2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008



# Heterogeneous Agent Models in Economics: A Study of Heterogenous Productivity of Sectors

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***Abstract*— Macroeconomic modeling is undergoing a change from the ground up. Previously models based on fully rational *representative agents* were constructed to give macroeconomics solid microeconomic foundations. However the representative agent models have been shown to be inconsistent with empirical evidence and a new method of approach has emerged, one based on heterogeneity of agents. Recently heterogenous models have been used to simulate expected outcomes but due to their complexity little analytic work has been done. In this paper a basic model of the macro economy, with heterogeneous sectors differentiated by productivity, and driven by a jump Markov process, is investigated and steady state solutions for a sector's output variance are discovered. We adjust the model to include a gain term, to represent a sector's reaction to its error signal, excess demand, and then linearize the transition rates and apply the fluctuation dissipation theorem to solve the model.**

## I. Introduction

The motivation for this paper is to show the potential for control theory to make progress in analyzing new models of macroeconomics and finance with heterogenous agents. The main result will be to show how the fluctuation dissipation theorem can be used to analyze a particular heterogenous jump Markov model of the macro economy. However, since some of the control audience might not be familiar with economic modeling, we will first give a brief introduction to macroeconomic modeling methodologies and in some part financial modeling. This will help to provide insight into why stochastic models with heterogenous agents are now being used in economics.

### A. Background

From the beginning of the industrial revolution, developed country's economic paths have experienced punctuated fluctuations of growth in output, income and employment. Since then the need to regulate these fluctuations has been generally agreed upon, however the method of solution has not. In fact, what is now commonly understood as the major cause of the Great Depression was poor policy that was designed to restore the economy to growth. Instead, the government's policy of reducing its expenditure in order to balance the budget in the midst of the 1930's depression is now clearly recognized as positive feedback which in the end induced a longer and deeper recession.

Most control engineers will tell you that positive feedback has great destabilizing effects, unfortunately policymakers were at first not aware that this is in fact what they were doing. Since then the concepts of negative feedback have been understood and applied by governments and economists for the past 70 years. This call for the use of negative feedback has helped reduce economic fluctuations, in magnitude and frequency, but it has not eliminated them.

The problem has been the conflict of two objectives; strong growth and stable prices. Add to this the fact that there are shock disturbances, uncertainty in the data and the model, and economic agents that can learn. It becomes increasingly more important to design better tools for analyzing a complex system such as the economy, to deliver increased performance in the face of uncertainty and constant shock disturbances. This is where control theory, traditionally used to study physical dynamical systems, is likely to also bode extremely useful for economists.

The system that we will be dealing with is the macroeconomy, which represents the aggregate behavior and choices made by individuals in a country. Because of the economies vast complexity coupled with the need to regulate it's fluctuations, the question of how to model the macro phenomenon of an economy has been a very challenging problem, one which has attracted the attention of a great many economists.

Ever since the 1930's the study of macro economic fluctuations did not separate individual from aggregate behavior. With these early models each had their own fundamental factor, whether it be the money supply or real factors such as technical progress. There were also discrepancies on whether to treat the economy as a system responding to random exogenous shocks or to purge the models of exogenous factors and construct models in which cycles are created endogenously.

John Maynard Keynes was one of the early central figures in macroeconomics for his Keynesian models of aggregate demand. In Keynesian theory, macro trends overwhelm individual behavior, and aggregate demand becomes the focal point of macroeconomic stabilization policy. Keynes's work, [1], was motivated by explaining the Great Depression, a period of high unemployment and deflation. Previous general equilibrium models, that incorporate market clearing dynamics, could not explain this. Keynesian models held their weight until Robert Lucas Jr., in the 1970's, changed the face of macroeconomics, with his "Lucas Critique," [2], by arguing that macroeconomic models should be built on microeconomic fundamentals.

After Lucas's call for involving microeconomic fundamentals to the study of macroeconomics a new branch of

This work was supported by the Gates Cambridge Trust.

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978-1-4244-2079-7/08/$25.00 ©2008 AACC.

economics based on the study of rational representative agents was created. This was typified by the Real Business Cycle theory, RBC, by Kydland and Prescott, [3]. In the 1980's, with focus now squarely on developing models with rational expectations, microfoundations became the focal point. In RBC theory the real business cycle is explained as the direct result of the behavior of individual agents. The motivation behind microeconomic foundations is to explain macro phenomenon as the total response of individual agents, termed the *representative agent*, to changes in their own environment. Current models based on the representative agent approach are characterized by an explicitly stated optimization problem of the representative agent. These individual behaviors are then assumed to sum up to the macroeconomic outcomes. This approach has been standard in macroeconomics for the last 25 years because it avoids the problem of having to deal with aggregation. However, this theory has recently been challenged for it's applicability.

Representative agent models imply rational expectations of agents and efficiency of markets and have been applied in both macroeconomics and finance. One implication of rational agents has been non-fat tailed distributions of stock returns. This however, has recently been seen to be untrue. Stock prices have been shown to follow fat-tailed distributions, meaning it is more likely to have windfall profits or disastrous market crashes than implied by gaussian distributions. This might be due to the fact that individuals are not perfectly rational, Keynes is famous for saying "markets can stay irrational longer then you can stay solvent." Also the efficient market hypothesis cannot explain volatility clustering and the large trading volumes we have seen since the 1990's. The representative agent models poor empirical performance, as well as the need to present a stronger theory on households preferences, created a need for the use of heterogeneous agents in a framework with micro foundations, as shown in [4]. Clearly, in the stock market, there are many different agents working at any given time using many different strategies, and presumably the same goes for the macroeconomy. It is this belief that has created a new way to study macroeconomics, one with heterogeneous agents. These models specifically try and handle the complex problem of aggregation and individuality of agents in an economy.

The idea is that the interaction of such a large number of consumers, firms and investors plays an important part in how economies work. Thus models should be built with this in mind and should be able to handle, diversity of and interaction between, players. The insight is that the outcome of interactions of a large number of agents facing random shocks is not adequately represented by the response of a representative agent and therefore the system must be treated stochastically.

Heterogeneous modeling is interested in is bridging the gap, creating models based on microeconomic foundations and behaviour of individuals but instead of representing each individual as the same agent, introduce heterogeneity and then when aggregating over these many different agents do not ignore how they interact, in so doing try to create models that capture what is being seen at the macro level. In this way, the models will take on a Keynesian approach, by respecting the aggregation process and fitting what is going on in the macro world but honour the Lucas Critique by seriously taking into account the behaviour of agents.

With the recent success of heterogeneous models it has become more apparent that there is also a need to determine analytical tools necessary to make claims about stationary distributions. Heterogeneous models will likely only stay popular if they can also provide the analytical results that representative agent models can. This leaves the door open for scientist and engineers who have worked with stochastic systems for some time now. Similar to the work done by control theorists in systems biology, see [5], analytic tools on jump Markov processes will be very useful in economics and finance.

It is one of these heterogenous jump Markov models that we will investigate. In this paper we investigate a model of the macroeconomy with heterogeneity of productivity of sectors supplied by [6], and linearize the model and then apply the fluctuation dissipation theorem to determine the steady state second moments as a function of a sectors control variable.

This paper is outlined as follows; Section 2 gives a description of the model, Section 3 analytically shows how to determine the stationary convaraince of employment in a particular sector. Section 4 shows the results of simulations of the stochastic system. Finally, Section 5 makes some concluding remarks.

## II. The Model

In this model, the output of an economy is broken down into a number of different sectors, each of which produces its own product, according to [6]. Each sector is represented by a different productivity coefficient and its share of overall demand.

The model is a multi-sector stochastic model of the economy. Each sector is differentiated by the goods or services it produces and the efficiency by which it produces them. The economy is assumed to have a fixed and finite number of sectors $K$. Each sector is indexed by $i (i = 1, ..., K)$ and employs $n_i(t)$ workers, which is assumed to be the only input to production. $n$ is a vector of length $K$. The productivity coefficient of sector $i$ is $c_i$, which can be thought of as a measure of goods produced per worker and is fixed and different for each sector. This is what describes the heterogeneity of the economy.

The output of a sector can then be determined by way of a linear production function, given below in (1)

$$y_i = p_i c_i n_i \tag{1}$$

where $p_i$ is the price of goods in the sector. From (1), if we sum over all sectors we can determine the aggregate output, $Y$, of the whole economy by summing over all sectors.

$$Y = \sum_{i=1}^{K} y_i = \sum_{i=1}^{K} p_i c_i n_i \tag{2}$$

The demand for the output of a sector, $d_i$, is assumed to be given by its share of the total economy, $s_i$, where $s_i > 0$ and

$$\begin{aligned} d_i =& s_i Y \\ \sum_{i=1}^{K} s_i =& 1. \end{aligned} \tag{3}$$

From this the excess demand, $f(n_i)$, can be determined for any given sector.

$$\begin{aligned} f(n_i) =& d_i - y_i \\ f(n_i) =& s_i Y - p_i c_i n_i \end{aligned} \tag{4}$$

When $f(n_i) > 0$ the sector is said to face *excess demand* and when $f(n_i) < 0$ the sector is said to face *excess supply*.

### A. The Dynamics

A continuous time jump Markov process is used to incorporate dynamics into the model. Sectors respond to their excess demand by adjusting their factor of production to meet output needs. When $f(n_i) > 0$ sectors will increase their input factor of production, labour, $n_i$, conversely when $f(n_i) < 0$ a sector will reduce $n_i$. It is assumed without loss of generality that $n_i$ jumps by one unit up or down at a given time. It is also assumed that each sector adjusts $n_i$ at a given time independent of another sector. For a given sector, the time between jumps is exponentially distributed with rates determined by the size of the sector's workforce. So, a larger sector is likely to hire more frequently than a smaller sector. Thus we can define the usual transition rates for this jump Markov process as follows:

$$\begin{aligned} Birth_i =& \begin{cases} \frac{n_i}{(n_1+n_2+\ldots+n_K)} = \frac{n_i}{N} & \text{if } f(n_i) > 0 \\ 0 & \text{if } f(n_i) \le 0 \end{cases} \\ Death_i =& \begin{cases} 0 & \text{if } f(n_i) \ge 0 \\ = \frac{n_i}{(n_1+n_2+\ldots+n_K)} = \frac{n_i}{N} & \text{if } f(n_i) < 0 \end{cases} \end{aligned} \tag{5}$$

where we define $N$ as the total number of employed workers, i.e. $N = \sum_{i=1}^{K} n_i$.

Equilibrium occurs when a birth and death are equally likely, which occurs when the birth and death rates are equal. This also makes intuitive sense, since equilibrium occurs when *excess demand* is zero. However, this model has the unfortunate characteristic of being discontinuous about its equilibrium point.

Transition rates are determined by the ratio of employed workers in each sector. This probability is reduced to $n_i/(n_1 + n_2 + ... + n_K) = \frac{n_i}{N}$.

In this model an *excess demand* function is used in order to determine if a sector increases or decreases labour, this means that the jump rate for a birth or death is discontinuous at its equilibrium value.

## III. Applying the Fluctuation Dissipation Theorem

The fluctuation dissipation theorem considers the following system with jumps:

$$x \xrightarrow{W_i(x)} x + r_i\ , \qquad i = 1, 2, \ldots, m \tag{6}$$

where $m$ is the number of interactions that take place and the random variable $x$, which follows a continuous time jump Markov chain with transition rate $W_i(x)$ and jump size $r_i$, is a vector with each index representing a population size, i.e.

$$x = [x_1, x_2, x_3, \cdots, x_K]^T. \tag{7}$$

If we were to fit the previous model to this framework we would have the following:

$$\begin{aligned} x = n =& [n_1, n_2, n_3, \cdots, n(K)]^T \\ r_{1i} =& [0, 0, 0, \cdots, 1, 0, \cdots, 0]^T \\ r_{-1i} =& [0, 0, 0, \cdots, -1, 0, \cdots, 0]^T \\ W_{1i}(n) =& \tau_i \cdot H_{n_{e,i}}(n_i) \\ W_{-1i}(n) =& \tau_i \cdot (1 - H_{n_{e,i}}(n_i)) \end{aligned} \tag{8}$$

where we slightly change the notation to make it more clear which jump is taking place. $r_{1i}$ is an increase in sector $i$ and $r_{-1i}$ is a decrease in sector $i$. And where $H$ is the heaviside function, where $H_{n_{e,i}}(n_i) = 1\ for\ n_i < n_{e,i},\ and\ H_{n_{e,i}}(n_i) = 0\ for\ n_i \ge n_{e,i}$, where $n_{e,i}$ is the value of $n$ that sets the excess demand to zero. $H_{n_{e,i}}(n_i)$ acts as our switch to represent the role that the sign of the excess demand plays.

From the fluctuation dissipation theorem we know that the covariance matrix of the above jump system, (6), follows the following time path when transition rates are linear, [5]:

$$\frac{d\Sigma}{dt} = A\Sigma + \Sigma A + D. \tag{9}$$

And so in the stationary case we have $\langle x \rangle = x_e$ and

$$0 = A\Sigma + \Sigma A + D \tag{10}$$

where $A = \frac{dg}{dx}$ is the *Drift* matrix and $D = \sum_{i=1}^{m} r_i W_i(x_e) r_i^T$ is the *Diffusion* matrix and $g(x) = \sum_{i=1}^{m} r_i W_i(x)$, when the $W_i(x)$ are linear.

Since our model is *nonlinear* it must be *linearized* about the equilibrium, $n = n_e$, however the transition rates are discontinuous at the equilibrium. So, we approximate the $H$ function by a continuous one, $H^C_{n_{e,i}}(n_i)$ about the equilibrium value $n_{e,i}$, and then linearize about that point. This will give an approximation for the covariance of the actual nonlinear system, however, as long as the system stays within the linear region, i.e. the fluctuations (variances) are small in equilibrium, then this approximation will be valid.

With this in mind fitting the above structure to our model and performing a little algebra we get:

$$\begin{aligned} g(n) =& \sum_{i=1}^{m} r_i W_i(n_e) \\ g(n) =& [\tau_1 \cdot X_1,\ \cdots,\ \tau_K \cdot X_K]^T \end{aligned} \tag{11}$$

where $X_i = 2H^C_{n_{e,i}}(n_i) - 1$.

And so the *Drift* matrix is:

$$A = \frac{dg(n)}{dn} = \begin{pmatrix} \frac{dg_1}{dn_1} & \cdots & \frac{dg_1}{dn_K} \\ \cdots & \frac{dg_i}{dn_i} & \cdots \\ \frac{dg_K}{dn_1} & \cdots & \frac{dg_K}{dn_K} \end{pmatrix}$$

where $g_i = \tau_1(2H^C_{n_{e,1}}(n_1) - 1)$ from (11). Therefore using the product rule we get the following:

$$A = \frac{dg(n)}{dn} =$$
$$\begin{pmatrix} \frac{d\tau_1}{dn_1}X_1 + 2\tau_1\frac{dH^C_{n_{e,1}}(n_1)}{dn_1} & \cdots & \frac{d\tau_1}{dn_K}X_1 \\ \cdots & \ddots & \cdots \\ \frac{d\tau_K}{dn_1}X_K & \cdots & \frac{d\tau_K}{dn_K}X_K + 2\tau_K\frac{dH^C_{n_{e,K}}(n_K)}{dn_K} \end{pmatrix}$$

Where $n = n_e$, i.e. the equations are evaluated at the equilibrium. The key to this equation is the terms of the form $\frac{dH^C_{n_{e,i}}(n_i)}{dn_i}$. Because as the approximation of $H$ gets better and better the slope at the equilibrium will tend to $\infty$.

The *Diffusion* matrix then becomes after a little algebra:

$$\begin{aligned} D &= \sum_{i=1}^{m} r_i W_i(x_e) r_i^T \\ &= \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_K \end{pmatrix} \\ &= \begin{pmatrix} \tau_1 & \cdots & 0 \\ 0 & \tau_i & 0 \\ 0 & \cdots & \tau_K \end{pmatrix} \end{aligned}$$

where $A_i = \tau_i H^C_{n_{e,i}}(n_i) + \tau_i(1 - H^C_{n_{e,i}}(n_i))$. Notice that $D$ very nicely doesn't not depend on the $H^C_{n_{e,i}}(n_i)$ function.

### A. Choosing an appropriate Heaviside Approximation

The approximation to the heaviside function we have chosen is the logistic sigmoid function. It is chosen to preserve the transition rates as much as possible away from the equilibrium point while at the same time making a smooth and more believable cutoff. The logistic sigmoid function also has the benefit of an inflection point at the equilibrium value thus preserving its symmetry.

$$H^C_{n_{e,i}}(n_i) = \frac{1}{1 + e^{-G(n_{e,i} - n_i)}} \qquad (12)$$

This means that at $n_i = n_{e,i}, H^C_{n_{e,i}}(n_i) = 1/2$ which gives a diagonal $A$ matrix. In this case, we have a diagonal $A$ and $D$ matrix which makes it easy to solve for the stationary $\Sigma$.

### B. Solving for the variance

From equation (10) we can see that if $A$ and $D$ are diagonal then so is $\Sigma$. The solution is therefore:

$$\begin{aligned} 0 &= 2\tau_i\frac{dH^C_{n_{e,i}}(n_i)}{dn_i} * \sigma_{i,i} + \sigma_{i,i} * 2\tau_i\frac{dH^C_{n_{e,i}}(n_i)}{dn_i} + \tau_i \\ \sigma_{i,i} &= -\frac{1}{4\frac{dH^C_{n_{e,i}}(n_i)}{dn_i}} \\ \sigma_{i,i} &= \frac{1}{G} \end{aligned} \qquad (13)$$

where $\sigma_{i,i} = var(n_i)$ and $\frac{dH^C_{n_{e,i}}(n_i)}{dn_i}$ is evaluated at $n_i = n_{e,i}$ and therefore $\frac{dH^C_{n_{e,i}}(n_i)}{dn_i} = -\frac{G}{4}$.

So, as the sigmoid function becomes a better approximation to the heaviside function the stationary variance reduces to zero.

We can see that there is an appropriate control theory interpretation to how *sharp* the approximation of $H$ is. The more *sharp* the approximation the more ***gain*** the sector uses when adjusting production to meet demand. There is also an economic interpretation, the gain describes how aggressive the sector is in its approach to production scheduling.

## IV. SIMULATIONS

In this section we simulate the actual nonlinear system, with continuous transition rates, so a sector can determine its own gain $G$, using the Gillespie algorithm. We show that as the parameter $G$, in equation (12), increases the steady state covariance matrix reduces to zero, as predicted by the linear approximation.

The model is shown running with only two sectors but can easily be extended for larger $K$. The two sectors are run with typical values for output and number of employees from recent data from the US, [7] [8]. The two sectors chosen were the private and public sector, which together make up the total GDP of the US economy. The simulations were run with the initial number of employees equal to the 2006 US employment numbers for the private and public sector, 111899 and 21804 respectively measured in thousands, therefore the jump size is assumed to be in thousands. The productivity and price numbers were calculated together as a best fit linear regression of historical output per sector against number of workers in the sector, remember in the above model the production function is assumed to be linear in its one input, labour.

Below are a series of plots from the simulations run for private sector employment run with different values of G, after the simulations had reached equilibrium. It is clear that the variance reduces as G increases.

The stationary covariance matrix for these simulations were the following:

G = 0.01

$$\Sigma = \begin{pmatrix} 79.0160 & -13.5935 \\ -13.5935 & 122.0308 \end{pmatrix}$$

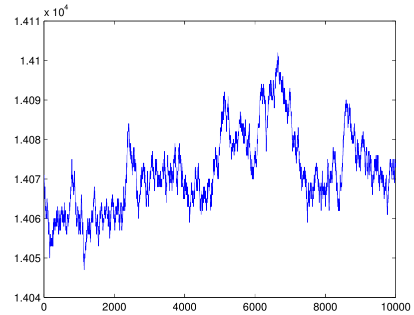


Fig. 1. Private Sector Employment Simulated with G = 0.01

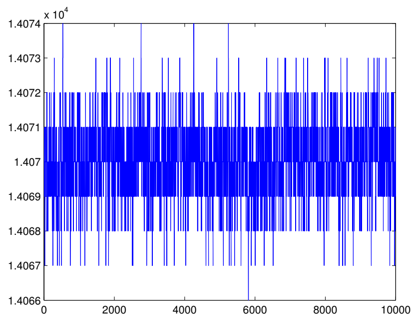


Fig. 2. Private Sector Employment Simulated with G = 1

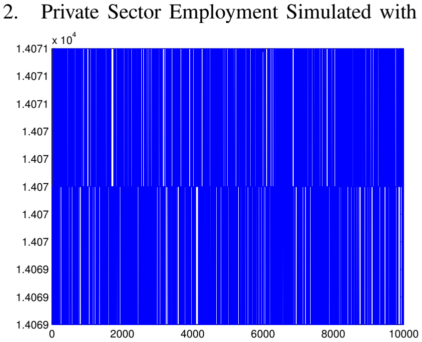


Fig. 3. Private Sector Employment Simulated with G = 100

G = 1

$$\Sigma = \begin{pmatrix} 1.2738 & 0.0672 \\ 0.0672 & 1.2262 \end{pmatrix}$$

G = 100

$$\Sigma = \begin{pmatrix} 0.4990 & 0.0049 \\ 0.0049 & 0.5013 \end{pmatrix}$$

From the model and from the given simulations we can see that the feedback comes into play through the transition rates and that G is the gain. It becomes more clear when we linearize the transition rates and see that G acts as the controllers natural gain. When a sector is facing *excess demand*, i.e. the sector is not producing enough output to meet demand, the system has a greater propensity to increase labour and thus output to meet demand and reach equilibrium, and vice versa when a sector faces *excess supply*. The propensity for change or increased reaction to a sectors *excess demand*, or in traditional control terminology the *error signal* (reference - output signal), is given by G, and we see from above that as expected when a sector acts with large negative feedback the employment variance and therefore output variance reduces.

## V. Conclusions

This paper has presented a stochastic model for the economy where output is divided up into different sectors which operate at different levels of productivity. We have shown that in this heterogenous model, feedback is a vital part of any system, no matter how complex, and its benefits and tradeoffs should be looked into. We have been able to show explicitly how negative feedback gain works in this model and shown how to reduce variance by increasing gain. Future work will include investigating minimum variance of the stochastic system when sectors react to old data. As we know feedback and delays are inherently bad and so we expect to see poor performance when agents are reacting to old data. Finally we will look at fitting the model to data that is already available for output and employment of a sector. This model has the benefit that sectors have different productivities which is already known to be true.

Heterogenous agent models are becoming a rich area of research in economics and finance because they have had success in explaining real phenomenon that was not previously explained. In the near future it is likely that there will be large demand for stochastic tools like the ones presented above capable of dealing with stochastic jump systems. Work on networks and fundamental limitations of such systems should be very useful to help further study issues such as global imbalances of trade, income inequality and sectoral unemployment.

As Keynes himself once said:

> Our criticism of the accepted classical theory of economics has consisted not so much in finding logical flaws in its analysis as in pointing out that its tacit assumptions are seldom or never satisfied, with the result that it cannot solve the economic problems of the actual world, [1].

This statement seems to also apply to previous models based on *representative agents*. With this in mind it would be wise to reconsider Keynes criticism together with Lucas's and build models with micro foundations but that take into account real phenomenon like differences in individuals and then test these models by how well they can solve real economic problems in the actual world. For this reason heterogenous agent models and tools used to analyze them should become more useful in the future.

## VI. ACKNOWLEDGMENTS

The authors would like to thank the members of the University of Cambridge Control Group for their support and insight and the reviewers for their comments. And again would like to thank the Cambridge Gates Trust for funding this research as well as the Department of Engineering and Hughes Hall both affiliates of the University of Cambridge.

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