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Lyapunov Functions and Robustness Analysis under Matrosov Conditions with an Application to Biological Systems

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Abstract— **We construct strict Lyapunov functions for broad classes of nonlinear systems satisfying Matrosov type conditions. Our new constructions are simpler than the designs available in the literature. We illustrate our designs using a model for an experimental anaerobic digester used to treat wastewater.**

I. INTRODUCTION

Lyapunov functions play an essential role in nonlinear systems analysis and controller design. In many important situations, *non*strict Lyapunov functions are readily available. However, *strict* Lyapunov functions are preferable since they can often be used to quantify the effects of disturbances; see the precise definitions in Section II below. Strict Lyapunov functions have been used in several biological contexts (e.g. to quantify the effects of actuator noise and other uncertainties in chemostats [13]) but their explicit construction can be quite challenging. One may sometimes transform non-strict Lyapunov functions into the required strict Lyapunov functions e.g. [4], [10], [11], [14], [15].

For systems satisfying conditions of Matrosov's type [7], [9], strict Lyapunov functions were constructed in [15], under very general conditions. However, the generality of [14], [15] makes their constructions complicated and therefore difficult to apply. Also, the Lyapunov function constructions in [15] are *nonexplicit*, unless the auxiliary functions are known. Moreover, the Lyapunov functions from [14], [15] are never locally lower bounded by positive definite quadratic functions, even for globally asymptotically stable linear systems, which have quadratic strict Lyapunov functions. The shape of the Lyapunov functions, their local properties and their simplicity matter when they are used to investigate robustness properties and construct feedbacks and gains.

In this note, we revisit the problem of constructing Lyapunov functions under Matrosov's conditions. Our results have these useful features. First, they lead to simple strict Lyapunov function constructions. For a large class of systems, our Lyapunov functions are locally lower bounded by positive definite quadratic functions. Second, we do not require a non-strict positive definite proper Lyapunov function. Rather, we only require a non-strict *positive definite* function whose derivative along the trajectories is nonpositive.

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One of our motivations is that for biological models, one can frequently find non-strict Lyapunov-like functions which are not proper but which make it possible to establish global asymptotic stability of an equilibrium point. For instance, the celebrated Lyapunov function in [5] (presented in [16]) for a multi-species chemostat is not proper. In such cases, the proof of stability is often based on the fact that the models are derived from mass balance properties [1] leading to the boundedness of the trajectories in compact sets.

Our work has the additional desirable property that it yields robustness in the sense of input-to-state stability (ISS). The ISS notion is a fundamental paradigm of nonlinear control that makes it possible to quantify the effects of uncertainty [17], [18]. While our assumptions are more restrictive than those in [7], [15], they are sufficiently general in the sense that, to the best of our knowledge, they are satisfied by all examples whose stability can be established by the generalized Matrosov's theorem. In Section IV, we use our results to construct a strict Lyapunov function and prove robustness for a wastewater treatment process stabilized through the adaptive feedback proposed in [8]. This illustrates the value added by our strict Lyapunov functions for systems that are of compelling engineering interest.

II. DEFINITIONS AND NOTATION

We omit the arguments of our functions when they are clear from the context. All (in)equalities should be understood to hold globally unless otherwise indicated. By C^ν, we mean ν times continuously differentiable.

A continuous function $k : [0, \infty) \rightarrow [0, \infty)$ is *of class* $\mathcal{K}_\infty$ (written $k \in \mathcal{K}_\infty$) provided it is zero at zero, strictly increasing and unbounded. A continuous function $\beta : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ is said to be *of class* $\mathcal{KL}$ (written $\beta \in \mathcal{KL}$) provided (a) $\beta(\cdot, t) \in \mathcal{K}_\infty$ for each $t \geq 0$, (b) $\beta(s, \cdot)$ is nonincreasing for each $s \geq 0$, and (c) for each $s \geq 0$, $\beta(s, t) \rightarrow 0$ as $t \rightarrow +\infty$. We always assume $D \subseteq \mathbb{R}^n$ is an open set for which $0 \in D$. A function $V : D \times \mathbb{R} \rightarrow \mathbb{R}$ is positive definite on D provided $V(0, t) = 0$ for all t and $\inf\{V(x, t) : t \in \mathbb{R}\} > 0$ for all $x \in D \setminus \{0\}$. A function V is negative definite provided $-V$ is positive definite. We always assume that our functions are sufficiently smooth.

Given an open set D that is diffeomorphic to $\mathbb{R}^n$, consider a time-varying nonlinear system

$$\dot{x} = f(x, t) \tag{1}$$

evolving on a forward invariant set D (meaning that all of its trajectories are defined on $[t_o, \infty)$ and valued in D for any initial condition $x(t_o) = x_o \in D$). Assume that $f \in C^1$ with

$f(0,t)=0$ for all t. A C^1 function $V:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ is a *Lyapunov-like function* for (1) provided V is positive definite and $L_fV(x,t):=V_t(x,t)+V_x(x,t)f(x,t)\le 0$ for all $x\in D$ and $t\in\mathbb{R}$. If in addition $L_fV(x,t)$ is negative definite, then we say that V is a *strict Lyapunov-like function* for (1). A function $W:D\times\mathbb{R}\to\mathbb{R}$ is *(uniformly) radially unbounded* (or *proper*) provided $\lim_{D\ni x,|x|\to+\infty}\inf\{W(x,t): t\in\mathbb{R}\}=+\infty$. A (strict) Lyapunov-like function is a *(strict) Lyapunov function* provided it is also proper. A function ϕ is *uniformly bounded in* t provided there exists a function $\Upsilon\in\mathcal{K}_\infty$ such that $|\phi(x,t)|\le\Upsilon(|x|)$ for all x and t.

The ISS paradigm was introduced by Sontag [17]; see also the recent survey [18]. The relevant definitions are these. Consider a general time-varying control system

$$\dot{x}=\mathcal{F}(x,t,\delta) \tag{2}$$

evolving on a forward invariant set $\mathcal{G}$ with disturbances δ in the set $\mathcal{L}_\infty(\mathcal{C})$ of all measurable essentially bounded functions valued in any subset $\mathcal{C}$ of Euclidean space. Let $t\mapsto\phi(t;t_o,x_o,\delta)$ denote the solution for (2) with any initial condition $x(t_0)=x_o$, which we always assume is uniquely defined on $[t_o,+\infty)$. We further assume that $\mathcal{F}(0,t,0)\equiv 0$. Let $M(\mathcal{G})$ denote the set of all continuous functions $\mathcal{M}:\mathcal{G}\to[0,\infty)$ for which (A) $\mathcal{M}(0)=0$ and (B) $\mathcal{M}(x)\to+\infty$ if $x\to\text{boundary}(\mathcal{G})$ or if $|x|\to+\infty$ while remaining in $\mathcal{G}$.

We say that (2) is *ISS on $\mathcal{G}$ with controls in $\mathcal{C}$* (or simply ISS when $\mathcal{G}$ and $\mathcal{C}$ are clear) provided there exist functions $\beta\in\mathcal{KL}$, $\mathcal{M}\in M(\mathcal{G})$ and $\gamma\in\mathcal{K}_\infty$ such that

$$|\phi(t;t_o,x_o,\delta)|\ \le\ \beta(\mathcal{M}(x_o),t-t_o)+\gamma(|\delta|_\infty) \tag{3}$$

for all $t\ge t_o\ge 0$, $x_o\in\mathcal{G}$, and $\delta\in\mathcal{L}_\infty(\mathcal{C})$, where $|\cdot|_\infty$ denotes the essential supremum. When $\mathcal{G}=\mathbb{R}^n$, this becomes the usual ISS definition when $\mathcal{M}(x)=|x|$. The ISS property reduces to the standard (uniformly) globally asymptotically stable condition when $\delta\equiv 0$, but is far more general because it quantifies the effects of the disturbance δ on the stability, in terms of the overshoot $\gamma(|\delta|_\infty)$.

III. Main Result

A. Assumptions, Result, and Remarks

For simplicity, we first state our main results for

$$\dot{x}=f(x) \tag{4}$$

evolving on D; see Remark 3 for the generalization to (1). We assume the following (but see [12] for methods for relaxing these assumptions):

Assumption 1: There exist an integer $j\ge 2$; known functions $V_i:D\to\mathbb{R}$, $\mathcal{N}_i:D\to[0,+\infty)$, and $\phi_i:[0,+\infty)\to(0,+\infty)$; and real numbers $a_i\in(0,1]$ such that $V_i(0)=0$ and $\mathcal{N}_i(0)=0$ for all i and

$$\begin{aligned}\nabla V_1(x)f(x)\ &\le\ -\mathcal{N}_1(x)\quad\text{and}\\ \nabla V_i(x)f(x)\ &\le\ -\mathcal{N}_i(x)\\ &\quad+\phi_i(V_1(x))\textstyle\sum_{l=1}^{i-1}\mathcal{N}_l^{a_i}(x)V_1^{1-a_i}(x)\end{aligned} \tag{5}$$

for $i=2,\ldots,j$ and for all $x\in D$. The function V_1 is assumed to be positive definite on D.

Assumption 2: i) There exists a function $\rho:[0,+\infty)\to(0,+\infty)$ such that

$$\sum_{l=1}^{j}\mathcal{N}_l(x)\ \ge\ \rho(V_1(x))V_1(x)\quad\forall x\in D. \tag{6}$$

ii) There exist $p_1,p_2,\ldots,p_j:[0,+\infty)\to[0,+\infty)$ such that

$$|V_i(x)|\ \le\ p_i(V_1(x))V_1(x)\quad\forall x\in D \tag{7}$$

holds for $i=2,3,\ldots,j$.

Our main theorem is:

Theorem 1: Assume that there exist $j\in\mathbb{N}$ and functions satisfying Assumptions 1-2. Then one can explicitly determine C^1 functions $k_l,\Omega_l\in\mathcal{K}_\infty$ such that the function

$$S(x)\ =\ \sum_{l=1}^{j}\Omega_l\left(k_l(V_1(x))+V_l(x)\right) \tag{8}$$

satisfies

$$S(x)\ \ge\ V_1(x)\quad\text{and} \tag{9}$$

$$\nabla S(x)f(x)\ \le\ -\frac{1}{4}\rho(V_1(x))V_1(x) \tag{10}$$

for all $x\in D$.

Remark 1: The differences between Assumptions 1-2 and the assumptions from [15] are as follows. First, while our Assumption 1 above ensures that V_1 is positive definite but not necessarily proper, [15] assumes that a radially unbounded non-strict Lyapunov function is known. Second, our Assumption 1 is a restrictive version of Assumption 2 from [15]. More precisely, our Assumption 1 above specifies the local properties of the functions which correspond to the χ_is of Assumption 2 in [15]. Finally, our Assumption 2 imposes relations between the functions $\mathcal{N}_i$ and V_1, which are not required in [15]. Note that our assumptions do not require the functions $V_2,\ldots,V_j$ to be nonnegative. Assumption 2 can often be relaxed; for details, see [12].

Remark 2: If $D=\mathbb{R}^n$ and V_1 is radially unbounded, then (9) implies that S is a strict Lyapunov function for (4). If V_1 is not radially unbounded, then S is not necessarily radially unbounded and therefore one cannot conclude from Lyapunov's theorem that the origin is globally asymptotically stable. However, in many cases, global asymptotic stability can be proved through a Lyapunov-like function and extra arguments, e.g. by proving that any trajectory belongs to a compact set included in D. This is often the case in biological models that are based on mass conservation properties. We illustrate these assertions in Section IV.

Remark 3: One can prove analogs of Theorem 1 for time-varying systems (1), as follows. We assume that f is uniformly bounded in t, and that time-varying versions of Assumptions 1-2 hold. The time-varying analogs of Assumptions 1-2 are obtained by (A) replacing their arguments x by (x,t) and (B) adding the assumption that the $V_i(x,t)$ and $\mathcal{N}_i(x,t)$ are uniformly bounded in t. We can further relax the time-varying version of (6) by replacing the lower bound

$\rho(V_1(x,t))V_1(x,t)$ with $\underline{p}(t)\rho(V_1(x,t))V_1(x,t)$ where $\underline{p}$ satisfies persistency of excitation conditions; see [12].

B. Proof of Theorem 1

Fix $j \geq 2$ and functions satisfying Assumptions 1-2. Fix $k_2, \ldots, k_j \in C^1 \cap \mathcal{K}_\infty$ such that

$$k_i(s) \ \geq \ s + p_i(s)s \quad \text{and} \quad k_i'(s) \ \geq \ 1 \tag{11}$$

for all $s \geq 0$ and $i = 2, 3, \ldots, j$.

Lemma 1: The functions $U_1(x) = V_1(x)$ and $U_i(x) = k_i(V_1(x)) + V_i(x)$ satisfy $2k_i(V_1(x)) \geq U_i(x) \geq V_1(x)$ for all $i = 1, 2, \ldots, j$ and all $x \in D$.

To check Lemma 1, note that (7) and (11) give $U_i(x) \geq V_1(x) + p_i(V_1(x))V_1(x) - p_i(V_1(x))V_1(x) = V_1(x)$ and $U_i(x) \leq k_i(V_1(x)) + p_i(V_1(x))V_1(x) \leq 2k_i(V_1(x))$ for $i = 2, ..., j$.

Returning to the proof of the theorem, define the functions U_i according to Lemma 1. Let $\Omega_1, \ldots, \Omega_j \in \mathcal{K}_\infty \cap C^1$ be functions for which $\Omega_j(s) \equiv s$ and $\Omega_i'(s) \geq 1$ and

$$\Omega_i'(U_i) \ \geq \ 2\Phi(V_1) \textstyle\sum_{l=1+i}^{j} \Omega_l'(U_l)^{\frac{1}{a_l}} \tag{12}$$

hold for $i = 1, 2, \ldots, j-1$ and all $s \geq 0$, where

$$\Phi(V_1) = \max_{i=2,..,j} \left\{ \phi_i(V_1)^{\frac{1}{a_i}} \left[\tfrac{4(j-1)(i-1)}{\rho(V_1)}\right]^{\frac{1-a_i}{a_i}} \right\}. \tag{13}$$

For example, we can take $\Omega_i(p) = \int_0^p \mu_i(r)dr$ where the functions $\mu_i : [0,\infty) \to [1,\infty)$ are from [12, Lemma A.1]. Since $\Omega_1'(s) \geq 1$ everywhere, $\Omega_1(U_1(x)) \geq U_1(x) = V_1(x)$ everywhere. Hence,

$$S(x) = \textstyle\sum_{i=1}^{j} \Omega_i(U_i(x)) \tag{14}$$

satisfies (9). To check (10), first note that (5) and (11), along with $\Omega_1'(U_1) \geq 0$ and $\Omega_i'(U_i)k_i'(V_1) \geq 0$, give

$$\begin{aligned}\nabla S(x)f(x) \ &\leq \ \textstyle\sum_{i=1}^{j} \Omega_i'(U_i)\dot{V}_i \ \leq -\sum_{i=1}^{j} \Omega_i'(U_i)\mathcal{N}_i \\ &+ \textstyle\sum_{i=2}^{j} \Omega_i'(U_i) \left(\phi_i(V_1) \sum_{l=1}^{i-1} \mathcal{N}_l^{a_i} V_1^{1-a_i}\right)\end{aligned} \tag{15}$$

along the trajectories of (4). Define the positive functions $\Gamma_2, \ldots, \Gamma_j$ by

$$\Gamma_i(x) \ = \ \tfrac{4(j-1)(i-1)\Omega_i'(U_i(x))\phi_i(V_1(x))}{\rho(V_1(x))} \ . \tag{16}$$

For any $i \geq 2$ for which $0 < a_i \neq 1$, Holder's Inequality $v_1 v_2 \leq v_1^p + v_2^q$ with $p = 1/a_i$, $q = 1/(1-a_i)$, $v_1 = \Gamma_i(x)^{1-a_i}\mathcal{N}_l(x)^{a_i}$, and $v_2 = \{V_1(x)/\Gamma_i(x)\}^{1-a_i}$ gives

$$\mathcal{N}_l(x)^{a_i} V_1(x)^{1-a_i} \ \leq \ \Gamma_i(x)^{\frac{1-a_i}{a_i}} \mathcal{N}_l(x) + \frac{1}{\Gamma_i(x)} V_1(x) \tag{17}$$

for all $x \in D$. The preceding inequality is also valid when $a_i = 1$. Substituting (17) into (15) gives

$$\begin{aligned}\nabla S(x)f(x) \ &\leq \ -\textstyle\sum_{i=1}^{j} \Omega_i'(U_i)\mathcal{N}_i \\ &+ \textstyle\sum_{i=2}^{j} \left(\Omega_i'(U_i)\phi_i(V_1)\Gamma_i^{\frac{1-a_i}{a_i}} \sum_{l=1}^{i-1} \mathcal{N}_l\right) \\ &+ \left(\textstyle\sum_{i=2}^{j} \Omega_i'(U_i) \frac{\phi_i(V_1)(i-1)}{\Gamma_i}\right) V_1 \ .\end{aligned} \tag{18}$$

Setting $\Lambda(i,j) := 4(j-1)(i-1)\Omega_i'(U_i)\phi_i(V_1)$, our choices (16) of the Γ_is then give

$$\begin{aligned}\nabla S(x)f(x) \ &\leq \ -\textstyle\sum_{i=1}^{j} \Omega_i'(U_i)\mathcal{N}_i + \frac{1}{4}\rho(V_1)V_1 \\ &+ \textstyle\sum_{i=2}^{j} \left(\Omega_i'(U_i)\phi_i(V_1)\left[\frac{\Lambda(i,j)}{\rho(V_1)}\right]^{\frac{1-a_i}{a_i}} \sum_{l=1}^{i-1} \mathcal{N}_l\right) \\ &\leq -\textstyle\sum_{i=1}^{j} \Omega_i'(U_i)\mathcal{N}_i + \frac{1}{4}\rho(V_1)V_1 \\ &+ \Phi(V_1) \textstyle\sum_{i=2}^{j} \left(\Omega_i'(U_i)^{\frac{1}{a_i}} \sum_{l=1}^{i-1} \mathcal{N}_l\right), \ \text{by (13)}.\end{aligned} \tag{19}$$

From the fact that $\Omega_i' \geq 1$ for all i and (6), we get

$$\textstyle\sum_{i=1}^{j} \Omega_i'(U_i)\mathcal{N}_i \ \geq \ \rho(V_1)V_1.$$

Hence, (19) gives

$$\begin{aligned}\nabla S(x)f(x) \ &\leq \ -\tfrac{1}{4}\rho(V_1)V_1 - \tfrac{1}{2}\textstyle\sum_{i=1}^{j} \Omega_i'(U_i)\mathcal{N}_i \\ &+ \Phi(V_1) \textstyle\sum_{i=2}^{j} \left(\Omega_i'(U_i)^{\frac{1}{a_i}} \sum_{l=1}^{i-1} \mathcal{N}_l\right) \ .\end{aligned} \tag{20}$$

By reversing the order of the double summation in (20), we easily deduce that

$$\begin{aligned}\nabla S(x)f(x) \ &\leq \ -\tfrac{1}{4}\rho(V_1)V_1 \\ &+ \sum_{i=1}^{j-1} \left[-\frac{1}{2}\Omega_i'(U_i) + \Phi(V_1)\left(\sum_{l=1+i}^{j} \Omega_l'(U_l)^{\frac{1}{a_l}}\right)\right]\mathcal{N}_i.\end{aligned} \tag{21}$$

Since the $\mathcal{N}_i$'s are nonnegative, it is now immediate from (12) that (10) holds. This proves Theorem 1.

IV. Biotechnological Application

Several adaptive control problems for bioreactors have been solved [1], [6, Section 3.4.2], [8]. The proofs in these works rely on the construction of a *non-strict* Lyapunov function. In this section, we show how to construct a *strict* Lyapunov-like function for the system and corresponding adaptive controller in [8]. We then use our strict Lyapunov function to demonstrate that our controller is robust to uncertainty in the input nutrient concentration s_{in} and to controller noise. Our strategy is to show that the corresponding error dynamics are ISS; see Section IV-D for details.

A. Problem Studied

Consider an experimental anaerobic digester used to treat wastewater [8], [19]. This process degrades a polluting organic substrate s with the anaerobic bacteria x and produces a methane flow rate y_1. In real applications, the methane and substrate can generally be measured. The system is therefore

$$\left\{ \begin{array}{rcl} \dot{s} &=& u(s_{in} - s) - kr(s,x,t) \ , \\ \dot{x} &=& r(s,x,t) - \alpha ux \ , \\ y &=& (\lambda r(s,x,t), s) \end{array} \right. \tag{22}$$

where the biomass growth rate r is any nonnegative C^1 function that admits positive functions $\underline{\Delta}$ and $\bar{\Delta}$ such that

$$s\bar{\Delta}(s,x) \ \geq \ r(s,x,t) \ \geq \ xs\underline{\Delta}(s,x); \tag{23}$$

u is the nonnegative input (i.e. dilution rate); α is a known positive real number representing the fraction of the biomass in the liquid phase; and λ, k, and s_{in} are positive constants representing methane production and substrate consumption yields and the influent substrate concentration, respectively. We wish to regulate the variable s to a prescribed positive real number $s_* \in (0, s_{in})$. We assume that there are known constants $\gamma_M > \gamma_m > 0$ such that

$$\gamma_* := \frac{k}{\lambda(s_{in} - s_*)} \in (\gamma_m, \gamma_M) \quad \text{and} \quad \frac{k}{\lambda s_{in}} < \gamma_m. \tag{24}$$

We introduce the notation

$$v_* = s_{in} - s_* \quad \text{and} \quad x_* = \frac{v_*}{k\alpha}. \tag{25}$$

Remark 4: Requirement (23) is very similar to but slightly different from the assumption on $r(\cdot)$ in [8]. We chose (23) because it is general and convenient for our illustration.

B. Feedback Stabilizer and Nonstrict Lyapunov Function

The results of [8] lead to a nonstrict Lyapunov function and an adaptive controller for a suitable error dynamics associated with (22). In this subsection, we review these earlier results. In Section IV-C, we use them to build a strict Lyapunov-like function for the error dynamics.

We introduce the dynamics $\dot{\gamma} = y_1(\gamma - \gamma_m)(\gamma_M - \gamma)\nu$ evolving on (γ_m, γ_M), where ν is to be selected and is independent of x. With $u = \gamma y_1$, the system (22) with its dynamic extension becomes

$$\left\{ \begin{array}{rcl} \dot{s} & = & y_1 \left[\gamma(s_{in} - s) - \frac{k}{\lambda}\right] \ , \\ \dot{x} & = & y_1 \alpha \left[\frac{1}{\alpha\lambda} - \gamma x\right] \ , \\ \dot{\gamma} & = & y_1(\gamma - \gamma_m)(\gamma_M - \gamma)\nu \end{array} \right. \tag{26}$$

by the definition of y_1, with the same output as before. The dynamics (26) evolves on the invariant domain $E = (0, +\infty) \times (0, +\infty) \times (\gamma_m, \gamma_M)$. One easily checks:

Lemma 2: For each initial value $(s(t_0), x(t_0), \gamma(t_0)) \in E$, we can find a compact set $K_o \subseteq E$ (depending on $s(t_o)$ and $x(t_o)$) so that the corresponding solution of (26) is such that $(s(t), x(t)) \in K_o$ for all $t \geq t_0$.

It follows from Lemma 2 and (23) that we can reparameterize (26) in terms of

$$\tau = \int_{t_o}^{t} y_1(l) dl. \tag{27}$$

Doing so and setting $\tilde{x} = x - x_*$, $\tilde{s} = s - s_*$, and $\tilde{\gamma} = \gamma - \gamma_*$ yields the error dynamics

$$\left\{ \begin{array}{rcl} \dot{\tilde{s}} & = & -\gamma\tilde{s} + \tilde{\gamma} v_* \ , \\ \dot{\tilde{x}} & = & \alpha\left[-\gamma\tilde{x} - \tilde{\gamma} x_*\right] \ , \\ \dot{\tilde{\gamma}} & = & (\gamma - \gamma_m)(\gamma_M - \gamma)\nu \end{array} \right. \tag{28}$$

for $t \mapsto (\tilde{s}, \tilde{x}, \tilde{\gamma})(\tau^{-1}(t))$. The state space of (28) is the invariant domain $D = (-s_*, +\infty) \times (-x_*, +\infty) \times (\gamma_m - \gamma_*, \gamma_M - \gamma_*)$. The system (28) has an uncoupled triangular structure; i.e., its $(\tilde{s}, \tilde{\gamma})$-subsystem does not depend on $\tilde{x}$ and the $\tilde{x}$-subsystem is globally ISS with respect to $\tilde{\gamma}$ with the ISS Lyapunov function $\tilde{x}^2$ [18]. Therefore (28) is globally asymptotically stable if and only if the system

$$\left\{ \begin{array}{rcl} \dot{\tilde{s}} & = & -\gamma\tilde{s} + \tilde{\gamma} v_* \ , \\ \dot{\tilde{\gamma}} & = & (\gamma - \gamma_m)(\gamma_M - \gamma)\nu \ , \end{array} \right. \tag{29}$$

with state space $F = (-s_*, +\infty) \times (\gamma_m - \gamma_*, \gamma_M - \gamma_*)$ is globally asymptotically stable to 0. Therefore, we may limit our analysis to (29) in the sequel.

The paper [8] uses the Lyapunov-like function

$$V_1(\tilde{s}, \tilde{\gamma}) = \frac{1}{2\gamma_m}\tilde{s}^2 + \frac{v_*}{K\gamma_m} \int_0^{\tilde{\gamma}} \frac{l}{(l + \gamma_* - \gamma_m)(\gamma_M - \gamma_* - l)} dl \tag{30}$$

for (29), where $K > 0$ is a tuning parameter. The function V_1 is positive definite on $D := (-s_*, +\infty) \times (\gamma_m - \gamma_*, \gamma_M - \gamma_*)$. Its time derivative along the trajectories of (29) satisfies

$$\dot{V}_1 \ = \ \frac{1}{\gamma_m}\left[-\gamma\tilde{s}^2 + \tilde{s}\tilde{\gamma} v_*\right] + \frac{v_*}{K\gamma_m}\tilde{\gamma}\nu \ . \tag{31}$$

By choosing

$$\nu(\tilde{s}) = -K\tilde{s} \tag{32}$$

and recalling that $\gamma(t) \in (\gamma_m, \gamma_M)$ for all t, we obtain

$$\dot{V}_1 = -\frac{\gamma}{\gamma_m}\tilde{s}^2 \ \leq \ -\mathcal{N}_1(\tilde{s}), \ \text{ where } \mathcal{N}_1(\tilde{s}) = \tilde{s}^2. \tag{33}$$

By applying the LaSalle Invariance Principle, it is proved in [8] that the origin of (29) is globally asymptotically stable when $\nu(\tilde{s}) = -K\tilde{s}$ with $K > 0$.

C. Strict Lyapunov-Like Function for System (29)

We can use Theorem 1 and the auxiliary function $V_2(\tilde{s}, \tilde{\gamma}) = -\tilde{s}\tilde{\gamma}$ to transform (30) into a strict Lyapunov-like function for (29). In fact, we can take

$$S(\tilde{s}, \tilde{\gamma}) = U_2(\tilde{s}, \tilde{\gamma}) + \left[\frac{\gamma_M^2}{2v_*} + \frac{K(\gamma_M - \gamma_m)^2}{4}\right] V_1(\tilde{s}, \tilde{\gamma}) \tag{34}$$

where $U_2(\tilde{s}, \tilde{\gamma}) = \Upsilon_1 V_1(\tilde{s}, \tilde{\gamma}) + V_2(\tilde{s}, \tilde{\gamma})$ and

$$\Upsilon_1 = 1 + \frac{\gamma_m\sqrt{K}(\gamma_M - \gamma_m)}{\sqrt{v_*}}. \tag{35}$$

Setting $\mathcal{N}_2(\tilde{\gamma}) = \frac{v_*}{2}\tilde{\gamma}^2$, this gives

$$\begin{array}{l} \dot{S} \ \leq \ -W(\tilde{s}, \tilde{\gamma}), \quad \text{where} \\ W(\tilde{s}, \tilde{\gamma}) = \mathcal{N}_2(\tilde{\gamma}) + \Upsilon_1\mathcal{N}_1(\tilde{s}) = \frac{v_*}{2}\tilde{\gamma}^2 + \Upsilon_1\tilde{s}^2 \ . \end{array} \tag{36}$$

along the trajectories of (29) with (32). For details, see [12].

D. Robustness Result

It is important to assess the robustness of a control design to bounded uncertainties before implementing it in a bioprocess. Indeed, biological systems are known to have highly uncertain dynamics. In [8], good performance of the controller was observed but could not be explained by theory. Here we prove that an appropriate adaptive controller gives ISS of the relevant error dynamics to disturbances; see Section II above for the definitions and motivations for ISS.

We focus on (22) for cases where s_{in} is replaced by $H_{in}(t) = s_{in} + \delta_1(t)$ and the adaptive control is given by

$$\begin{array}{l} u = (\gamma + \delta_2(t))y_1, \\ \dot{\gamma} = -Ky_1(\gamma - \gamma_m)(\gamma_M - \gamma)(\tilde{s} + \delta_3(t)) \end{array} \tag{37}$$

where the disturbances $\delta_1(t)$ and $\delta_3(t)$ are bounded in absolute value by a constant $\overline{\delta}_1$ and the disturbance $\delta_2(t)$ is bounded in absolute value by a constant $\overline{\delta}_2$; we specify the $\overline{\delta}_i$s below, and $K > 0$ is an arbitrary tuning constant.

We continue to use the assumptions and notation from the preceding subsections. We also assume

$$\frac{k}{\lambda} < (\gamma_m - \overline{\delta}_2)(s_{in} - \overline{\delta}_1), \ \overline{\delta}_1 < s_{in}, \text{ and } \overline{\delta}_2 < \gamma_m. \tag{38}$$

In particular, we keep the definitions of x_* and γ_* from (24) and (25) unchanged; we do not replace s_{in} by the (unknown) time-varying function $H_{in}(t)$ in the expressions for x_* and γ_*. Our analysis will use the function S from (34) extensively. To specify our bounds $\bar{\delta}_i$, we use the constants

$$\begin{array}{l}\Xi = \frac{1}{\gamma_m}\left[\Upsilon_1 + \frac{\gamma_M^2}{2v_*} + \frac{K(\gamma_M-\gamma_m)^2}{4}\right] \ \text{ and} \\ \Upsilon_2 = \min\{\gamma_* - \gamma_m, \gamma_M - \gamma_*\}\end{array} \tag{39}$$

where Υ_1 is from (35). See Section IV-E for a concrete example with specific bounds $\bar{\delta}_i$.

Replacing s_{in} with H_{in} in (22) and using the expressions for y_1 and u from (37) gives the system

$$\left\{\begin{array}{lcl}\dot{s} & = & y_1\left[(\gamma+\delta_2(t))(s_{in}+\delta_1(t)-s) - \frac{k}{\lambda}\right] \ , \\ \dot{x} & = & y_1\left[\frac{1}{\lambda} - \alpha(\gamma+\delta_2(t))x\right] \ , \\ \dot{\gamma} & = & -Ky_1(\gamma-\gamma_m)(\gamma_M-\gamma)(\tilde{s}+\delta_3(t)) \ .\end{array}\right. \tag{40}$$

For simplicity, we restrict our analysis to the attractive and invariant domain where $0 < s < 2s_{in}$ which, in practice, is the domain of interest. Using (38) and arguing as we did to obtain Lemma 2, one can apply the rescaling (27) to get

$$\left\{\begin{array}{l}\dot{\tilde{s}} = -\gamma\tilde{s} + \tilde{\gamma}v_* + \delta_2(t)(s_{in}-s) + (\gamma+\delta_2(t))\delta_1(t), \\ \dot{\tilde{x}} = -\alpha(\gamma+\delta_2(t))\tilde{x} - \alpha(\delta_2(t)+\tilde{\gamma})x_\star, \\ \dot{\tilde{\gamma}} = -K(\gamma-\gamma_m)(\gamma_M-\gamma)(\tilde{s}+\delta_3(t))\end{array}\right. \tag{41}$$

evolving on the invariant domain $\hat{D} = (-x_*, +\infty) \times (-s_*, 2s_{in} - s_*) \times (\gamma_m - \gamma_*, \gamma_M - \gamma_*)$. We now state our ISS result for the error dynamics (41). The result implies that the trajectories of (40) satisfy $(x(t), s(t)) \to (x_\star, s_\star)$ in a globally uniformly way, with an overflow that can be quantified to be small when the disturbances δ_i are small; see our ISS estimate (3) with $\mathcal{G} = \hat{D}$. Set

$$\Delta_1 = \frac{v_*}{4\left(\Xi v_* + \frac{5}{4}\gamma_M\right)} \ \text{ and } \Delta_2 = \frac{4\sqrt{v_*}\sqrt{\Upsilon_1}}{K(\gamma_M-\gamma_m)^2 + 5\Xi\gamma_M}.$$

Theorem 2: Let the preceding assumptions hold. Assume that δ_1 and δ_3 are bounded in absolute value by

$$\bar{\delta}_1 \ = \ \frac{99}{100}\Upsilon_2 \min\{\Delta_1, \Delta_2\} \tag{42}$$

and that the disturbance δ_2 is bounded in absolute value by

$$\bar{\delta}_2 \ = \ \frac{\gamma_M\bar{\delta}_1}{4(2s_{in}+\bar{\delta}_1)}. \tag{43}$$

Then the closed loop error dynamics (41) is ISS on $\hat{D}$.

Proof: Since the $\tilde{x}$ subdynamics is ISS when $(\tilde{\gamma}, \delta_2)$ is viewed as its disturbance (using the fact that $(d/dt)\tilde{x}^2 \le -\underline{b}\tilde{x}^2 + \bar{b}(|\delta_2| + |\tilde{\gamma}|)$ along its trajectories for appropriate constants $\underline{b}, \bar{b} > 0$ combined with standard ISS Lyapunov function arguments [18]), it suffices to check that the $(\tilde{s}, \tilde{\gamma})$ subdynamics is ISS with respect to δ [18]. Hence, we focus on the $(\tilde{s}, \tilde{\gamma})$ subdynamics in the rest of the proof.

It follows from (36) that along the trajectories of (41),

$$\begin{array}{rcl}\dot{S} & \le & -W(\tilde{s},\tilde{\gamma}) + \frac{\partial S}{\partial\tilde{s}}(\tilde{s},\tilde{\gamma})[\delta_2(t)(s_{in}-s) \\ & & +(\gamma+\delta_2(t))\delta_1(t)] \\ & & -\frac{\partial S}{\partial\tilde{\gamma}}(\tilde{s},\tilde{\gamma})K(\gamma-\gamma_m)(\gamma_M-\gamma)\delta_3(t) \\ & \le & -W(\tilde{s},\tilde{\gamma}) + T_1(\tilde{s},\tilde{\gamma}) + T_2(\tilde{s},\tilde{\gamma})\end{array} \tag{44}$$

with W defined in (36) and

$$\begin{array}{rcl}T_1(\tilde{s},\tilde{\gamma}) & = & \left|\frac{\partial S}{\partial\tilde{s}}(\tilde{s},\tilde{\gamma})\right|\left[\bar{\delta}_2|s_{in}-\tilde{s}| + (|\gamma|+\bar{\delta}_2)\bar{\delta}_1\right] \ , \\ T_2(\tilde{s},\tilde{\gamma}) & = & \left|\frac{\partial S}{\partial\tilde{\gamma}}(\tilde{s},\tilde{\gamma})\right|K(\gamma-\gamma_m)(\gamma_M-\gamma)\bar{\delta}_1 \ .\end{array} \tag{45}$$

Computing $\frac{\partial S}{\partial\tilde{s}}$ and $\frac{\partial S}{\partial\tilde{\gamma}}$ (from (34)) readily gives

$$\begin{array}{rcl}T_1(\tilde{s},\tilde{\gamma}) & \le & |\tilde{\gamma} - \Xi\tilde{s}|[\bar{\delta}_2|s_{in}-\tilde{s}| + \gamma\bar{\delta}_1 + \bar{\delta}_1\bar{\delta}_2] \\ & \le & |\tilde{\gamma} - \Xi\tilde{s}|[\bar{\delta}_2(2s_{in}+\bar{\delta}_1) + \gamma_M\bar{\delta}_1] \ , \\ T_2(\tilde{s},\tilde{\gamma}) & \le & \left|\tilde{s} - \Xi\frac{v_*}{K}\frac{\tilde{\gamma}}{(\tilde{\gamma}+\gamma_*-\gamma_m)(\gamma_M-\gamma_*-\tilde{\gamma})}\right| \\ & & \times K(\gamma-\gamma_m)(\gamma_M-\gamma)\bar{\delta}_1 \\ & \le & \frac{K}{4}(\gamma_M-\gamma_m)^2\bar{\delta}_1|\tilde{s}| + \Xi v_*\bar{\delta}_1|\tilde{\gamma}| \ .\end{array} \tag{46}$$

Therefore, $T_1(\tilde{s},\tilde{\gamma}) + T_2(\tilde{s},\tilde{\gamma}) \le E_1|\tilde{\gamma}| + E_2|\tilde{s}|$ with

$$\begin{array}{rcl}E_1 & = & \Xi v_*\bar{\delta}_1 + (2s_{in}+\bar{\delta}_1)\bar{\delta}_2 + \gamma_M\bar{\delta}_1 \ , \\ E_2 & = & \frac{K}{4}(\gamma_M-\gamma_m)^2\bar{\delta}_1 + \Xi[\bar{\delta}_2(2s_{in}+\bar{\delta}_1) + \gamma_M\bar{\delta}_1].\end{array} \tag{47}$$

From (42) and (43), we deduce that

$$\begin{array}{rcl}E_1 & \le & \frac{99}{100}\Upsilon_2\frac{v_*}{4} \ , \\ E_2 & \le & \left(\frac{K}{4}(\gamma_M-\gamma_m)^2 + \Xi\frac{5\gamma_M}{4}\right)\bar{\delta}_1 \\ & \le & \frac{99}{100}\Upsilon_2\sqrt{v_*}\sqrt{\Upsilon_1} \ .\end{array} \tag{48}$$

Therefore, (44) and (46) give

$$\begin{array}{rl}\dot{S} \ \le & -\frac{v_*}{2}\tilde{\gamma}^2 - \Upsilon_1\tilde{s}^2 + \frac{99}{100}\Upsilon_2\frac{v_*}{4}|\tilde{\gamma}| \\ & +\frac{99}{100}\Upsilon_2\sqrt{v_*}\sqrt{\Upsilon_1}|\tilde{s}| \ .\end{array} \tag{49}$$

We consider two cases. <u>Case 1</u>: If $\tilde{\gamma} \in \left(\gamma_m - \gamma_*, \frac{199}{200}(\gamma_m - \gamma_*)\right]$ or $\tilde{\gamma} \in \left[\frac{199}{200}(\gamma_M - \gamma_*), \gamma_M - \gamma_*\right)$, then $|\tilde{\gamma}| > 0.995\Upsilon_2$. From the relation $pq \le p^2 + q^2/4$ with $p = \sqrt{\Upsilon_1}|\tilde{s}|$ and $q = .99\Upsilon_2\sqrt{v_\star}$, we deduce from (49) that

$$\begin{array}{rcl}\dot{S} & \le & -\frac{v_*}{4}\left(\frac{199}{200}\right)^2\Upsilon_2^2 - \Upsilon_1\tilde{s}^2 + \frac{99}{100}\Upsilon_2\sqrt{v_*}\sqrt{\Upsilon_1}|\tilde{s}| \\ & \le & -\frac{v_*}{4}\left[\left(\frac{199}{200}\right)^2 - \left(\frac{99}{100}\right)^2\right]\Upsilon_2^2 \ \le \ -\Upsilon_3 \\ & & \text{where} \quad \Upsilon_3 = \frac{397v_*}{160000}\Upsilon_2^2 \ .\end{array} \tag{50}$$

<u>Case 2</u>: If $\tilde{\gamma} \in \left[\frac{199}{200}(\gamma_m - \gamma_*), \frac{199}{200}(\gamma_M - \gamma_*)\right]$, then

$$\begin{array}{l}\int_0^{\tilde{\gamma}} \frac{l}{(l+\gamma_*-\gamma_m)(\gamma_M-\gamma_*-l)}dl \\ \le \ \frac{2(10^4)\tilde{\gamma}^2}{(\gamma_*-\gamma_m)(\gamma_M-\gamma_*)} \ .\end{array} \tag{51}$$

In this case, (34) gives

$$\begin{array}{l}S(\tilde{s},\tilde{\gamma}) \ \le \ \frac{1+\Xi}{2}\tilde{s}^2 + \Upsilon_4\tilde{\gamma}^2 \\ \quad \le \ \max\left\{\frac{1+\Xi}{2}, \Upsilon_4\right\}\left[\tilde{s}^2 + \tilde{\gamma}^2\right]\end{array} \tag{52}$$

with

$$\Upsilon_4 = \frac{1}{2} + \Xi\frac{v_*}{K}\frac{2(10^4)}{(\gamma_*-\gamma_m)(\gamma_M-\gamma_*)} \ . \tag{53}$$

Also, $W(\tilde{s},\tilde{\gamma}) \ge \min\left\{\frac{v_*}{2}, \Upsilon_1\right\}\left(\tilde{s}^2 + \tilde{\gamma}^2\right)$. Therefore

$$\begin{array}{rcl}\frac{1}{2}W(\tilde{s},\tilde{\gamma}) & \ge & \Upsilon_5 S(\tilde{s},\tilde{\gamma}), \quad \text{where} \\ \Upsilon_5 & = & \frac{\min\left\{\frac{v_*}{2}, \Upsilon_1\right\}}{2\max\left\{\frac{1+\Xi}{2}, \Upsilon_4\right\}} \ .\end{array} \tag{54}$$

It follows from (44) and our choices (47) of the E_is that

$$\begin{array}{rcl}\dot{S} & \le & -\Upsilon_5 S(\tilde{s},\tilde{\gamma}) - \frac{1}{2}W(\tilde{s},\tilde{\gamma}) + E_1|\tilde{\gamma}| + E_2|\tilde{s}| \\ & \le & -\Upsilon_5 S(\tilde{s},\tilde{\gamma}) - \frac{v_*}{4}\tilde{\gamma}^2 - \frac{1}{2}\Upsilon_1\tilde{s}^2 \\ & & +\left\{\frac{E_1}{\sqrt{v_*}}\right\}\left\{|\tilde{\gamma}|\sqrt{v_\star}\right\} + \left\{\frac{E_2}{\sqrt{\Upsilon_1}}\right\}\left\{|\tilde{s}|\sqrt{\Upsilon_1}\right\} .\end{array} \tag{55}$$

Applying the general inequality $pq \leq p^2 + \frac{1}{4}q^2$ to the terms in braces in (55), we deduce that

$$\begin{array}{l} \dot{S} \;\; \leq \;\; -\Upsilon_5 S(\tilde{s},\tilde{\gamma}) + \frac{E_1^2}{v_*} + \frac{E_2^2}{\Upsilon_1} \\ \leq \; -\Upsilon_5 S(\tilde{s},\tilde{\gamma}) + \Upsilon_6 \overline{\delta}_1^2, \\ \text{where } \; \Upsilon_6 = \frac{1}{v_*}\left(\Xi v_* + \frac{5\gamma_M}{4}\right)^2 \\ +\frac{1}{\Upsilon_1}\left[\frac{K}{4}(\gamma_M - \gamma_m)^2 + \Xi\frac{5\gamma_M}{4}\right]^2 \end{array} \tag{56}$$

and we used (43) and (47). Therefore, in the first case we get (50), while in the second case, we get (56). Using the construction of (34), one easily checks [12] that S admits a constant $Q_o > 0$ such that $Q_o(\tilde{s}^2 + \tilde{\gamma}^2) \leq S(\tilde{s},\tilde{\gamma})$ on $\hat{D}$. We conclude by applying [12, Lemma A.2], combined with the relation $\sqrt{p+q} \leq \sqrt{2p} + \sqrt{2q}$ (by taking square roots of both sides of the conclusion of the lemma with $M = S$). ■

E. Numerical Example

We simulated an anaerobic digestion process used to process wastewater and produce biogas. We calibrated the model using real experimental data [2] in order to get realistic parameter values. Finally, we simulated white noises for the δ_i, introducing a 1% noise in the controller u and 10% noises to perturb s_{in} and γ; see (37). We took the influent concentration s_{in} to be piecewise constant with respect to time, following the real profile in [3]; see the figure below. Our setpoint s^* was 1.5 g/l. We applied our controller successively on the intervals on which s_{in} is constant.

We report our noisy controlled simulation in the upper figure below as a solid curve, with circles to indicate experimental results obtained without control, i.e., $u \equiv 0$. The controller was implemented in [8] which asserted that the feedback lent itself to realistic settings in which there are noisy signals. However, [8] did not provide any rigorous robustness theory (e.g. ISS results) to prove this assertion. Our theory and simulation confirm and validate the robustness of the controller under realistic values of noise.

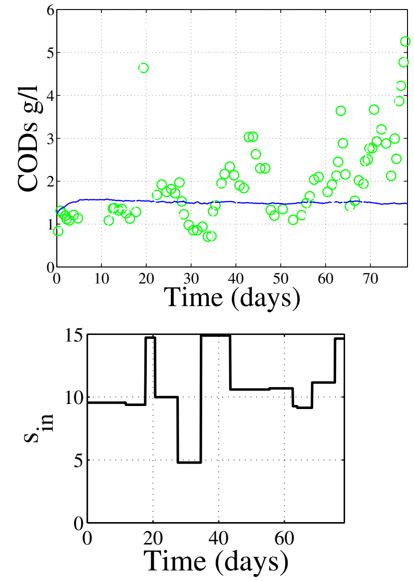

Fig. 1. Controlled substrate s (upper panel) and influent substrate concentration s_{in} (lower panel). Circles are experimental points obtained without control. Units are g per liter of chemical oxygen demand.

V. Conclusions

We provided new constructions of strict Lyapunov functions for nonlinear systems that satisfy appropriate variants of Matrosov's conditions. The advantages of our methods lie in the explicitness and simplicity of our Lyapunov functions and their applicability to the various examples whose stability can be established by the generalized Matrosov theorem. Using our Lyapunov constructions and ISS, we provided control laws that are robust to uncertainties. We validated our methods through a biotechnological model which is of compelling engineering interest.

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