frac{1}{(d_i-1)!}\frac{d^{d_i-1}}{d\lambda_i^{d_i-1}} e^{\lambda_i t} \end{bmatrix} = \begin{bmatrix} e^{\lambda_i t} \\ t e^{\lambda_i t} \\ \vdots \\ \frac{t^{(d_i-1)}}{(d_i-1)!} e^{\lambda_i t} \end{bmatrix}. \tag{11}$$

We can express e^{tA} in the matrix polynomial form[12]

$$e^{tA} = \sum_{i=0}^{n-1} \alpha_i(t) A^i, \tag{12}$$

$$= \sum_{i=0}^{n-1} (\gamma_i^T \mathcal{E}(t)) A^i, \tag{13}$$

consistent with the Cayley Hamilton theorem, where γ_i, $i = 0, 1, \ldots, n-1$, are $n \times 1$ vectors, which can be computed recursively as

$$\gamma_{n-1} = \kappa, \quad \gamma_{n-k-1} = J\gamma_{n-k} + a_{n-k}\kappa, \quad k = 1, 2, \ldots, n-1. \tag{14}$$

The derivatives of $y_b(t)$ are given by

$$y_b^{(k)}(t) = (J^k \kappa)^T \mathcal{E}(t), \ k = 1, 2, \ldots. \tag{15}$$

Remark 1: Let $y_b(t)$ be defined as in (8) and define

$$Y_b(t) = \begin{bmatrix} y_b(t) \\ y_b^{'}(t) \\ \vdots \\ y_b^{(n-2)}(t) \\ y_b^{(n-1)}(t) \end{bmatrix}, \tag{16}$$

then it follows from (15) and (14) that:

$$\begin{bmatrix} \gamma_0^T \\ \gamma_1^T \\ \vdots \\ \gamma_{n-2}^T \\ \gamma_{n-1}^T \end{bmatrix} \mathcal{E}(t) = \begin{bmatrix} a_1 & \cdots & a_{n-1} & 1 \\ \vdots & \cdot & \cdot & 0 \\ a_{n-1} & \cdot & \cdot & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} Y_b(t). \tag{17}$$

III. A CLOSED FORM SOLUTION OF THE SYLVESTER EQUATION

Consider the Gramian

$$X \equiv \int_0^\infty e^{t\mathcal{A}} \mathcal{F}\mathcal{G}^H e^{t\mathcal{B}^H} dt, \tag{18}$$

where X is an $m \times n$ matrix, $\mathcal{A}$ is $m \times m$, $\mathcal{B}$ is $n \times n$, $\mathcal{F}$ is $m \times p$ and $\mathcal{G}$ is $n \times p$. If $Re(\lambda_i + \mu_j) < 0$ for all eigenvalues λ_i of $\mathcal{A}$ and μ_j of $\mathcal{B}$. X satisfies the general Sylvester equation

$$\mathcal{A}X + X\mathcal{B}^H + \mathcal{F}\mathcal{G}^H = 0. \tag{19}$$

This is easily shown by observing that

$$\begin{aligned} &\mathcal{A}X + X\mathcal{B}^H = \int_0^\infty \left(\mathcal{A}e^{t\mathcal{A}}\mathcal{F}\mathcal{G}^H e^{t\mathcal{B}^H} + e^{t\mathcal{A}}\mathcal{F}\mathcal{G}^H e^{t\mathcal{B}^H}\mathcal{B}^H \right) dt \\ &= \int_0^\infty d\left(e^{t\mathcal{A}}\mathcal{F}\mathcal{G}^H e^{t\mathcal{B}^H} \right) = -\mathcal{F}\mathcal{G}^H, \end{aligned} \tag{20}$$

where the combined strict stability condition is used in the last step.

Theorem 1: The Gramian X can be expressed as

$$X = \sum_{j=0}^{n-1} \sum_{i=0}^{m-1} \pi_{\mathcal{A}\mathcal{B},ij} \mathcal{A}^i \mathcal{F}\mathcal{G}^H \left(\mathcal{B}^H\right)^j, \tag{21}$$

where

$$\pi_{\mathcal{A}\mathcal{B},ij} = \gamma_{\mathcal{A},i}^T \int_0^\infty \mathcal{E}_\mathcal{A}(t)\mathcal{E}_\mathcal{B}(t)^H dt \bar{\gamma}_{\mathcal{B},j}, \tag{22}$$

with the γ_i's defined as in (14) and the $\mathcal{E}(t)$'s defined as in (10) for the strictly stable matrices $\mathcal{A}$ and $\mathcal{B}$.

Proof: We have from (13) that

$$e^{t\mathcal{A}}\mathcal{F} = \sum_{i=0}^{m-1} \left(\gamma_{\mathcal{A},i}^T \mathcal{E}_\mathcal{A}(t)\right) \mathcal{A}^i \mathcal{F} \tag{23}$$

and

$$\mathcal{G}^H e^{t\mathcal{B}^H} = \mathcal{G}^H \sum_{j=0}^{n-1} \left(\gamma_{\mathcal{B}^H,j}^T \mathcal{E}_{\mathcal{B}^H}(t)\right) \left(\mathcal{B}^H\right)^j. \tag{24}$$

Noting that $\gamma_i^T \mathcal{E}(t)$ is a scalar function, the result follows directly.

Q.e.d.

Remark 2: The (ρ, σ)–th element of the (k, j)–th subblock of $\int_0^\infty \mathcal{E}_\mathcal{A}(t)\mathcal{E}_\mathcal{B}(t)^H dt$, i.e., of the matrix $\int_0^\infty \mathcal{E}_{\mathcal{A},k}(t)\mathcal{E}_{\mathcal{B},j}(t)^H dt$ is given by

$$\int_0^\infty \mathcal{E}_{\mathcal{A},k}(t)\mathcal{E}_{\mathcal{B},j}(t)^H dt_{\rho,\sigma} = \frac{\binom{\rho+\sigma-2}{\rho-1}}{(-\lambda_{\mathcal{A},k} - \bar{\lambda}_{\mathcal{B},j})^{\rho+\sigma-1}}.$$

The subsequent evaluation of the $\pi_{\mathcal{A}\mathcal{B},ij}$ coefficients then requires $\sim 2mn(m+n)$ operations. Having calculated the coefficients $\pi_{\mathcal{A}\mathcal{B},ij}$, X can be calculated in three steps:

1) Calculate the matrices $\mathcal{A}^i F$ for $i = 0, 1, \ldots, m-1$ and $\mathcal{B}^j \mathcal{G}$ for $j = 0, 1, \ldots, n-1$ recursively, $\sim 2p(m^3+n^3)$ operations;
2) Calculate $\sum_{i=0}^{m-1} \pi_{\mathcal{A}\mathcal{B},ij} \mathcal{A}^i \mathcal{F}$ for $j = 0, 1, \ldots, n-1$, $\sim 2pm^2n$ operations;
3) Calculate $\sum_{j=0}^{n-1} \left(\sum_{i=0}^{m-1} \pi_{\mathcal{A}\mathcal{B},ij} \mathcal{A}^i \mathcal{F}\right) \left(\mathcal{B}^j \mathcal{G}\right)^H$, $\sim 2pmn^2$ operations;

i.e. a total of $\sim 2p(m^3 + m^2n + mn^2 + n^3)$ operations.

Remark 3: Assuming that we know both the eigenvalues of $\mathcal{A}$, $\lambda_1, \lambda_2, \ldots, \lambda_m$ and of $\mathcal{B}$, $\mu_1, \mu_2, \ldots, \mu_n$ as well as the corresponding coefficients of the characteristic equations, the calculation of the partial fraction coefficients, the vectors $\gamma_{\mathcal{A},i}$, $i = 0, 1, \ldots, m-1$ and $\gamma_{\mathcal{B},j}$, $j = 0, 1, \ldots, n-1$ require $\mathcal{O}(m^2)$ and $\mathcal{O}(n^2)$ operations, respectively, and the calculation of the matrix $\int_0^\infty \mathcal{E}_\mathcal{A}(t)\mathcal{E}_\mathcal{B}(t)^H dt$ requires $\mathcal{O}(mn)$ operations. Thus the main remaining computational task requires $\sim 2(pm^3 + (p+1)m^2n + (p+1)mn^2 + pn^3)$ operations. By contrast the solution of the Sylvester equation by the Hessenberg-Schur method requires $\sim 10m^3/3 + 10m^2n + 5mn^2 + 26n^3$ operations regardless of the value of p and does not depend on knowing either the eigenvalues nor the coefficients of the characteristic equation[16].

IV. SOLUTION OF THE LYAPUNOV EQUATION AND COMPUTATION OF HANKEL SINGULAR VALUES

Consider the input Gramian

$$P = \int_0^\infty e^{tA} BB^H e^{tA^H} dt \tag{25}$$

which satisfies the Lyapunov equation

$$AP + PA^H + BB^H = 0. \tag{26}$$

We can now write directly from *Theorem 1*

$$P = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \pi_{ij} A^i BB^H \left(A^H\right)^j, \tag{27}$$

where

$$\pi_{ij} = \gamma_i^T \int_0^\infty \mathcal{E}(t)\mathcal{E}(t)^H dt \bar{\gamma}_j, \tag{28}$$

with γ_i defined as in (14) and $\mathcal{E}(t)$ defined as in (10).

Remark 4: Making use of the symmetry of P, the main computational task requires in this case, following the same procedure as above, $\sim (5p+3)n^3$ operations, comparing with $\sim 32n^3$ operations required by the Schur algorithm[16].

Remark 5: We can make use of (17) and express

$$\pi_{ij} = \tilde{a}_{i+1}^T \int_0^\infty Y_b(t)Y_b(t)^H dt \tilde{a}_{j+1}, \tag{29}$$

where $\tilde{a}_i$ denotes the i-th column vector of the matrix on the right hand side of (17). Here we note that the matrix $\int_0^\infty Y_b(t)Y_b(t)^H dt$ will have the following plaid like structure [17]

$$\mathcal{Y} = \begin{bmatrix} \mathcal{Y}_0 & 0 & -\mathcal{Y}_1 & 0 & \mathcal{Y}_2 & \cdots \\ 0 & \mathcal{Y}_1 & 0 & -\mathcal{Y}_2 & 0 & \\ -\mathcal{Y}_1 & 0 & \mathcal{Y}_2 & 0 & -\mathcal{Y}_3 & \\ 0 & -\mathcal{Y}_2 & 0 & \mathcal{Y}_3 & 0 & \\ \mathcal{Y}_2 & 0 & -\mathcal{Y}_3 & 0 & \ddots & \\ \vdots & & \ddots & & & \mathcal{Y}_{n-1} \end{bmatrix}, \quad (30)$$

where

$$\begin{aligned} \mathcal{Y}_i &= \int_0^\infty \left(y_b^{(i)}(t)\right)^2 dt \\ &= (J^i\kappa)^T \int_0^\infty \mathcal{E}(t)\mathcal{E}^H(t)dt\overline{J^i\kappa}. \end{aligned} \quad (31)$$

This follows from the fact that

$$\begin{array}{lll} y_b^{(i)}(0) = 0, & i = 0, 1, \ldots, n-2, & y_b^{(n-1)}(0) = 1 \\ \lim_{t\to\infty} y_b^{(i)}(t) = 0, & i = 0, 1, \ldots, n-1, & \end{array} \quad (32)$$

since assuming the system is strictly stable. In addition, it follows from Lyapunov's stability theorem, that $\mathcal{Y}$ is positive definite, also easily noted by the fact that for any nonzero vector c, we have that

$$c^T\mathcal{Y}c = \int_0^\infty (c^TY_b)^2 dt > 0. \quad (33)$$

Remark 6: Let (A^c, B^c) denote the controller (companion) form

$$A^c = \begin{bmatrix} 0_{(n-1)\times 1} & I_{(n-1)\times(n-1)} \\ -a_0 & -a_1 \ \cdots \ -a_{n-1} \end{bmatrix}, \ B^c = \begin{bmatrix} 0_{(n-1)\times 1} \\ 1 \end{bmatrix}. \quad (34)$$

Then it follows from the observation $Y_b(t) = e^{tA^c}B^c$ that $\mathcal{Y}$ satisfies the Lyapunov equation

$$A^c\mathcal{Y} + \mathcal{Y}(A^c)^T + B^c(B^c)^T = 0. \quad (35)$$

The last line in the Lyapunov equation can be written as

$$(A^c\mathcal{Y})_{n\cdot} + ((A^c\mathcal{Y})_{\cdot n})^T + \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} = 0_{1\times n}. \quad (36)$$

Then, transposing, rearranging and noting that the last element in the first two vectors will be the same, we can write

$$\mathcal{Y}\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} - \begin{bmatrix} 0 \\ (-1)^{\frac{n+1}{2}}\mathcal{Y}_{\frac{n+1}{2}} \\ 0 \\ \vdots \\ 0 \\ \mathcal{Y}_{n-1} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1/2 \end{bmatrix} \ n \text{ odd}, \quad (37)$$

and

$$\mathcal{Y}\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} - \begin{bmatrix} (-1)^{\frac{n-2}{2}}\mathcal{Y}_{\frac{n}{2}} \\ 0 \\ \vdots \\ 0 \\ \mathcal{Y}_{n-1} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1/2 \end{bmatrix} \ n \text{ even}. \quad (38)$$

Rewriting, we can solve for $\mathcal{Y}$ directly as a function of the a-coefficients when n is odd:

$$\begin{bmatrix} a_0 & a_2 & \cdots & \cdots & a_{n-1} & 0 & \cdots & 0 \\ 0 & a_1 & a_3 & \cdots & a_{n-2} & 1 & \cdots & 0 \\ 0 & a_0 & a_2 & \cdots & \cdots & a_{n-1} & \cdots & 0 \\ 0 & 0 & a_1 & \cdots & \cdots & a_{n-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & & & & & \vdots \\ \vdots & \vdots & & 0 & a_1 & a_3 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & a_0 & a_2 & \cdots & a_{n-1} \end{bmatrix} \times \begin{bmatrix} \mathcal{Y}_0 \\ -\mathcal{Y}_1 \\ \mathcal{Y}_2 \\ -\mathcal{Y}_3 \\ \vdots \\ \vdots \\ \mathcal{Y}_{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1/2 \end{bmatrix} \quad (39)$$

and when n is even:

$$\begin{bmatrix} a_0 & a_2 & \cdots & \cdots & 1 & 0 & \cdots & 0 \\ 0 & a_1 & a_3 & \cdots & a_{n-1} & 0 & \cdots & 0 \\ 0 & a_0 & a_2 & \cdots & \cdots & 1 & \cdots & 0 \\ 0 & 0 & a_1 & \cdots & \cdots & & \cdots & 0 \\ \vdots & \vdots & \ddots & & & & & \vdots \\ \vdots & \vdots & & a_0 & a_2 & \cdots & \cdots & 1 \\ 0 & 0 & \cdots & 0 & a_1 & a_3 & \cdots & a_{n-1} \end{bmatrix} \times \begin{bmatrix} \mathcal{Y}_0 \\ -\mathcal{Y}_1 \\ \mathcal{Y}_2 \\ -\mathcal{Y}_3 \\ \vdots \\ \vdots \\ -\mathcal{Y}_{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -1/2 \end{bmatrix} \quad (40)$$

The implication of these facts is that we can evaluate the coefficients π_{ij}, from (29), (39) and (40) without having to evaluate the eigenvalues λ_i, i.e. only by knowing the coefficients of the characteristic equation and by making use of the zero structure in (17), (30), (39) and (40), the computational task will in fact be slightly less than applying (13).

A dual result may be derived for the output Gramian and subsequently the Hankel singular values may be computed.

Consider the output Gramian

$$Q = \int_0^\infty e^{tA^H} C^H C e^{tA} dt \tag{41}$$

which satisfies the Lyapunov equation

$$A^H Q + QA + C^H C = 0. \tag{42}$$

It then follows by substituting (13) into (41), similarly as in the proof of *Theorem 1*, and then making use of (28)

$$Q = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \pi_{ij} (A^H)^i C^H C A^j. \tag{43}$$

Similarly, using duality $A \longleftrightarrow A^H$, $B \longleftrightarrow C^H$, we can directly write (43) from (27), by noting that the characteristic equations of the dual systems and thus (28) stay the same.

Now, having P and Q, the Hankel singular values are easily computed as

$$\sigma_i^2 = \lambda_i(PQ). \tag{44}$$

Finally, the cross Gramian

$$X = \int_0^\infty e^{tA} BC e^{tA} dt, \tag{45}$$

which is the solution to the Sylvester equation where $p = r$

$$AX + XA + BC = 0 \tag{46}$$

can by a similar argument be expressed as

$$X = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \pi_{ij} A^i BC A^j. \tag{47}$$

V. Examples

Example 1:

First, consider a fictitious example to demonstrate the solution of the Sylvester equation (19) where

$$\mathcal{A} = \begin{bmatrix} -9 & -4 & -4 \\ -2 & -8 & 0 \\ -6 & -7 & -8 \end{bmatrix}, \quad \mathcal{F} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \tag{48}$$

and

$$\mathcal{B} = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix}, \quad \mathcal{G} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}. \tag{49}$$

The eigenvalues of the $\mathcal{A}$ matrix are given by $-14.8574, -4.4303, -5.7123$ and the corresponding characteristic equation is given by $s^3+25s^2+176s+376 = 0$. Now the $\gamma_{\mathcal{A},i}$ vectors can be computed recursively as in (14), where $\gamma_{\mathcal{A},2}$ contains the partial fraction coefficients of the unity numerator transfer function computed from (6)

$$\gamma_{\mathcal{A},2} = \kappa_{\mathcal{A}} = \begin{bmatrix} 6.3490 \\ -5.6144 \\ 0.2654 \end{bmatrix}, \quad \gamma_{\mathcal{A},1} = \begin{bmatrix} 1.5388 \\ -1.6452 \\ 0.1064 \end{bmatrix},$$

$$\gamma_{\mathcal{A},0} = \begin{bmatrix} 0.0748 \\ -0.0853 \\ 0.0105 \end{bmatrix}. \tag{50}$$

Similarly, the eigenvalues of the $\mathcal{B}$ matrix are given by $-1, -5$ and the corresponding characteristic equation is given by $s^2+6s+5 = 0$. Then, the $\gamma_{\mathcal{B},i}$ vectors (14) result in

$$\gamma_{\mathcal{B},1} = \kappa_{\mathcal{B}} = \begin{bmatrix} 1.25 \\ -0.25 \end{bmatrix}, \quad \gamma_{\mathcal{B},0} = \begin{bmatrix} 0.25 \\ -0.25 \end{bmatrix}. \tag{51}$$

We now compute

$$\int_0^\infty \mathcal{E}(t)\mathcal{E}(t)^H dt = \begin{bmatrix} 0.1842 & 0.1060 \\ 0.1490 & 0.0934 \\ 0.0631 & 0.0504 \end{bmatrix} \tag{52}$$

utilizing *Remark 2*. We then compute $\pi_{\mathcal{AB}}$ from (22) resulting in

$$\pi_{\mathcal{AB}} = \begin{bmatrix} 0.3962 & 0.0467 \\ 0.0525 & 0.0075 \\ 0.0020 & 0.0003 \end{bmatrix}. \tag{53}$$

We finally obtain the solution to the Sylvester equation (19) from (22) applying steps 1)-3) in *Remark 2* resulting in

$$X = \begin{bmatrix} -0.2690 & -0.1777 \\ 0.2383 & 0.3688 \\ 0.3224 & 0.6337 \end{bmatrix}. \tag{54}$$

Example 2:

We will now use a well known MIMO model of a jet (see e.g. Matlab's Control Toolbox help) in order to illustrate the application of some of the formulae presented above. The inputs to the system, u_1 and u_2, symbolize the rudder and aileron deflections, respectively, in degrees. The outputs y_1 and y_2 represent the yaw rate and bank angle. The corresponding system matrices are given by

$$A = \begin{bmatrix} -0.0558 & -0.9968 & 0.0802 & 0.0415 \\ 0.5980 & -0.1150 & -0.0318 & 0 \\ -3.0500 & 0.3880 & -0.4650 & 0 \\ 0 & 0.0805 & 1.0000 & 0 \end{bmatrix}, \tag{55}$$

$$B = \begin{bmatrix} 0.0073 & 0 \\ -0.4750 & 0.0077 \\ 0.1530 & 0.1430 \\ 0 & 0 \end{bmatrix}, \tag{56}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{57}$$

The systems eigenvalues are given by $-0.0329 \pm 0.9467i, -0.5627, -0.0073$ and the corresponding characteristic equation is given by $s^4 + 0.6358s^3 + 0.9389s^2 + 0.5116s + 0.0037 = 0$. Now the γ_i vectors can be computed recursively as in (14) where γ_3 contains the partial fraction coefficients of the unity numerator transfer function,

$$\gamma_3 = \kappa = \begin{bmatrix} -0.2388 + 0.4553i \\ -0.2388 - 0.4553i \\ -1.5301 \\ 2.0078 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} -0.5750 + 0.0484i \\ -0.5750 - 0.0484i \\ -0.1119 \\ 1.2619 \end{bmatrix},$$

$$\gamma_1 = \begin{bmatrix} -0.2511 - 0.1184i \\ -0.2511 + 0.1184i \\ -1.3736 \\ 1.8759 \end{bmatrix}, \quad \gamma_0 = \begin{bmatrix} -0.0018 - 0.0009i \\ -0.0018 + 0.0009i \\ -0.0100 \\ 1.0136 \end{bmatrix}. \tag{58}$$

In order to compute directly the output Gramian, the solution to the Lyapunov matrix equation, we first compute

$$\int_0^\infty \mathcal{E}(t)\mathcal{E}(t)^H dt =$$

$$\begin{bmatrix} 15.1812 & 0.0184 + 0.5275i & 0.4761 + 0.7568i & 0.0448 + 1.0545i \\ 0.0184 - 0.5275i & 15.1812 & 0.4761 - 0.7568i & 0.0448 - 1.0545i \\ 0.4761 - 0.7568i & 0.4761 + 0.7568i & 0.8887 & 1.7546 \\ 0.0448 - 1.0545i & 0.0448 + 1.0545i & 1.7546 & 68.7005 \end{bmatrix} \tag{59}$$

utilizing *Remark 2*.

Now it is straightforward to compute the solution to the Lyapunov equation (22) applying steps 1)-3) in *Remark 2* and substituting $\mathcal{A} = \mathcal{B} = A$ and $\mathcal{F} = \mathcal{G} = B$ resulting in

$$P = \begin{bmatrix} 1.8663 & -0.0066 & -2.9371 & 8.0258 \\ -0.0066 & 1.7427 & -2.8782 & 18.3346 \\ -2.9371 & -2.8782 & 16.9103 & -1.4759 \\ 8.0258 & 18.3346 & -1.4759 & 524.8139 \end{bmatrix}. \tag{60}$$

We also get the same result using (27) and (28), or (29)-(31) and (39)-(40).

The output Gramian Q can be computed in an analogous manner, using (22) in steps 1)-3) in *Remark 2* and substituting $\mathcal{A} = \mathcal{B} = A^H$ and $\mathcal{F} = \mathcal{G} = C^H$ resulting in

$$Q = \begin{bmatrix} 73.3 & -85.0 & -18.0 & -12.0 \\ -85.0 & 2624.0 & 492.7 & 314.8 \\ -18.0 & 492.7 & 98.5 & 62.9 \\ -12.0 & 314.8 & 62.9 & 41.0 \end{bmatrix}. \tag{61}$$

We also get the same result using (43) and (28). The Hankel singular values can finally be computed from P and Q, i.e. using (44) resulting in

$$\begin{aligned} \sigma_1^2 &= 36034.81, \\ \sigma_2^2 &= 1.96, \\ \sigma_3^2 &= 131.41, \\ \sigma_4^2 &= 151.84. \end{aligned} \tag{62}$$

VI. CONCLUSIONS AND FUTURE WORK

In this paper, closed form expressions of linear MIMO system responses are used to express the solution of the Sylvester equation in closed form. The solution makes use of matrix polynomial formulations of e^{tA}. The input and the output Lyapunov equations and solutions of the input and the output Gramians are then presented as a special case of the Sylvester equation. From these the Hankel singular values can be computed. Two examples are then shown. The final expressions are presented in a form that emphasizes efficient computational implementation and the resulting time complexity.

The emphasis in this work has been on the derivation of computationally efficient formulations of closed form expressions. The short term motivation has simply been to provide another tool in the linear systems toolbox to be used along with methods that have already been developed based on numerical approaches. The aim is to develop criteria based on numerical efficiency and stability to aid in the choice of appropriate solution tools. It is finally hoped that such expressions may enhance the understanding of linear systems as such and provide new approaches to solving control problems. For example it may be of interest to make use of the closed form solutions of the Lyapunov equation in the solution of the Riccati equation by Newtons method, where a Lyapunov equation needs to be solved in each iteration (see e.g. [16], pp. 567).

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