2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Lyapunov Tools for Predictor Feedbacks for Delay Systems: Inverse Optimality and Robustness to Delay Mismatch

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Abstract— **We consider LTI finite-dimensional, completely controllable, but possibly open-loop unstable, plants, with arbitrarily long actuator delay, and the corresponding predictor-based feedback for delay compensation. We study the problem of inverse-optimal re-design of the predictor-based feedback law. We obtain a simple modification of the basic predictor-based controller, which employs a low-pass filter, and has been proposed previously by Mondie and Michiels for achieving robustness to discretization of the integral term in the predictor feedback law. The key element in our work is the employment of an infinite-dimensional "backstepping" transformation, and the resulting complete Lyapunov function, for the infinite dimensional systems consisting of the state of the ODE plant and the delay state. The Lyapunov function allows us to establish inverse optimality of the modified feedback and its disturbance attenuation properties. For the basic predictor feedback, the availability of the Lyapunov function also allows us to prove robustness to small delay mismatch (in both positive and negative directions).**

I. Introduction

We consider control systems of the form

$$\dot{X}(t) = AX(t) + BU(t-D)\,, \tag{1}$$

where $X \in \mathbb{R}^n$, (A, B) is a completely controllable pair, and the input signal $U(t)$ is delayed by D units of time. We allow A to be unstable and the delay D to be arbitrarily large. In [8], [9], [1] the following controller was developed which achieves asymptotic stabilization for any $D > 0$:

$$U(t) = K\left[e^{AD}X(t) + \int_{t-D}^{t} e^{A(t-\theta)}BU(\theta)d\theta\right]\,. \tag{2}$$

and which is viewed as a "delay-compensated" version of the 'nominal controller'

$$U(t) = KX(t)\,. \tag{3}$$

The controller (2) employs the history of the control input $U(t)$ over the last D time units, so it is infinite-dimensional. The controller (2) is known under several names in the literature, including "finite-spectum assignment," "predictor-based controller" or "Smith predictor [14] for unstable systems," and "reduction-based controller." The properties of this controller have been widely studied in the literature [3], [11] and it has been extended to the parameter-adaptive case [2], [12].

This work was supported by NSF.

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In [7], using the backstepping method for PDEs, we have constructed the Lyapunov function for the closed-loop system (1), (2), which is in the form

$$\begin{aligned} V(t) &= X(t)^T PX(t) \\ &\quad +\frac{a}{2}\int_{t-D}^{t}(1+\theta+D-t)\,W(\theta)^2 d\theta\,, \end{aligned} \tag{4}$$

where P is the solution of the Lyapunov equation

$$P(A+BK) + (A+BK)^{\mathrm{T}}P = -Q\,, \tag{5}$$

P and Q are positive definite and symmetric, the constant $a > 0$ is sufficiently large, and $W(\theta)$ is defined as

$$\begin{aligned} W(\theta) &= U(\theta) - K\left[\int_{t-D}^{\theta} \mathrm{e}^{A(\theta-\sigma)}BU(\sigma)d\sigma \right.\\ &\quad \left. +\mathrm{e}^{A(\theta+D-t)}X(t)\right]\,, \end{aligned} \tag{6}$$

with $-D \le t-D \le \theta \le t$.

In this note we highlight some of the benefits of constructing the transformation (6) and of the Lyapunov function (4). The first benefit is the ability to derive an inverse-optimal controller, which incorporates a penalty not only on the ODE state $X(t)$ and the input $U(t)$ but also on the delay state. Inverse optimality, as an objective in designing controllers for delay systems, was pursued by Jankovic [4], [5]. The predictor-based feedback (2) is not inverse optimal, even if the nominal feedback (3) is optimal, for example, if $K = -B^T P$, where P solves a Riccati equation, in other words, there does not exist (in general) a positive definite functional in $X(t)$, $U(t)$ and $U(\theta), \theta \in [t-D, t]$, which the feedback (2) minimizes. The inverse-optimal feedback that we design in the note is of the form

$$\begin{aligned} U(t) &= \frac{c}{s+c}\left\{K\left[e^{AD}X(t)\right.\right. \\ &\quad \left.\left. +\int_{t-D}^{t} e^{A(t-\theta)}BU(\theta)d\theta\right]\right\}\,, \end{aligned} \tag{7}$$

where $c > 0$ is sufficiently large, i.e., the inverse-optimal feedback is of the form of a low-pass filtered version of (2). As it turns out, the low pass modification, proposed here for inverse optimality, has already been proposed in [11] as a tool for ensuring robustness to the discretization of the integral term in (2).

The second benefit of constructing the transformation (6) and of the Lyapunov function (4) is that one can prove robustness of exponential stability of the predictor feedback to *small* mismatch in the actuator delay, both in the positive and in the negative direction.

In Section II we establish inverse optimality of the feedback law (7) and its stabilization property for sufficiently large c. In Section III we consider the plant (1) in the presence of an additive disturbance and establish the inverse optimality of the feedback (7) in the sense of solving a meaningful differential game problem and we quantify its L_∞ disturbance attenuation property. Finally, for the basic predictor feedback, in Section IV we use our Lyapunov function to prove robustness to small delay mismatch.

II. Inverse Optimal Re-Design

In the formulation of the inverse optimality problem we will consider $\dot{U}(t)$ as the input to the system, whereas $U(t)$ is still the actuated variable. Hence, our inverse optimal design will be implementable after integration in time, i.e., as dynamic feedback. Treating $\dot{U}(t)$ as an input is the same as adding an integrator, which has been observed as being beneficial in the control design for delay systems in [4].

Theorem 1: There exists c^* such that the feedback system (1), (7) is exponentially stable in the sense of the norm

$$N(t) = \left(|X(t)|^2 + \int_{t-D}^{t} U(\theta)^2 d\theta + U(t)^2\right)^{1/2} \tag{8}$$

for all $c > c^*$. Furthermore, there exists $c^{**} > c^*$ such that, for any $c \geq c^{**}$, the feedback (7) minimizes the cost functional

$$J = \int_0^\infty \left(\mathcal{Q}(t) + \dot{U}(t)^2\right) dt\,, \tag{9}$$

where $\mathcal{Q}(t) \geq \mu N(t)^2$ for some $\mu(c) > 0$, which is such that $\mu(c) \to \infty$ as $c \to \infty$.

Proof: We start by writing (1) as the ODE-PDE system

$$\dot{X} = AX + Bu(0,t). \tag{10}$$

$$u_t(x,t) = u_x(x,t) \tag{11}$$

$$u(D,t) = U(t)\,, \tag{12}$$

where $u(x,t) = U(t+x-D)$ and therefore the output $u(0,t) = U(t-D)$ gives the delayed input (see Fig. 1).

Consider the infinite-dimensional backstepping transformation of the delay state [7]

$$\begin{aligned} w(x,t) &= u(x,t) - \left[\int_0^x Ke^{A(x-y)}Bu(y,t)dy \right. \\ &\quad \left. + Ke^{Ax}X(t)\right]. \end{aligned} \tag{13}$$

It is readily verified that

$$\dot{X} = (A+BK)X + Bw(0,t) \tag{14}$$

$$w_t(x,t) = w_x(x,t)\,. \tag{15}$$

Let us now consider $w(D,t)$. It is easily seen that

$$\begin{aligned} w_t(D,t) &= u_t(D,t) - K\left[Bu(D,t)\right. \\ &\quad + \int_0^D e^{A(D-y)}Bu(y,t)dy \\ &\quad \left.+ Ae^{AD}X(t)\right]. \end{aligned} \tag{16}$$

Note that $u_t(D,t) = \dot{U}(t)$, which is designated as the control input penalized in (9). The inverse of (13) can be derived as

$$\begin{aligned} u(x,t) = w(x,t) &+ \int_0^x Ke^{(A+BK)(x-y)}Bw(y,t)\,dy \\ &+ Ke^{(A+BK)x}X(t)\,. \end{aligned} \tag{17}$$

Plugging (17) into (16), after a lengthy calculation that involves a change of the order of integration in a double integral, we get

$$\begin{aligned} w_t(D,t) &= u_t(D,t) - KBw(D,t) \\ &\quad - K(A+BK)\left[\int_0^D M(y)Bw(y,t)dy\right. \\ &\quad \left.+ M(0)X(t)\right], \end{aligned} \tag{18}$$

where

$$\begin{aligned} M(y) &= \int_y^D e^{A(D-\sigma)}BKe^{(A+BK)(\sigma-y)}d\sigma \\ &\quad + e^{A(D-y)} \\ &= e^{(A+BK)(D-y)} \end{aligned} \tag{19}$$

is a matrix-valued function defined for $y \in [0,D]$. Note that $N : [0,D] \to \mathbb{R}^{n\times n}$ is in both $L_\infty[0,D]$ and in $L_2[0,D]$.

Consider now a Lyapunov function

$$\begin{aligned} V(t) &= X(t)^T PX(t) + \frac{a}{2}\int_0^D (1+x)w(x,t)^2\,dx \\ &\quad + \frac{1}{2}w(D,t)^2\,, \end{aligned} \tag{20}$$

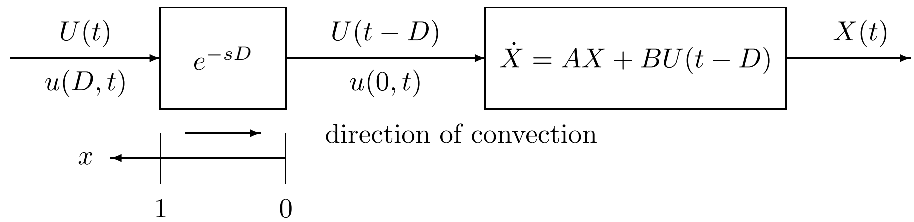

Fig. 1. Linear system $\dot{X} = AX + BU(t-D)$ with actuator delay D.

where $P > 0$ is defined in (5) and the parameter $a > 0$ is to be chosen later. We have

$$\begin{aligned}
\dot{V} &= (X^T((A+BK)^T P + P(A+BK))X \\
&\quad +2X^T PBw(0,t) \\
&\quad +\frac{a}{2}\int_0^D (1+D)w(x,t)w_x(x,t)dx \\
&\quad +2w(D,t)w_t(D,t) \\
&= -X^T QX + 2X^T PBw(0,t) \\
&\quad +\frac{a}{2}(1+D)w(D,t)^2 - \frac{a}{2}w(0,t)^2 \\
&\quad -\frac{a}{2}\int_0^D w(x,t)^2\,dx \\
&\quad +w(D,t)w_t(D,t) \qquad (21) \\
&\leq -X^T QX + \frac{2}{a}\|X^T PB\|^2 \\
&\quad -\frac{a}{2}\int_0^D w(x,t)^2\,dx \\
&\quad +w(D,t)\left(w_t(D,t) + \frac{a(1+D)}{2}w(D,t)\right), \qquad (22)
\end{aligned}$$

and finally,

$$\begin{aligned}
\dot{V} &\leq -\frac{1}{2}X^T QX - \frac{a}{2}\int_0^D w(x,t)^2\,dx \\
&\quad +w(D,t)\left(w_t(D,t) + \frac{a(1+D)}{2}w(D,t)\right), \qquad (23)
\end{aligned}$$

where we have chosen

$$a = 4\frac{\lambda_{\max}(PBB^T P)}{\lambda_{\min}(Q)}, \qquad (24)$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are minimum and maximum eigenvalues of the corresponding matrices. Now we consider (23) along with (18). With a completion of squares, we obtain

$$\begin{aligned}
\dot{V} &\leq -\frac{1}{4}X^T QX - \frac{a}{4}\int_0^D w(x,t)^2\,dx \\
&\quad +\frac{|K(A+BK)M(0)|^2}{\lambda_{\min}(Q)}w(D,t)^2 \\
&\quad +\frac{\|K(A+BK)MB\|^2}{a}w(D,t)^2 \\
&\quad +\left(\frac{a(1+D)}{2} - KB\right)w(D,t)^2 \\
&\quad +w(D,t)u_t(D,t). \qquad (25)
\end{aligned}$$

Choosing

$$u_t(D,t) = -cw(D,t), \qquad (26)$$

we arrive at

$$\begin{aligned}
\dot{V} &\leq -\frac{1}{4}X^T QX - \frac{a}{4}\int_0^D w(x,t)^2\,dx \\
&\quad -(c-c^*)w(D,t)^2, \qquad (27)
\end{aligned}$$

where

$$\begin{aligned}
c^* &= \frac{a(1+D)}{2} - KB \\
&\quad +\frac{|K(A+BK)M(0)|^2}{\lambda_{\min}(Q)} \\
&\quad +\frac{\|K(A+BK)MB\|^2}{a}. \qquad (28)
\end{aligned}$$

Using (13) for $x = D$ and the fact that $u(D,t) = U(t)$, from (26) we get (7). Hence, from (27), the first statement of the theorem is proved if we can show that there exist positive numbers α_1 and α_2 such that

$$\alpha_1 N^2 \leq V \leq \alpha_2 N^2, \qquad (29)$$

where

$$N(t)^2 = |X(t)|^2 + \int_0^D u(x,t)^2\,dx + u(D,t)^2. \qquad (30)$$

This is straightforward to establish by using (13), (17), and (20), and employing the Cauchy-Schwartz inequality and other calculations, following a pattern of a similar computation in [15]. Thus, the first part of the theorem is proved.

The second part of the theorem is established in a manner very similar to the lengthy proof of Theorem 6 in [15], which is based on the idea of the proof of Theorem 2.8 in [6]. We choose $c^{**} = 4c^*$ and

$$\begin{aligned}\mathcal{Q}(t) &= -2c\dot{V}\Big|_{(21) \text{ with } (18), (26), \text{ and } c=2c^*} \\ &\quad + c(c-4c^*)w(D,t)^2 \\ &\geq c\left(\frac{1}{2}X^TQX + \frac{a}{2}\int_0^D w(x,t)^2\,dx \right.\\ &\quad \left. + (c-2c^*)w(D,t)^2\right). \end{aligned} \tag{31}$$

We have that $\mathcal{Q}(t) \geq \mu N(t)^2$ for the same reason that (29) holds. This completes the proof of inverse optimality. ■

Remark 2.1: Having obtained inverse optimality, one would be tempted to conclude that the controller (7) has an infinite gain margin and a phase margin of 60°. This is unfortunately not true. These properties can be claimed only for the feedback law (26), i.e., $\dot{U}(t) = -cW(t)$. Hence, the only gain margin one has is the stability robustness to varying the parameter c from some large value c^* to ∞, recovering in the limit the basic, unfiltered predictor-based feedback (2). This robustness property might be intuitively expected from a singular perturbation idea, though an off-the-shelf theorem for establishing this property would be highly unlikely to be found in the literature due to the infinite dimensionality and the special hybrid (ODE-PDE-ODE) structure of the system at hand. The feedback (2) is not inverse optimal, however the feedback (7) is, for any $c \in [c^{**}, \infty)$. Its optimality holds for a highly relevant cost functional, which is underbounded by the temporal $L_2[0,\infty)$ norm of the ODE state $X(t)$, the norm of the control $U(t)$, as well as the norm of its derivative $\dot{U}(t)$ (in addition to $\int_0^D U(\theta)^2 d\theta$ which is fixed because feedback has no influence on it). The controller (7) is stabilizing for $c = \infty$, namely, in its nominal form (2), however, since $\mu(\infty) = \infty$, it is not optimal with respect to a cost functional that includes a penalty on $\dot{U}(t)$.

III. Disturbance Attenuation

Consider the following system

$$\dot{X}(t) = AX(t) + BU(t-D) + Gd(t)\,, \tag{32}$$

where $d(t)$ is an unmeasurable disturbance signal and G is a vector. In this section, the availability of the Lyapunov function (20) lets us establish the disturbance attenuation properties of the controller (7).

Theorem 2: There exists c^* such that, for all $c > c^*$, the feedback system (32), (7) is L_∞-stable, i.e., there exist positive constants $\beta_1, \beta_2, \gamma_1$, such that

$$N(t) \leq \beta_1 \mathrm{e}^{-\beta_2 t} N(0) + \gamma_1 \sup_{\tau\in[0,t]} |d(\tau)|\,. \tag{33}$$

Furthermore, there exists $c^{**} > c^*$ such that, for any $c \geq c^{**}$, the feedback (7) minimizes the cost functional

$$\begin{aligned} J &= \sup_{d\in\mathcal{D}} \lim_{t\to\infty} \Big[2cV(t) \\ &\quad + \int_0^t \left(Q(\tau) + \dot{U}(t)^2 - c\gamma_2 d(\tau)^2\right) d\tau\Big]\,, \end{aligned} \tag{34}$$

for any

$$\gamma_2 \geq \gamma_2^{**} = 8\frac{\lambda_{\max}(PBB^TP)}{\lambda_{\min}(Q)}\,, \tag{35}$$

where $\mathcal{Q}(t) \geq \mu N(t)^2$ for some $\mu(c,\gamma_2) > 0$, which is such that $\mu(c,\gamma_2) \to \infty$ as $c \to \infty$, and $\mathcal{D}$ is the set of linear scalar-valued functions of X.

Proof: First, with a slight modification of the calculations leading to (27) we get that

$$\begin{aligned}\dot{V} &\leq -\frac{1}{8}X^TQX - \frac{a}{4}\int_0^D w(x,t)^2\,dx \\ &\quad -(c-c^*)w(D,t)^2 + \frac{b}{2}d^2\,, \end{aligned} \tag{36}$$

From here, a straightforward, though lengthy, calculation gives the L_∞ stability result.

The proof of inverse optimality is obtained by specializing the proof of Theorem 2.8 in [6] to the present case. The function $\mathcal{Q}(t)$ is defined as

$$\begin{aligned}\mathcal{Q}(t) &= -2c\Omega(t) \\ &\quad + 2c\frac{\gamma_2-\gamma_2^{**}}{\gamma_2}\frac{8|PG|^2}{b}|X(t)|^2 \\ &\quad + c(c-4c^*)w(D,t)^2\,, \end{aligned} \tag{37}$$

where $\Omega(t)$ is defined as the value of $\dot{V}(t)$ in (21) with (18), (26), $c = 2c^*$, and

$$d(t) = \frac{1}{b}G^TPX(t)\,, \tag{38}$$

where $b = 2\gamma_2^{**}$. It is easy to see that $\Omega(t) \leq -\frac{1}{8}X^TQX - \frac{a}{4}\int_0^D w(x,t)^2\,dx - 2c^*w(D,t)^2$. Therefore,

$$\begin{aligned}\mathcal{Q}(t) &\geq c\left(\frac{\gamma_2-\gamma_2^{**}/2}{\gamma_2}\lambda_{\min}(Q)|X(t)|^2 \right.\\ &\quad + \frac{a}{2}\int_0^D w(x,t)^2\,dx \\ &\quad \left. + (c-2c^*)w(D,t)^2\right)\,, \end{aligned} \tag{39}$$

which is lower-bounded by $\mu N(t)^2$ as in the proof of Theorem 1. This completes the proof of inverse optimality. ■

Remark 3.1: Similar to the last point in Remark 2.1, the nominal predictor feedback (2), though not inverse optimal, is L_∞ stabilizing. This is seen with a different Lyapunov function, $V(t) = X(t)^TPX(t) + \frac{a}{2}\int_0^D (1+x)w(x,t)^2\,dx$, which yields $dV(t)/dt \leq -\frac{1}{4}X^TQX - \frac{a}{2}\int_0^D w(x,t)^2\,dx + \frac{b}{4}d^2$.

IV. Robustness to Delay Mismatch

Predictor-based feedbacks are known to be sensitive to errors in the knowledge of the value of actuator delay. This problem is discussed in [10], [3], [11] and other references. Despite the sensitivity, the predictor feedbacks are an 'irreplaceble and widely used tool' [13].

The existing studies of robustness to delay mismatch are frequency domain studies. We are not aware of robustness analyses using Lyapunov techniques. The result in [16] answers a similar question for ODEs, however it does not apply to the present case where the nominal case (without delay mismatch) is infinite dimensional and the feedback law is also infinite dimensional.

We consider the feedback system

$$\dot{X} = AX + BU(t - D_0 - \Delta D)\,, \tag{40}$$

$$U(t) = K\left[e^{AD_0}X(t) + \int_{t-D_0}^{t} e^{A(t-\theta)}BU(\theta)d\theta\right]. \tag{41}$$

The reader should note that the actual actuator delay has a mismatch of ΔD, which can be either positive or negative, relative to the assumed plant delay $D_0 > 0$, with the obvious necessary condition that $D_0 + \Delta D \geq 0$. Being in the possession of a complete Lyapunov function, we are able to prove the following result.

Theorem 3: There exists $\delta > 0$ such that for all $\Delta D \in (-\delta, \delta)$ the system (40), (41) is exponentially stable in the sense of the state norm

$$N_2(t) = \left(|X(t)|^2 + \int_{t-\bar{D}}^{t} U(\theta)^2 d\theta\right)^{1/2}, \tag{42}$$

where $\bar{D} = D_0 + \max\{0, \Delta D\}$.

Proof: We use the same transport PDE formalism as in Theorem 1 and the transformations (13), (17). First, we note that the feedback (41) is written as

$$u(D_0 + \Delta D, t) = K\left[e^{AD_0}X(t) + \int_{\Delta D}^{D_0+\Delta D} \mathrm{e}^{A(D_0+\Delta D-y)}Bu(y,t)dy\right], \tag{43}$$

which, using (13) for $x = D_0 + \Delta D$, gives us

$$w(D_0 + \Delta D, t) = Ke^{AD_0}\left[\left(I - \mathrm{e}^{A\Delta D}\right)X(t) - \int_0^{\Delta D} \mathrm{e}^{A(\Delta D-y)}Bu(y,t)dy\right]. \tag{44}$$

Then, employing (17) under the integral, and performing certain calculations, we obtain

$$w(D_0 + \Delta D, t) = Ke^{AD_0}\left[\left(I - \mathrm{e}^{(A+BK)\Delta D}\right)X(t) - \int_0^{\Delta D} \mathrm{e}^{(A+BK)(\Delta D-y)}Bw(y,t)dy\right]. \tag{45}$$

One then shows that

$$w(D_0 + \Delta D, t)^2 \leq 2q_1|X|^2 + 2q_2\int_{\min\{0,\Delta D\}}^{\max\{0,\Delta D\}} w(x,t)^2 dx\,, \tag{46}$$

where the functions $q_1(\Delta D)$ and $q_2(\Delta D)$ are

$$q_1 = \left|Ke^{AD_0}\left(I - \mathrm{e}^{(A+BK)\Delta D}\right)\right|^2 \tag{47}$$

$$q_2 = \int_{\min\{0,\Delta D\}}^{\max\{0,\Delta D\}} \left(Ke^{AD_0}e^{(A+BK)(\Delta D-y)}B\right)^2 dy \tag{48}$$

Note that $q_1(0) = q_2(0) = 0$ and that q_1 and q_2 are continuous functions of ΔD (the two integral terms in q_2 are both zero at zero, and continuous in ΔD).

The cases $\Delta D > 0$ and $\Delta D < 0$ are considered separately. The case $\Delta D > 0$ is easier; the state of the system is $X(t), u(x,t), x \in [0, D_0 + \Delta D]$, i.e., $X(t), U(\theta), \theta \in [t - D_0 - \Delta D, t]$. The case $\Delta D < 0$ is more intricate, as the state of the system is $X(t), u(x,t), x \in [\Delta D, D_0 + \Delta D]$, i.e., $X(t), U(\theta), \theta \in [t - D_0, t]$.

Case $\Delta D > 0$. We take the Lyapunov function

$$V(t) = X(t)^T PX(t) + \frac{a}{2}\int_0^{D_0+\Delta D} (1+x)w(x,t)^2\,dx\,. \tag{49}$$

A calculation similar to that at the beginning of the proof of Theorem 1 gives

$$\begin{aligned}\dot{V} &= -X^TQX + 2X^TPBw(0,t) \\ &\quad + \frac{a}{2}(1+D)w(D_0+\Delta D, t)^2 \\ &\quad - \frac{a}{2}w(0,t)^2 - \frac{a}{2}\int_0^{D_0+\Delta D} w(x,t)^2\,dx \qquad (50)\\ &\leq -\left(\frac{\lambda_{\min}(Q)}{2} - a(1+D)q_1(\Delta D)\right)|X|^2 \\ &\quad - a\left(\frac{1}{2} - (1+D)q_2(\Delta D)\right) \\ &\quad \times \int_0^{D_0+\Delta D} w(x,t)^2\,dx\,, \qquad (51)\end{aligned}$$

where a is chosen as in (24), and where we have denoted $D = D_0 + \Delta D$ for brevity. This proves exponential stability of the origin of the $(X(t), w(x,t), x \in [0, D_0 + \Delta D])$ system. Exponential stability in the norm $N_2(t)$ is obtained using the standard procedures for over- and under-bounding $V(t)$ by a linear function of $N_2^2(t)$.

Case $\Delta D < 0$. In this case we use a different Lyapunov function,

$$V(t) = X(t)^T PX(t) + \frac{a}{2}\int_0^{D_0+\Delta D} (1+x)w(x,t)^2\,dx + \frac{1}{2}\int_{\Delta D}^{0} (D_0+x)w(x,t)^2 dx\,. \tag{52}$$

At the end of the Lyapunov calculation we obtain:

$$
\begin{aligned}
\dot{V} \leq \\
& -\left(\frac{\lambda_{\min}(Q)}{2} - a(1+D)q_1(\Delta D)\right)|X|^2 \\
& -\left(\frac{a}{2} - \frac{D_0}{2} - \frac{2|PB|^2}{\lambda_{\min}(Q)}\right)w(0,t)^2 \\
& -\left(\frac{1}{2} - a(1+D)q_2(\Delta D)\right)\int_{\Delta D}^{0} w(x,t)^2 dx \\
& -\frac{D}{2}w(\Delta D,t)^2 - \frac{\max\{a,1\}}{4}\int_{\Delta D}^{D_0+\Delta D} w(x,t)^2 dx
\end{aligned} \tag{53}
$$

This quantity is made negative definite by first choosing $a > D_0 + \frac{4|PB|^2}{\lambda_{\min}(Q)}$, and then choosing ΔD sufficiently small so that $\frac{\lambda_{\min}(Q)}{2} > a(1+D)q_1(\Delta D)$ and $\frac{1}{2} > a(1+D)q_2(\Delta D)$. One thus gets exponential decay estimates in terms of $|X(t)|^2 + \int_{\Delta D}^{D_0+\Delta D} w(x,t)^2 dx$, and with some further work also in terms of $|X(t)|^2 + \int_{\Delta D}^{D_0+\Delta D} u(x,t)^2 dx$, i.e., in terms of $|X(t)|^2 + \int_{t-D_0}^{t} U(\theta)^2 d\theta$. ■

Remark 4.1: The result of Theorem 3 is fairly subtle. The case when $\Delta D > 0$ is clear, the robustness to a "surplus" of actuator delay is a result that already holds for ODEs [16]. The case $\Delta D < 0$ is more tricky. The controller, which overestimates the delay to be $D_0 > D_0 + \Delta D$, introduces the delayed inputs from the time interval $[t - D_0, t - D_0 - \Delta D]$ into the overall dynamic system, making its state consist of control inputs $U(\theta)$ from the entire interval $\theta \in [t - D_0, t]$, even though the actual actuator delay $D_0 + \Delta D$ is shorter. This peculiarity results in more complicated analysis for $\Delta D < 0$, with different weights on the Krassovskii functionals for the different parts of the delay interval (with lesser weight on the subinterval that represents the delay "mismatch"). The greater difficulty in proving the result for $\Delta D < 0$, leads us to conjecture that the predictor-based controllers would exhibit greater sensitivity to delay mismatch in the cases where the delay is "over-estimated" (and thus "overcompensated") rather than when it is "underestimated." This means that, while there is no question that predictor-based delay compensation is indispensable for dealing with long actuator delay, and thus, that some "some amount" of delay compensation is better than none, when faced with a delay of uncertain length, "less" may be better than "more," i.e., it may be better to err on the side of caution and design for the lower end of the delay range expected.

V. Conclusions

In this note we derived inverse optimality results for stabilization and disturbance attenuation with the low-pass filtered modification of the predictor-based feedback for actuator delay compensation. It would be interesting to explore direct optimality results, both for stabilization and H_∞ disturbance attenuation, however, one would expect that direct optimal control formulations yield operator Riccati equations and lack the simplicity of (2).

Having established robustness to small delay mismatch, as the most critical form of robustness in the predictor-feedback problem (as well as robustness to 'bandwidth limitation,' in the form of a low-pass filter), other forms of robustness are worth studying next, using the complete Lyapunov functions, with the help of the backtepping transformation and its inverse, (13), (17).

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