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A Kalman-Yakubovich-Popov Lemma with Affine Dependence on the Frequency

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Abstract— **This paper revisits the problem of checking feasibility of a given matrix inequality with rational dependence on a real variable, ω, often interpreted as frequency. In the case the frequency variable is allowed to assume arbitrary values, the Kalman-Yakubovich-Popov (KYP) Lemma provides an equivalent formulation of this problem as a Linear Matrix Inequality (LMI). When the frequency lies within a finite or semi-infinite range, generalizations of the KYP Lemma provide equivalent formulations as a pair of LMIs. All such tests have a particular form in which a constant, i.e. frequency independent, coefficient matrix, Θ, is used to parametrize the Frequency Domain Inequality (FDI). Previous results showed how one of these LMI tests can be modified to render a sufficient test for a given FDI in which $\Theta(\omega)$ is an affine function of ω. The main contribution of the present paper is to present a construction that proves such test is also necessary. Many interesting results are presented along the way related to the case when $\Theta(\omega)$ is quadratic.**

I. Introduction

The Kalman-Yakubovich-Popov (KYP) Lemma has played a major role in systems and control theory (see for instance the survey paper [1]). In [2], [3] an extension to the KYP Lemma was presented that established the equivalence between the frequency domain inequality (FDI)

$$\begin{bmatrix}(j\omega I - A)^{-1}B \\ I\end{bmatrix}^* \Theta \begin{bmatrix}(j\omega I - A)^{-1}B \\ I\end{bmatrix} \prec 0, \quad \text{for all } \omega_1 \leq \omega \leq \omega_2, \quad (1)$$

which is evaluated only on the finite frequency interval, with the pair of Linear Matrix Inequalities (LMIs)

$$\begin{bmatrix}A & B \\ I & 0\end{bmatrix}^* \begin{bmatrix}-Q & P + j\omega_c Q \\ P - j\omega_c Q & -\omega_1\omega_2 Q\end{bmatrix} \begin{bmatrix}A & B \\ I & 0\end{bmatrix} + \Theta \prec 0, \quad (2)$$
$$Q \succeq 0,$$

in Hermitian matrix variables P, Q of appropriate dimension where $\omega_c := (\omega_1 + \omega_2)/2$. In [4], [5] equivalence was also established with the pair of LMIs

$$\begin{bmatrix}A & B \\ I & 0\end{bmatrix}^* \begin{bmatrix}I \\ j\omega_i I\end{bmatrix} K^* + K \begin{bmatrix}I & -j\omega_i I\end{bmatrix} \begin{bmatrix}A & B \\ I & 0\end{bmatrix} + \Theta \prec 0, \quad (3)$$

in complex matrix variable K of appropriate dimension for $i = \{1, 2\}$. Many problems in systems and control theory can be posed in the form (1) where appropriate choices for the matrix Θ represent the analysis of various system properties. Being LMIs, the set of feasible solutions to inequalities (2) and (3) are convex and suitable variables P, Q and respectively K can be computed (or proved that none exists) in polynomial time using interior-point methods [6].

An interesting feature of the inequality (3) is that the class of coefficient matrices Θ can be extended to be affine on the frequency variable ω without incurring extra computational cost. In [4], [5], solving a pair of LMIs of the form (3) was shown to be sufficient in checking a FDI of the form (1) with an affine frequency dependent coefficient matrix $\Theta(\omega)$. The main contribution of this paper is to show necessity for the same affine frequency dependent coefficient matrix $\Theta(\omega)$ resulting in an alternative formulation of the Generalized KYP Lemma [3].

Necessity for the alternative formulation of Generalized KYP Lemma LMI conditions is shown through a constructive approach, which on its own offers interesting developments and insight. The affine frequency dependent coefficient matrix $\Theta(\omega)$ is incorporated into the Generalized KYP Lemma conditions by way of an augmented system realization. That produces a FDI in which the coefficient matrix Θ is constant. Indeed, arbitrary polynomial dependence of Θ on ω could be obtained at the expense of an exponential growth in the number of variables and size of the inequalities, see for instance [7], [8], [9]. This particular augmented system considered when incorporating the affine frequency dependent coefficient matrix $\Theta(\omega)$ is then projected back to the proposed alternative formulation of the Generalized KYP Lemma.

In [4], [5], searching simultaneously for a frequency dependent coefficient matrix while satisfying the FDI (1) has applications, for instance, in robustness analysis via the structured singular value (μ-analysis) [10]. Allowing Θ to be frequency dependent can give tighter upper bounds for μ (see for example [11], [12]).

A. Notation

The following notation will be used throughout the paper. The scalar $j = \sqrt{-1}$. For a matrix $X \in \mathbb{C}^{n\times n}$: $\overline{X}$, X^* are the complex-conjugate and complex-conjugate transpose of the matrix X respectively and X^{-1}, $X_\perp$ are full rank matrices

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such that $XX^{-1} = I$ and $XX_\perp = 0$. $\mathrm{He}\{X\}$ is short-hand notation for $X + X^*$. We denote by $\mathbb{HC}^n$ the space of $\mathbb{C}^{n\times n}$ Hermitian matrices.

II. Main Result

The main result is presented in the following theorem, which can be viewed as an alternative KYP Lemma that holds over finite frequency intervals whereby the coefficient matrix depends affinely on the frequency variable ω.

Theorem 1: Let matrices $A \in \mathbb{C}^{n\times n}$ with no eigenvalues on the imaginary axis, $B \in \mathbb{C}^{n\times m}$ and $\Theta_1, \Theta_2 \in \mathbb{HC}^{n+m}$ be given. Let scalars ω_1, $\omega_2 \in \mathbb{R}$ be also given, then the following statements are equivalent.

(i) The FDI

$$\begin{bmatrix}(j\omega I - A)^{-1}B\\ I\end{bmatrix}^* \Theta(\omega) \begin{bmatrix}(j\omega I - A)^{-1}B\\ I\end{bmatrix} \prec 0, \quad (4)$$

holds for all $\omega_1 \le \omega \le \omega_2$, where $Theta(\omega)$ has the form

$$\Theta(\omega) = \frac{\omega_2 - \omega}{\omega_2 - \omega_1}\,\Theta_1 + \frac{\omega - \omega_1}{\omega_2 - \omega_1}\,\Theta_2. \quad (5)$$

(ii) There exists a matrix $K \in \mathbb{C}^{n+m\times n}$ such that the pair of LMI hold

$$\mathrm{He}\left\{K\begin{bmatrix}I & -j\omega_i I\end{bmatrix}\begin{bmatrix}A & B\\ I & 0\end{bmatrix}\right\} + \Theta_i \prec 0, \quad (6)$$

for $i = \{1, 2\}$.

Sufficiency was previously established in [4], [5] and is based on the following idea. To show that (ii) implies (i), assume that the pair of inequalities (6) have feasible solutions and consider the affine function

$$\Theta(\omega) := \lambda(\omega)\Theta_1 + [1 - \lambda(\omega)]\Theta_2, \quad (7)$$

where $\Theta_1, \Theta_2 \in \mathbb{HC}^{n+m}$ and

$$\lambda(\omega) := \frac{\omega_2 - \omega}{\omega_2 - \omega_1}, \qquad \omega_1 \le \omega \le \omega_2, \quad (8)$$

so that $\lambda \in [0, 1]$. Note that (5) and (7) are equal with the definition of $\lambda(\omega)$ plugged in. The sum of (6) for $i = 1$ multiplied by $\lambda(\omega) := (\omega_2 - \omega)/(\omega_2 - \omega_1) \in [0, 1]$ and of (6) for $i = 2$ multiplied by $(1 - \lambda(\omega))$ implies that

$$\mathrm{He}\left\{K\begin{bmatrix}I & -j\omega I\end{bmatrix}\begin{bmatrix}A & B\\ I & 0\end{bmatrix}\right\} + \Theta(\omega) \prec 0$$

is feasible for all $\omega_1 \le \omega \le \omega_2$. Set

$$\mathcal{N}(\omega) := \begin{bmatrix}(j\omega I - A)^{-1}B\\ I\end{bmatrix} \quad (9)$$

which is a well defined matrix for all $\omega \in \mathbb{R}$ due to the assumption that A has no eigenvalue on the imaginary axis. Multiply the above inequality by $\mathcal{N}(\omega)$ on the right and by its transpose conjugate on the left to obtain

$$\mathcal{N}^*(\omega)\,\Theta(\omega)\,\mathcal{N}(\omega) = \begin{bmatrix}(j\omega I - A)^{-1}B\\ I\end{bmatrix}^* \Theta(\omega) \begin{bmatrix}(j\omega I - A)^{-1}B\\ I\end{bmatrix} \prec 0,$$

which is the FDI in (i).

Remark 1: In item (ii) one has to solve two inequalities (6) of dimension $n + m$ in $2n(n + m)$ real optimization variables in the matrix $K \in \mathbb{C}^{n+m\times n}$.

In forming the class of affine frequency dependent coefficient matrices (5) via the parameter $\lambda(\omega) \in [0, 1]$ in (7), one can recognize concepts from the analysis of polytopic systems [13] treating the frequency as a real uncertain parameter. Furthermore, the implication from (ii) to (i) demonstrates a clear connection with the Elimination Lemma (see for instance [14]). Indeed this is a connection that will appear again in proving necessity of (6). The proof of necessity will be developed in the remainder of the paper.

III. Generalized KYP Lemma

The Generalized KYP Lemma gives necessary and sufficient conditions for replacing FDI constraints, specified over finite frequency interval $\omega_1 \le \omega \le \omega_2$, with pairs of LMIs for a constant coefficient matrix Θ.

Lemma 2: (Generalized KYP Lemma) Let the matrices $H \in \mathbb{C}^{2p\times q}$ with $q < 2p$ and $\Theta \in \mathbb{HC}^q$ be given. The following statements are equivalent.

(i) The FDI

$$\left(\begin{bmatrix}I & -j\omega I\end{bmatrix}H\right)_\perp^* \Theta \left(\begin{bmatrix}I & -j\omega I\end{bmatrix}H\right)_\perp \prec 0, \quad (10)$$

holds for all $\omega_1 \le \omega \le \omega_2$.

(ii) There exists matrices $P, Q \in \mathbb{HC}^p$ with $Q \succ 0$ such that the LMI holds

$$H^*\begin{bmatrix}-Q & P + j\omega_c Q\\ P - j\omega_c Q & -\omega_1\omega_2 Q\end{bmatrix}H + \Theta \prec 0, \quad (11)$$

where

$$\omega_c := (\omega_1 + \omega_2)/2. \quad (12)$$

(iii) There exists a matrix $K \in \mathbb{C}^{q\times p}$ such that the pair of LMI holds

$$\mathrm{He}\left\{K\begin{bmatrix}I & -j\omega_i I\end{bmatrix}H\right\} + \Theta \prec 0, \quad i = \{1, 2\}. \quad (13)$$

Equivalence between item (i) and (ii) was established in [3] for the case $Q \succeq 0$. Note that one can impose a strict positive definite condition $Q \succ 0$ in item (ii) without loss. For instance consider the matrix

$$\begin{bmatrix}1 & -j\omega_c\\ j\omega_c & \omega_1\omega_2\end{bmatrix},$$

which is indefinite since it can be reduced via congruent transformation to the diagonal matrix $\mathrm{diag}\left(-\omega_c^2, (\omega_1\omega_2)^2\right)$. Now suppose that (11) holds, then there exists a sufficiently small $\varepsilon > 0$ such that

$$H^*\left(\begin{bmatrix}-Q & P + j\omega_c Q\\ P - j\omega_c Q & -\omega_1\omega_2 Q\end{bmatrix} + \varepsilon\begin{bmatrix}-I & j\omega_c I\\ -j\omega_c I & -\omega_1\omega_2 I\end{bmatrix}\right)H + \Theta \prec 0,$$

holds and where $\tilde{Q} = Q + \varepsilon I$ is strictly positive definite. Equivalence between items (i),(ii) and (iii) was established in [4], [5].

Note that as a particular case of Lemma 2, the FDI (1) can be replaced with either of the pairs of LMIs (2) or (3) by specifying

$$H = \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \tag{14}$$

and recognizing that

$$\left(\begin{bmatrix} I & -j\omega I \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}\right)_{\perp} = \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}$$

holds for all ω since A has no eigenvalue on the imaginary axis.

IV. Augmented System and Equivalent FDIs

This section introduces an augmented system that is useful in replacing FDIs where the coefficient matrix depends on frequency with FDIs where the coefficient matrix is constant. Consider the frequency dependent coefficient matrix

$$\Theta(\omega) = \begin{bmatrix} j\omega I \\ I \end{bmatrix}^* \Sigma \begin{bmatrix} j\omega I \\ I \end{bmatrix}, \tag{15}$$

where Σ is partitioned appropriately

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^* & \Sigma_{22} \end{bmatrix}, \tag{16}$$

with $\Sigma_{ii} \in \mathbb{HC}^{n+m}$ for $i = \{1, 2\}$ and $\Sigma_{12} \in \mathbb{C}^{n+m}$. Note that the frequency dependent coefficient matrix can also be written in the form

$$\Theta(\omega) = \Sigma_{22} + j\omega\left(\Sigma_{12}^* - \Sigma_{12}\right) + \omega^2\Sigma_{11}, \tag{17}$$

which clearly displays a quadratic dependence on the frequency variable when $\Sigma_{11} \neq 0$.

In case $\Sigma_{11} = 0$, it is possible to relate the matrix Σ with Θ_1 and Θ_2 of the affine coefficient matrix (5). Note that in Theorem 1 the form for the affine coefficient matrix (5) is developed constructively from the proof of sufficiency, that is from (7) with (8). The next result gives equivalent forms for the affine frequency dependent coefficient matrix, one of which relates the general quadratic form (15).

Lemma 3: Let $\Theta_i \in \mathbb{HC}^{n+m}$ and the scalar $\omega_i \in \mathbb{R}$ for $i = \{1, 2\}$ be given. Define the affine frequency dependent coefficient matrix $\Theta(\omega)$ as in (5) over the finite frequency range $\omega_1 \leq \omega \leq \omega_2$. Then $\Theta(\omega)$ can also be written as

$$\Theta(\omega) = \begin{bmatrix} j\omega I \\ I \end{bmatrix}^* \Pi \begin{bmatrix} j\omega I \\ I \end{bmatrix}, \tag{18}$$

where

$$\Pi = \begin{bmatrix} 0 & j\Pi_2/2 \\ -j\Pi_2/2 & \Pi_1 \end{bmatrix}, \tag{19}$$

and Π_1 and Π_2 are given by

$$\Pi_1 = \frac{\omega_2\Theta_1 - \omega_1\Theta_2}{\omega_2 - \omega_1}, \qquad \Pi_2 = \frac{\Theta_2 - \Theta_1}{\omega_2 - \omega_1},$$

respectively.

Proof: Equivalence between (5) and (18) can be established by plugging in the definition for Π_1 and Π_2. ■

The general quadratic form (15) can be used in replacing a FDI of the form (4) where the coefficient matrix $\Theta(\omega)$ depends quadratically on frequency with an equivalent augmented FDI where the coefficient matrix is not frequency dependent. The following lemma states this equivalence.

Lemma 4: Let matrices $A \in \mathbb{C}^{n \times n}$ with no eigenvalues on the imaginary axis, $B \in \mathbb{C}^{n \times m}$ and $\Sigma \in \mathbb{HC}^{2(n+m)}$ be given. The following statements are equivalent.

(i) The FDI

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \Theta(\omega) \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \prec 0, \tag{20}$$

holds for all frequencies $\omega_1 \leq \omega \leq \omega_2$ where $\Theta(\omega)$ has the form (15).

(ii) The FDI

$$\left(\begin{bmatrix} I & -j\omega I \end{bmatrix} H\right)_{\perp}^* H^*\Sigma H \left(\begin{bmatrix} I & -j\omega I \end{bmatrix} H\right)_{\perp} \prec 0, \tag{21}$$

holds for all $\omega_1 \leq \omega \leq \omega_2$ where

$$H = \begin{bmatrix} 0 & A & B \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}. \tag{22}$$

Proof: Note that

$$\begin{aligned} \left(\begin{bmatrix} I & -j\omega I \end{bmatrix} H\right)_{\perp} &= \begin{bmatrix} 0 & A - j\omega I & B \\ I & 0 & -j\omega I \end{bmatrix}_{\perp} \\ &= \begin{bmatrix} j\omega I \\ (j\omega I - A)^{-1}B \\ I \end{bmatrix}, \end{aligned}$$

which exists for all ω due to the assumption that A has no eigenvalue on the imaginary axis. Now note that

$$\begin{aligned} H\left(\begin{bmatrix} I & -j\omega I \end{bmatrix} H\right)_{\perp} &= \begin{bmatrix} A(j\omega I - A)^{-1}B + B \\ j\omega I \\ (j\omega I - A)^{-1}B \\ I \end{bmatrix} \\ &= \begin{bmatrix} j\omega I & 0 \\ 0 & j\omega I \\ I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \end{aligned}$$

where the final inequality comes from the fact that

$$\begin{aligned} A(j\omega I - A)^{-1}B + B &= [A + (j\omega I - A)](j\omega I - A)^{-1}B \\ &= j\omega I(j\omega I - A)^{-1}B. \end{aligned}$$

Hence the FDI (21) is equivalent to (4). ■

Feasibility of the FDI (20) with coefficient matrices that depend quadratically on the frequency variable, such as (15), can be verified by checking feasibility of the augmented FDI (21) with H given by (22). Using an augmented systems realization is not a new technique for incorporating coefficient matrices that depend rationally or polynomially on a scalar variable (see [8], [9] for instance). Traditionally this has been done by noticing that although $\Theta(\omega)$ given in (17) is not a proper rational function of ω one can always introduce fixed poles and augment the system realization for considering proper rational coefficient matrices [7], [15]. Here a different technique is used in which the augmented system can be interpreted as a descriptor system. That is, the FDI (21) can be viewed as the FDI

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Sigma \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \prec 0,$$

where the system $G(s)$ has a descriptor system realization given by

$$G(s) = \begin{bmatrix} 0 & A \\ I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & sI - A \\ -I & 0 \end{bmatrix}^{-1} \begin{bmatrix} B \\ -sI \end{bmatrix} + \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix}.$$

Furthermore, feasibility of the augmented FDI (21) for all frequencies $\omega_1 \le \omega \le \omega_2$ can be verified using the results of Lemma 2 with the constant coefficient matrix

$$\Theta = H^* \Sigma H. \tag{23}$$

This result is presented next as a corollary to Lemma 2.

Corollary 5: Let matrices $A \in \mathbb{C}^{n\times n}$ with no eigenvalues on the imaginary axis, $B \in \mathbb{C}^{n\times m}$, $\Sigma_{11}, \Sigma_{22} \in \mathbb{HC}^{n+m}$ and $\Sigma_{12} \in \mathbb{C}^{n+m}$ be given. Let scalars $\omega_1, \omega_2 \in \mathbb{R}$ be also given, then the following statements are true.

(i) The FDI (20) holds for all $\omega_1 \le \omega \le \omega_2$, where $\Theta(\omega)$ has the form (15).

(ii) There exists matrices $P, Q \in \mathbb{HC}^{n+m}$ with $Q \succ 0$ such that the LMI holds

$$H^* \begin{bmatrix} -Q & P + j\omega_c Q \\ P - j\omega_c Q & -\omega_1\omega_2 Q \end{bmatrix} H + H^* \Sigma H \prec 0, \tag{24}$$

where H is given by (22).

(iii) There exists a matrix $K \in \mathbb{C}^{(n+2m)\times(n+m)}$ such that the pair of LMI holds

$$\mathrm{He}\left\{K \begin{bmatrix} I & -j\omega_i I \end{bmatrix} H\right\} + H^* \Sigma H \prec 0, \tag{25}$$

for $i = \{1, 2\}$ where H is given by (22).

Proof: Using Lemma 4, the FDI (4) with $\Theta(\omega)$ given by (17) is equivalent to (21) with H given by (22). Application of Lemma 2 gives the desired result. ■

Remark 2: In Corollary 5 item (ii), one has to solve inequality (24) of dimension $n + 2m$ and inequality $Q \succ 0$ of dimension $n+m$ in $2(n+m)^2$ real optimization variables, namely the matrices $P, Q \in \mathbb{HC}^{n+m}$. In item (iii), one has to solve two inequalities (25) of dimension $n + 2m$ in $n^2 + 3nm + 2m^2$ real optimization variables, namely the matrix $K \in \mathbb{C}^{(n+2m)\times(n+m)}$.

Note that Lemma 3 allows to change between equivalent forms of the affine frequency dependent coefficient matrix, that is (5) can be written in the general quadratic form (15). Similar to Corollary 5, one can write equivalent LMI conditions for the FDI (4) where $\Theta(\omega)$ has affine frequency dependence (5). That is, one could use the augmented FDI (21) with H given by (22) and the resulting LMI conditions (24) and (25) with $\Sigma = \Pi$ where Π is given by (19). These conditions however have larger dimension and more optimization variables than what is posed by the conditions of Theorem 1.

The FDI (21) reveals an interesting structure, namely that the matrix H appears in both the constant coefficient matrix (23) as well as in the nullspace realization $\left(\begin{bmatrix} I & -j\omega I \end{bmatrix} H\right)_\perp$. The following section explores the structure present in the FDI (21) in the particular case of an affine frequency dependent coefficient matrix of the form (15) with (19).

V. Exploring Structure in the Affine Case

As mentioned in Section II, sufficiency of Theorem 1 demonstrates a connection with the Elimination Lemma. Indeed the Elimination Lemma plays an additional role in exploiting the structure of the FDI (21) for the particular case when $\Theta(\omega)$ depends affinely of the frequency variable.

Lemma 6 (Elimination Lemma): Let matrices $\mathcal{Q} \in \mathbb{HC}^n$, $\mathcal{B} \in \mathbb{C}^{k\times n}$ such that $\mathrm{rank}(\mathcal{B}) < n$, and $\mathcal{C} \in \mathbb{C}^{m\times n}$ such that $\mathrm{rank}(\mathcal{C}) < n$ be given. Then the following statements are equivalent.

(i) The two conditions hold

$$\begin{gathered} \mathcal{B}_\perp^* \mathcal{Q} \mathcal{B}_\perp \prec 0, \\ \text{and} \\ \mathcal{C}_\perp^* \mathcal{Q} \mathcal{C}_\perp \prec 0. \end{gathered} \tag{26}$$

(ii) There exist a matrix $\mathcal{X} \in \mathbb{C}^{m\times k}$ such that

$$\mathcal{C}^* \mathcal{X} \mathcal{B} + \mathcal{B}^* \mathcal{X}^* \mathcal{C} + \mathcal{Q} \prec 0. \tag{27}$$

There are several proofs available in the literature for this lemma, see for instance [16], [6].

Recall that FDIs of the form (5) with affine frequency dependent coefficient matrices $\Theta(\omega)$ can be equivalently transformed into FDIs of the form (21) where the coefficient matrix is constant and given by (23). Motivated by this fact, the following lemma presents an extension of the Generalized KYP Lemma (Lemma 2 items (i) and (ii)) where the FDI has the particular structure given in (21) with the additional constraint that $\Sigma_{11} = 0$. That is, Σ takes the form of Π given in (19).

Lemma 7: Let matrices $H \in \mathbb{C}^{2p\times q}$ with $q < 2p$ and $\Pi \in \mathbb{HC}^{2p}$ with the form (19) be given. Define the matrix

$$L := (H^*)_\perp \in \mathbb{C}^{2p\times(2p-q)}. \tag{28}$$

The following are equivalent statements.

(i) The FDI

$$\left(\begin{bmatrix} I & -j\omega I \end{bmatrix} H\right)_{\perp}^{*} H^{*}\Sigma H \left(\begin{bmatrix} I & -j\omega I \end{bmatrix} H\right)_{\perp} \prec 0, \quad (29)$$

holds for all $\omega_1 \le \omega \le \omega_2$.

(ii) There exists matrices $P, Q \in \mathbb{HC}^{p}$ where $Q \succ 0$, and $K \in \mathbb{C}^{p\times(2p-q)}$ such that the LMI

$$\begin{bmatrix} -Q & P + j\omega_c Q \\ P - j\omega_c Q & -\omega_1\omega_2 Q \end{bmatrix} + \mathrm{He}\left\{\begin{bmatrix} 0 \\ K \end{bmatrix} L^{*}\right\} + \Pi \prec 0, \quad (30)$$

where ω_c is defined in (12).

Proof: Recall from Lemma 2 the FDI (29) is equivalent to the existence of $P, Q \in \mathbb{HC}^{p}$ with $Q \succ 0$ such that

$$H^{*}\begin{bmatrix} -Q & P + j\omega_c Q \\ P - j\omega_c Q & -\omega_1\omega_2 Q \end{bmatrix} H + H^{*}\Sigma H \prec 0, \quad (31)$$

holds. Let us rewrite (31) as

$$H^{*}\left(\begin{bmatrix} -Q & P + j\omega_c Q \\ P - j\omega_c Q & -\omega_1\omega_2 Q \end{bmatrix} + \Pi\right) H \prec 0. \quad (32)$$

Also note the condition $Q \succ 0$ provides feasibility of the additional inequality

$$\begin{aligned} 0 &\succ -Q, \\ &= \begin{bmatrix} I & 0 \end{bmatrix}\left(\begin{bmatrix} -Q & P + j\omega_c Q \\ P - j\omega_c Q & -\omega_1\omega_2 Q \end{bmatrix} + \Pi\right)\begin{bmatrix} I \\ 0 \end{bmatrix}. \end{aligned} \quad (33)$$

With both inequalities (32) and (33) feasible, application of the Elimination Lemma (Lemma 6) where

$$\mathcal{B}_{\perp} = \begin{bmatrix} I \\ 0 \end{bmatrix}, \qquad \mathcal{C}_{\perp} = H,$$

gives an equivalence with the existence of a matrix variable $\mathcal{X} \in \mathbb{C}^{(2p-q)\times p}$ such that

$$\mathrm{He}\left\{L\mathcal{X}\begin{bmatrix} 0 & I \end{bmatrix}\right\} + \begin{bmatrix} -Q & P + j\omega_c Q \\ P - j\omega_c Q & -\omega_1\omega_2 Q \end{bmatrix} + \Pi \prec 0.$$

Finally, defining $K := \mathcal{X}^{*}$ gives the desired result. ■

In the above lemma, an extra multiplier variable K has been introduced as in the extended LMI conditions derived for robustness analysis [13], [17], [18]. The inequality (30) remains an LMI in the optimization variables P, Q and K, however, on a space of larger dimension and with more optimization variables. Also note that Lemma 7 holds for any matrix H such that (28) exists.

Remark 3: Consider the constant coefficient matrix (23) with Σ having the form (19) and H given by (22). In Lemma 7 item (ii), one has to solve inequality (30) of dimension $2(n+m)$ and inequality $Q \succ 0$ of dimension $n+m$ in $2(n+m)^2 + 2m(n+m)$ real optimization variables, namely the matrices $P, Q \in \mathbb{C}^{n+m}$ and $K \in \mathbb{C}^{(n+m)\times m}$.

Now that the essential preliminary results have been introduced, the proof of necessity for Theorem 1 is completed in the following section.

VI. Proof of the Main Result: Necessity

To show necessity of Theorem 1, that is the implication from item (i) to item (ii), suppose that (4) holds with $\Theta(\omega)$ given by (5). Recall from Lemma 3 that $\Theta(\omega)$ can be written in the general quadratic form (18) where Π given by (19). It is important to note that the (1,1)-block of Π is zero. From Lemma 4 the FDI (4) is equivalent to the FDI (21) where H is given by (22).

Note that for H given by (22)

$$L^{*} = (H^{*})_{\perp}^{*} = \begin{bmatrix} -I & 0 & A & B \end{bmatrix}.$$

Then from Lemma 7 there exist matrices $P, Q \in \mathbb{HC}^{n+m}$ with $Q \succ 0$ and $K \in \mathbb{C}^{(n+m)\times n}$ such that

$$\begin{aligned} &\begin{bmatrix} -Q & P + j\omega_c Q \\ P - j\omega_c Q & -\omega_1\omega_2 Q \end{bmatrix} + \Pi \\ &\quad + \mathrm{He}\left\{\begin{bmatrix} 0 \\ K \end{bmatrix}\begin{bmatrix} -I & 0 & A & B \end{bmatrix}\right\} \prec 0, \end{aligned} \quad (34)$$

holds.

Define the matrix

$$X(\omega) := \begin{bmatrix} 1 & -j\omega \\ j\omega & \hat{\omega} + 2\omega\omega_c - \omega_c \end{bmatrix} \otimes Q,$$

where $\hat{\omega} := (\omega_2 - \omega_1)/2$ and ω_c is given in (12). Note that the matrix $X(\omega)$ is positive semidefinite for all $\omega_1 \le \omega \le \omega_2$, for details see [4], [5]. Add the matrix $X(\omega) \succeq 0$ to the right hand side of (34) so that

$$\begin{aligned} &\Pi + \mathrm{He}\left\{\begin{bmatrix} 0 \\ K \end{bmatrix}\begin{bmatrix} -I & 0 & A & B \end{bmatrix}\right\} \prec \\ &\quad \left(\begin{bmatrix} Q & -P - j\omega_c Q \\ -P + j\omega_c Q & \omega_1\omega_2 Q \end{bmatrix} + \right. \\ &\qquad \left. \begin{bmatrix} Q & -j\omega Q \\ j\omega Q & (\hat{\omega} + 2\omega\omega_c - \omega_c) Q \end{bmatrix}\right), \end{aligned}$$

where the right hand side of the inequality above can be factored as

$$\begin{aligned} &\Pi + \mathrm{He}\left\{\begin{bmatrix} 0 \\ K \end{bmatrix}\begin{bmatrix} -I & 0 & A & B \end{bmatrix}\right\} \prec \\ &\quad \left(\begin{bmatrix} -Q \\ -P + j\omega_c Q \end{bmatrix}\begin{bmatrix} I & -j\omega I \end{bmatrix} + \right. \\ &\qquad \left. \begin{bmatrix} I \\ j\omega I \end{bmatrix}\begin{bmatrix} Q & -P - j\omega_c Q \end{bmatrix}\right). \end{aligned}$$

Now choose

$$Y = \begin{bmatrix} -Q \\ P - j\omega_c Q \end{bmatrix}, \quad (35)$$

so that

$$\mathrm{He}\left\{Y\begin{bmatrix} I & -j\omega I \end{bmatrix} + \begin{bmatrix} 0 \\ K \end{bmatrix}\begin{bmatrix} -I & 0 & A & B \end{bmatrix}\right\} + \Pi \prec 0,$$

holds for all $\omega_2 \le \omega \le \omega_2$.

Multiply the above inequality on the left hand side by $\begin{bmatrix} -j\omega I & I \end{bmatrix}$ and on the right hand side by the conjugate transpose,

$$0 \succ \begin{bmatrix} j\omega I \\ I \end{bmatrix}^* \Big(\mathrm{He}\Big\{ Y \begin{bmatrix} I & -j\omega I \end{bmatrix} + \begin{bmatrix} 0 \\ K \end{bmatrix} \begin{bmatrix} -I & 0 & A & B \end{bmatrix} \Big\} + \Pi \Big) \begin{bmatrix} j\omega I \\ I \end{bmatrix},$$

which reduces to

$$0 \succ \mathrm{He}\left\{ K \begin{bmatrix} -j\omega I + A & B \end{bmatrix} \right\} + \begin{bmatrix} j\omega I \\ I \end{bmatrix}^* \Pi \begin{bmatrix} j\omega I \\ I \end{bmatrix}.$$

Finally using Lemma 3 the above inequality can be rewritten as

$$\mathrm{He}\left\{ K \begin{bmatrix} I & -j\omega I \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \right\} + \Theta(\omega) \prec 0,$$

which should hold for any $\omega_1 \leq \omega \leq \omega_2$, in particular, for $\omega = \omega_1$ and $\omega = \omega_2$ to imply that the pair of inequalities (6) are feasible.

VII. Conclusions

The KYP Lemma converts FDIs with a constant coefficient matrix into computationally tractable LMI conditions on additional matrix variables. Under the original KYP Lemma formulation, including frequency dependent coefficient matrices required augmentation of the original system realization. The main result of Theorem 1 is to include an affine frequency dependent coefficient matrix in the FDI without the need for augmenting the original system.

Sufficiency of Theorem 1 was shown in [4], [5], where constructing the convex combination of LMI conditions dictates the affine frequency dependent form of the coefficient matrix. Necessity requires a constructive approach, which on its own offers interesting insight into the proposed alternative formulation of the Generalized KYP Lemma conditions. The affine frequency dependent matrix $\Theta(\omega)$ can be incorporated to the FDIs through particular augmented system realizations. However, the particular form of augmented systems allows for a projection of the augmented LMI conditions back to the original system realization.

The results in this paper can be extended to more general classes of systems and curves on the complex plane as in the Generalized KYP Lemma of [3]. This will be presented in the paper [19].

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