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Recursive Computation of Stochastic Nash Games with State-Dependent Noise for Weakly-Coupled Large-Scale Systems

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Abstract— **This paper discusses the infinite horizon stochastic Nash games with state-dependent noise. After establishing the asymptotic structure along with the positive semidefiniteness for the solutions of the cross-coupled stochastic algebraic Riccati equation (CSARE), recursive algorithm for solving the CSARE is derived. As a result, it is shown that the proposed algorithm attains linear convergence and the reduced-order computations for sufficiently small parameter ε. As another important feature, the high-order approximate strategy that is based on the iterative solutions is proposed. Using such strategy, the degradation of the cost functional is established. Moreover, it is shown that the exponentially mean-square stable is guaranteed. Finally, in order to demonstrate the efficiency of the proposed algorithm, numerical example is given.**

I. INTRODUCTION

The stochastic control problems governed by Itô's differential equation have become a popular research topic in a past decade. It is demonstrated that such system appears in the flexible structure comprising a mass-spring system [1]. Recently, stochastic H_∞ control problem with state- and control dependent noise was considered [1], [2]. It has attracted much attention and has been widely applied to various fields. Particularly, the stochastic H_2/H_∞ control with state-dependent noise has been addressed [3]. Although this approach is based on two-players Nash game, uniqueness of the solution has not been investigated. It is well-known that it is very hard to find the condition of such uniqueness.

Recently, linear quadratic Nash games and their applications have been widely investigated in many literatures. Particularly, the linear quadratic Nash games and related topics for weakly coupled large-scale systems have been discussed in [6], [7], [8], [9]. These results are based on the deterministic systems. Very recently, the stochastic Nash games for weakly-coupled large scale systems have been tackled [10]. Specifically, the uniqueness of the solution has been proved in the field of the stochastic systems for the first time. However, the considered algorithms for solving the cross-coupled stochastic algebraic Riccati equation (CSARE) consist of Newton's method and two fixed-point iteration. Therefore, many tedious algebra and the CPU time are needed. Moreover, the eligible proof of the exponentially mean-square stable under the proposed strategy has not been considered.

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In this paper, the stochastic Nash games governed by Itô differential equations with state-dependent noise are discussed. Specifically, for solving CSARE, this paper focuses on the development of a new numerical algorithm in which fast convergence and less procedure are both attained. First, the uniqueness and boundedness of the solution to the CSARE and their asymptotic structure are investigated. Second, the recursive algorithm for solving the CSARE is given. Since the proposed numerical computation is based on recursive algorithm [6], linear convergence and the reduced-order computation are both guaranteed for sufficiently small parameter ε. It should be noted that since the stochastic Nash games have not been addressed in [6], the considered cross-coupled algebraic Riccati equation is quite different. Thus, a new recursive algorithm should be formulated. Moreover, the reduction of the CPU time and the direct algebraic manipulation with reduced-order dimension can be attained by computing the new recursive algorithm as compared with [10]. As another important features, the high-order approximate strategy that is based on the iterative solutions is proposed. As a result, the degradation of the cost functional using the proposed strategy set is shown. Furthermore, it is rigorously shown for the first time that the closed loop stochastic systems with the proposed high-order strategy are the exponentially mean-square stable. Finally, in order to demonstrate the efficiency of the proposed algorithm, a numerical example is included.

Notation: The notations used in this paper are fairly standard. Superscript T denotes the matrix transpose. I_n denotes an $n \times n$ identity matrix. **block diag** denotes a block diagonal matrix. $\| \cdot \|$ denotes the Euclidean norm of a matrix. E denotes the expectation. $\otimes$ denotes the Kronecker product. $\mathrm{vec} M$ denotes the column vector of matrix M. $\lambda(M)$ denotes the eigenvalue of a matrix M.

II. STOCHASTIC NASH GAMES

Consider linear time-invariant weakly-coupled large-scale systems.

$$dx(t) = [A_\varepsilon x(t) + B_{1\varepsilon} u_1(t) + B_{2\varepsilon} u_2(t)]dt + A_{1\varepsilon} x(t) dw(t),\ x(0) = x^0, \quad (1)$$

where

$$x(t) := \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

$$A_\varepsilon := \begin{bmatrix} A_{11} & \varepsilon A_{12} \\ \varepsilon A_{21} & A_{22} \end{bmatrix},\ A_{1\varepsilon} := \begin{bmatrix} A_{111} & \varepsilon A_{112} \\ \varepsilon A_{121} & A_{122} \end{bmatrix},$$

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$$B_{1\varepsilon} := \begin{bmatrix} B_{11} \\ \varepsilon B_{21} \end{bmatrix},\ B_{2\varepsilon} := \begin{bmatrix} \varepsilon B_{12} \\ B_{22} \end{bmatrix}.$$

$x_i(t) \in \mathbf{R}^{n_i},\ i = 1,\ 2$ represent the i-th state vectors. $u_i(t) \in \mathbf{R}^{m_i},\ i = 1,\ 2$ represent the i-th control inputs. $w(t) \in \mathbf{R}$ is a one-dimensional standard Wiener process defined in the filtered probability space [1], [2], [3], [4]. Here, ε denotes a relatively small positive coupling parameter that relates the linear system with the other subsystems.

The cost function for each strategy subset is defined by

$$J_i(u_1,\ u_2,\ x(0)) = E\int_0^{\infty}\Big[x^T(t)Q_{i\varepsilon}x(t) + u_i^T(t)R_{ii}u_i(t)\Big]dt, \quad (2)$$

where $i = 1,\ 2$, $Q_{i\varepsilon} = Q_{i\varepsilon}^T \geq 0$,

$$Q_{1\varepsilon} = \begin{bmatrix} Q_{111} & \varepsilon Q_{112} \\ \varepsilon Q_{112}^T & \varepsilon Q_{122} \end{bmatrix},\ Q_{2\varepsilon} = \begin{bmatrix} \varepsilon Q_{211} & \varepsilon Q_{212} \\ \varepsilon Q_{212}^T & Q_{222} \end{bmatrix},$$

$R_{ii} = R_{ii}^T > 0 \in \mathbf{R}^{m_i \times m_i}$.

It should be noted that in this study, the strategies $u_i(t) := F_{i\varepsilon}x(t)$ are restricted as the linear feedback strategies [5].

As an essential assumption, stabilizability is introduced [3], [4].

Definition 1: The stochastically controlled system governed by the Itô's equation $dx = (Fx + Gu)dt + G_1 x dw_1$, $x(0) = x_0$ is called stabilizable in the mean-square sense if there exists a feedback law $u = Kx$ such that for any x_0, the closed-loop system $dx = (F + GK)xdt + G_1 x dw_1$, $x(0) = x_0$ is asymptotically mean-square stable, i.e., $\lim_{t\to\infty} Ex^T(t)x(t) = 0$, where K is a constant matrix.

For the matrices A_ε, $B_{j\varepsilon}$, $j = 1,\ \ldots,\ N$ and $A_{1\varepsilon}$, the set $\mathbf{F}_N$ is defined as $\mathbf{F}_N := \Big\{ (F_{1\varepsilon},\ \ldots,\ F_{N\varepsilon}) \mid$ The closed-loop system $dx(t) = [A_\varepsilon + \sum_{p=1}^{N} B_{p\varepsilon}F_{p\varepsilon}]x(t)dt + A_{1\varepsilon}x(t)dw(t)$ is asymptotically mean-square stable. $\Big\}$.

The stochastic Nash equilibrium strategy pair $(u_1^*,\ u_2^*)$, $u_i^*(t) := F_{i\varepsilon}^* x(t)$ is defined such that it satisfies the following conditions [10].

$$J_1(F_{1\varepsilon}^* x,\ F_{2\varepsilon}^* x,\ x(0)) \leq J_1(F_{1\varepsilon}x,\ F_{2\varepsilon}^* x,\ x(0)), \quad (3a)$$
$$J_2(F_{1\varepsilon}^* x,\ F_{2\varepsilon}^* x,\ x(0)) \leq J_2(F_{1\varepsilon}^* x,\ F_{2\varepsilon}x,\ x(0)), \quad (3b)$$

for all $x(0)$ and for all $(F_{1\varepsilon},\ F_{2\varepsilon})$ that satisfy $(F_{1\varepsilon}^*,\ F_{2\varepsilon}) \in \mathcal{F}_2$, $(F_{1\varepsilon},\ F_{2\varepsilon}^*) \in \mathcal{F}_2$, and $(F_{1\varepsilon}^*,\ F_{2\varepsilon}^*) \in \mathcal{F}_2$.

The stochastic Nash games are given below [10].

Lemma 1: Suppose real symmetric matrices $P_{i\varepsilon}$ exist.

$$\begin{aligned} &\mathcal{G}_i(\varepsilon,\ P_{1\varepsilon},\ P_{2\varepsilon}) \\ &= P_{i\varepsilon}(A_\varepsilon - S_{j\varepsilon}P_{j\varepsilon}) + (A_\varepsilon - S_{j\varepsilon}P_{j\varepsilon})^T P_{i\varepsilon} + A_{1\varepsilon}^T P_{i\varepsilon} A_{1\varepsilon} \\ &\quad - P_{i\varepsilon}S_{i\varepsilon}P_{i\varepsilon} + Q_{i\varepsilon} = 0, \end{aligned} \quad (4)$$

where $i,\ j = 1,\ 2, i \neq j$, $S_{i\varepsilon} := B_{i\varepsilon}R_{ii}^{-1}B_{i\varepsilon}^T$.

The strategy set $(F_{1\varepsilon}^*,\ F_{2\varepsilon}^*)$ is defined by

$$u_i^*(t) := F_{i\varepsilon}^* x(t) = -R_{ii}^{-1}B_{i\varepsilon}^T P_{i\varepsilon}x(t),\ i = 1,\ 2. \quad (5)$$

Then, $(F_{1\varepsilon}^*,\ F_{2\varepsilon}^*) \in \mathcal{F}_2$ and this strategy set denote stochastic Nash equilibrium. Furthermore, $J_i(F_{1\varepsilon}^*,\ F_{2\varepsilon}^*,\ x(0)) = x^T(0)P_{i\varepsilon}x(0)$.

III. ASYMPTOTIC STRUCTURE OF CSARE

Firstly, in order to obtain the strategy set based on numerical solutions, the asymptotic structure of CSARE (4) is established. Since A_ε, $A_{1\varepsilon}$, $S_{i\varepsilon}$ and $Q_{i\varepsilon}$ include the term of the parameter ε, the solution $P_{i\varepsilon}$ of CSARE (4)-if it exists-should contain the parameter ε. By considering this fact, the solution $P_{i\varepsilon}$ of CSARE (4) is assumed to have the following structure.

$$P_{1\varepsilon} = \begin{bmatrix} P_{111} & \varepsilon P_{112} \\ \varepsilon P_{112}^T & \varepsilon P_{122} \end{bmatrix},\ P_{2\varepsilon} = \begin{bmatrix} \varepsilon P_{211} & \varepsilon P_{212} \\ \varepsilon P_{212}^T & P_{222} \end{bmatrix}. \quad (6)$$

Substituting the matrices A_ε, $A_{1\varepsilon}$, $S_{i\varepsilon}$, $Q_{i\varepsilon}$ and $P_{i\varepsilon}$ into CSARE (4), letting $\varepsilon = 0$, and partitioning CSARE (4), the following reduced-order stochastic algebraic Riccati equation (SARE) is obtained, where $\bar{P}_{iii},\ i = 1,\ 2$ is the 0-order solutions of CSARE (4) as $\varepsilon = 0$.

$$\begin{aligned} &\bar{P}_{iii}A_{ii} + A_{ii}^T\bar{P}_{iii} + A_{1ii}^T\bar{P}_{iii}A_{1ii} \\ &\quad -\bar{P}_{iii}S_{iii}P_{iii} + Q_{iii} = 0, \end{aligned} \quad (7a)$$

$$\begin{aligned} &\bar{P}_{ijj}(A_{jj} - S_{jjj}\bar{P}_{jjj}) + (A_{jj} - S_{jjj}\bar{P}_{jjj})^T\bar{P}_{ijj} \\ &\quad +A_{1jj}^T\bar{P}_{ijj}A_{1jj} + Q_{ijj} = 0, \end{aligned} \quad (7b)$$

$$\begin{aligned} &\bar{P}_{112}(A_{11} - S_{111}\bar{P}_{111}) + (A_{22} - S_{222}\bar{P}_{222})^T\bar{P}_{112} \\ &\quad +A_{122}^T\bar{P}_{112}A_{122} + \bar{P}_{111}A_{12} \\ &\quad +A_{111}^T\bar{P}_{111}A_{112} - \bar{P}_{111}S_{212}\bar{P}_{222} + Q_{112} = 0, \end{aligned} \quad (7c)$$

$$\begin{aligned} &\bar{P}_{212}(A_{11} - S_{111}\bar{P}_{111}) + (A_{22} - S_{222}\bar{P}_{222})^T\bar{P}_{212} \\ &\quad +A_{122}^T\bar{P}_{212}A_{122} + A_{21}^T\bar{P}_{222} \\ &\quad +A_{121}^T\bar{P}_{222}A_{122} - \bar{P}_{111}S_{112}\bar{P}_{222} + Q_{212} = 0, \end{aligned} \quad (7d)$$

where $S_{iii} := B_{ii}R_{ii}^{-1}B_{ii}^T$, $S_{ijj} := B_{jj}R_{ii}^{-1}B_{jj}^T$, $S_{112} := B_{11}R_{11}^{-1}B_{21}^T$, $S_{212} := B_{12}R_{11}^{-1}B_{22}^T$, $i,\ j = 1,\ 2,\ i \neq j$.

The Nash equilibrium strategies for the stochastic large-scale systems will be studied under the following basic assumption.

Assumption 1: $(A_{ii},\ B_{ii})$ is stabilizable, $(\sqrt{Q_{iii}},\ A_{ii})$ is observable.

Assumption 2:

$$\inf_{K_{ii}}\left\| \int_0^{\infty} \exp[(A_{ii} - B_{ii}K_{ii})^T t]A_{1ii}^T A_{1ii} \exp[(A_{ii} - B_{ii}K_{ii})t]dt \right\| < 1. \quad (8)$$

Assumption 3: The matrices T_{11}, T_{12} and T_{22} are nonsingular, where $D_{ii} := A_{ii} - S_{iii}\bar{P}_{iii},\ i = 1,\ 2,$

$$T_{11} := I_{n_1} \otimes D_{11}^T + D_{11}^T \otimes I_{n_1} + A_{111}^T \otimes A_{111}^T,$$
$$T_{12} := I_{n_2} \otimes D_{11}^T + D_{22}^T \otimes I_{n_1} + A_{122}^T \otimes A_{111}^T,$$
$$T_{22} := I_{n_2} \otimes D_{22}^T + D_{22}^T \otimes I_{n_2} + A_{122}^T \otimes A_{122}^T.$$

It should be noted that Assumptions 1 and 2 are standard [6]. On the other hand, Assumption 3 seems to be conservative. However, in order to guarantee the uniqueness of the stochastic algebraic Lyapunov equation (7), this assumption should be needed.

If the above assumption holds, there exists the unique positive definite stabilizing solution $\bar{P}_{iii} > 0$ of SARE (7a) such that D_{ii} is stable.

The asymptotic expansion of CSARE (4) for $\varepsilon = 0$ is described by the following theorem.

Theorem 1: Under Assumptions 1-3, there exists a small constant σ^* such that for all $\varepsilon \in (0,\ \sigma^*)$, CSARE (4) admits a positive semidefinite solution $P_{i\varepsilon}^*$ that can be expressed as

$$P_{i\varepsilon} := P_{i\varepsilon}^* = \bar{P}_i + O(\varepsilon), \tag{9}$$

where

$$\begin{aligned} \bar{P}_1 &= \textbf{block diag}\begin{pmatrix} \bar{P}_{111} & 0 \end{pmatrix}, \\ \bar{P}_2 &= \textbf{block diag}\begin{pmatrix} 0 & \bar{P}_{222} \end{pmatrix}. \end{aligned}$$

Proof: This can be proved by performing the implicit function theorem on CSARE (4). It should be noted that the present proof is discussed in detail as compared with the existing one [10]. To do so, it is sufficient to show that the corresponding Jacobian is nonsingular at $\varepsilon = 0$. The derivative of the function $\mathcal{G}_i(\varepsilon,\ P_{1\varepsilon},\ P_{2\varepsilon})$ at matrix $P_{i\varepsilon}$ is given by

$$\begin{aligned} J_P &= \begin{bmatrix} \frac{\partial[\mathrm{vec}G_1]^T}{\partial\mathrm{vec}P_{111}} & \frac{\partial[\mathrm{vec}G_1]^T}{\partial\mathrm{vec}P_{112}} & \cdots & \frac{\partial[\mathrm{vec}G_1]^T}{\partial\mathrm{vec}P_{222}} \\ \frac{\partial[\mathrm{vec}G_2]^T}{\partial\mathrm{vec}P_{111}} & \frac{\partial[\mathrm{vec}G_2]^T}{\partial\mathrm{vec}P_{112}} & \cdots & \frac{\partial[\mathrm{vec}G_2]^T}{\partial\mathrm{vec}P_{222}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial[\mathrm{vec}G_6]^T}{\partial\mathrm{vec}P_{111}} & \frac{\partial[\mathrm{vec}G_6]^T}{\partial\mathrm{vec}P_{112}} & \cdots & \frac{\partial[\mathrm{vec}G_6]^T}{\partial\mathrm{vec}P_{222}} \end{bmatrix} \\ &= \begin{bmatrix} J_{11} & 0 & 0 & 0 & 0 & 0 \\ \star & J_{22} & 0 & 0 & 0 & \star \\ 0 & 0 & J_{33} & 0 & 0 & \star \\ \star & 0 & 0 & J_{44} & 0 & 0 \\ \star & 0 & 0 & 0 & J_{55} & \star \\ 0 & 0 & 0 & 0 & 0 & J_{66} \end{bmatrix}. \end{aligned} \tag{10}$$

where

$$\begin{aligned} & J_{11} = J_{44} = T_{11},\ J_{22} = J_{55} = T_{12},\ J_{33} = J_{66} = T_{22}, \\ & D_{ii} := A_{ii} - S_{iii}\bar{P}_{iii},\ i = 1,\ 2, \\ & \mathcal{G}_1(\varepsilon,\ P_{1\varepsilon},\ P_{2\varepsilon}) := \begin{bmatrix} G_1 & \varepsilon G_2 \\ \varepsilon G_2^T & G_3 \end{bmatrix} = 0, \\ & \mathcal{G}_2(\varepsilon,\ P_{1\varepsilon},\ P_{2\varepsilon}) := \begin{bmatrix} G_4 & \varepsilon G_5 \\ \varepsilon G_5^T & G_6 \end{bmatrix} = 0. \end{aligned}$$

The Jacobian (10) can be expressed as

$$\det J_P = \prod_{i=1}^{6} \det J_{ii}. \tag{11}$$

Obviously, J_{ii}, $i = 1,\ 2$ are nonsingular under Assumption 3. By the nonsingularity assumption of the matrix $\det J_P \neq 0$, i.e., J_P is nonsingular at $\varepsilon = 0$. The conclusion of Theorem 1 is obtained directly by using the implicit function theorem. ∎

IV. RECURSIVE ALGORITHM

In order to reduce the dimension of the calculation, a new algorithm for solving CSARE (4) which is based on the recursive algorithm is established.

It is assumed that the exact solutions can further decompose into the parameter independent solution $\bar{P}_{ikl}$, $kl = 11,\ 12,\ 22$ and the error matrices E_{ikl}. This is a standard assumption in the theory of weakly-coupled systems [6]. Defining approximation errors as

$$\begin{aligned} & P_{i11} = \bar{P}_{i11} + \varepsilon E_{i11},\ P_{i12} = \bar{P}_{i12} + \varepsilon E_{i12}, \\ & P_{i22} = \bar{P}_{i22} + \varepsilon E_{i22},\ i = 1,\ 2. \end{aligned} \tag{12}$$

Substituting E_{ipq}, $pq = 11,\ 12,\ 22$ of (12) into CSARE (4) and using (7), the following expressions for the error equations (13) are obtained.

$$\begin{aligned} H_i := {} & E_{iii}(A_{ii} - S_{iii}\bar{P}_{iii}) + (A_{ii} - S_{iii}\bar{P}_{iii})^T E_{iii} \\ & + A_{1ii}^T E_{iii} A_{1ii} + \varepsilon h_i = 0,\ i = 1,\ 2, \end{aligned} \tag{13a}$$

$$\begin{aligned} H_3 := {} & E_{122}(A_{22} - S_{222}\bar{P}_{222}) + (A_{22} - S_{222}\bar{P}_{222})^T E_{122} \\ & + A_{122}^T E_{122} A_{122} \\ & - E_{222} S_{222}\bar{P}_{222} - \bar{P}_{222} S_{222} E_{222} + \varepsilon h_3 = 0, \end{aligned} \tag{13b}$$

$$\begin{aligned} H_4 := {} & E_{211}(A_{11} - S_{111}\bar{P}_{111}) + (A_{11} - S_{111}\bar{P}_{111})^T E_{211} \\ & + A_{111}^T E_{211} A_{111} \\ & - E_{111} S_{111}\bar{P}_{211} - \bar{P}_{211} S_{111} E_{111} + \varepsilon h_4 = 0, \end{aligned} \tag{13c}$$

$$\begin{aligned} H_5 := {} & E_{212}(A_{22} - S_{222}\bar{P}_{222}) + (A_{11} - S_{111}\bar{P}_{111})^T E_{212} \\ & + A_{111}^T E_{212} A_{122} + (A_{21} - S_{112}^T\bar{P}_{111} - S_{222}\bar{P}_{212}^T)^T E_{222} \\ & - E_{111}(S_{112}\bar{P}_{222} + S_{111}\bar{P}_{212}) \\ & + A_{121}^T E_{222} A_{122} + \varepsilon h_5 = 0, \end{aligned} \tag{13d}$$

$$\begin{aligned} H_6 := {} & E_{112}(A_{22} - S_{222}\bar{P}_{222}) + (A_{11} - S_{111}\bar{P}_{111})^T E_{112} \\ & + A_{111}^T E_{112} A_{122} + E_{111}(A_{12} - S_{111}\bar{P}_{112} - S_{212}\bar{P}_{222}) \\ & - (S_{212}^T\bar{P}_{111} + S_{222}\bar{P}_{112}^T)^T E_{222} \\ & + A_{111}^T E_{111} A_{112} + \varepsilon h_6 = 0, \end{aligned} \tag{13e}$$

where $h_i := h_i(\varepsilon,\ E_{111},\ \ldots,\ E_{222})$, $H_i := H_i(\varepsilon,\ E_{111},\ \ldots,\ E_{222})$, $i = 1,\ \ldots,\ 6$.

Hence, the following recursive algorithm for solving CSARE (4) is given below.

$$\begin{aligned} & E_{111}^{(n+1)}(A_{11} - S_{111}\bar{P}_{111}) + (A_{11} - S_{111}\bar{P}_{111})^T E_{111}^{(n+1)} \\ & + A_{111}^T E_{111}^{(n+1)} A_{111} \\ = {} & -\varepsilon(P_{112}^{(n)} A_{21} + A_{21}^T P_{112}^{(n)T}) \\ & + \varepsilon(P_{111}^{(n)} S_{212} P_{212}^{(n)T} + P_{212}^{(n)} S_{212}^T P_{111}^{(n)} + P_{112}^{(n)} S_{222} P_{212}^{(n)T} \\ & + P_{212}^{(n)} S_{222} P_{112}^{(n)T}) + \varepsilon^2(P_{111}^{(n)} S_{211} P_{211}^{(n)} + P_{211}^{(n)} S_{211} P_{111}^{(n)} \\ & + P_{112}^{(n)} S_{212}^T P_{211}^{(n)} + P_{211}^{(n)} S_{212} P_{112}^{(n)T}) - \varepsilon(A_{121}^T P_{112}^{(n)T} A_{111} \\ & + A_{111}^T P_{112}^{(n)} A_{121}) - \varepsilon^2 A_{121}^T P_{122}^{(n)} A_{121} + \varepsilon E_{111}^{(n)} S_{111} E_{111}^{(n)} \\ & + \varepsilon(P_{112}^{(n)} S_{112}^T P_{111}^{(n)} + P_{111}^{(n)} S_{112} P_{112}^{(n)T}) \\ & + \varepsilon^3 P_{112}^{(n)} S_{122} P_{112}^{(n)T}, \end{aligned} \tag{14a}$$

$$\begin{aligned} & E_{222}^{(n+1)}(A_{22} - S_{222}\bar{P}_{222}) + (A_{22} - S_{222}\bar{P}_{222})^T E_{222}^{(n+1)} \\ & + A_{122}^T E_{222}^{(n+1)} A_{122} \end{aligned}$$

$$
\begin{aligned}
&= -\varepsilon(P_{212}^{(n)T} A_{12} + A_{12}^T P_{212}^{(n)}) \\
&\quad +\varepsilon(P_{112}^{(n)T} S_{111} P_{212}^{(n)} + P_{212}^{(n)T} S_{111} P_{112}^{(n)} + P_{112}^{(n)T} S_{112} P_{222}^{(n)} \\
&\quad +P_{222}^{(n)} S_{112}^T P_{112}^{(n)}) + \varepsilon^2(P_{122}^{(n)} S_{112}^T P_{212}^{(n)} + P_{212}^{(n)T} S_{112} P_{122}^{(n)} \\
&\quad +P_{122}^{(n)} S_{122} P_{222}^{(n)} + P_{222}^{(n)} S_{122} P_{122}^{(n)}) - \varepsilon(A_{122}^T P_{212}^{(n)T} A_{112} \\
&\quad +A_{112}^T P_{212}^{(n)} A_{122}) - \varepsilon^2 A_{112}^T P_{211}^{(n)} A_{112} + \varepsilon E_{222}^{(n)} S_{222} E_{222}^{(n)} \\
&\quad +\varepsilon(P_{222}^{(n)} S_{212}^T P_{212}^{(n)} + P_{212}^{(n)T} S_{212} P_{222}^{(n)}) \\
&\quad +\varepsilon^3 P_{212}^{(n)T} S_{211} P_{212}^{(n)}, \qquad (14b)
\end{aligned}
$$

$$
\begin{aligned}
&E_{122}^{(n+1)}(A_{22} - S_{222}\bar{P}_{222}) + (A_{22} - S_{222}\bar{P}_{222})^T E_{122}^{(n+1)} \\
&\quad +A_{122}^T E_{122}^{(n+1)} A_{122} \\
&= E_{222}^{(n+1)} S_{222} \bar{P}_{222} + \bar{P}_{222} S_{222} E_{222}^{(n+1)} \\
&\quad +\varepsilon(E_{122}^{(n)} S_{222} E_{222}^{(n)} + E_{222}^{(n)} S_{222} E_{122}^{(n)}) \\
&\quad -(P_{112}^{(n)T} A_{12} + A_{12}^T P_{112}^{(n)}) \\
&\quad +P_{112}^{(n)T} S_{212} P_{222}^{(n)} + P_{222}^{(n)} S_{212}^T P_{112}^{(n)} \\
&\quad +\varepsilon^2(P_{112}^{(n)T} S_{211} P_{212}^{(n)} + P_{212}^{(n)T} S_{211} P_{112}^{(n)}) \\
&\quad +\varepsilon(P_{122}^{(n)} S_{212}^T P_{212}^{(n)} + P_{212}^{(n)T} S_{212} P_{122}^{(n)}) \\
&\quad -(A_{112}^T P_{111}^{(n)} A_{112} + A_{122}^T P_{112}^{(n)T} A_{112} + A_{112}^T P_{112}^{(n)} A_{122}) \\
&\quad +P_{112}^{(n)T} S_{111} P_{112}^{(n)} + \varepsilon^2 P_{122}^{(n)} S_{122} P_{122}^{(n)} \\
&\quad +\varepsilon(P_{122}^{(n)} S_{112}^T P_{112}^{(n)} + P_{112}^{(n)T} S_{112} P_{122}^{(n)}), \qquad (14c)
\end{aligned}
$$

$$
\begin{aligned}
&E_{211}^{(n+1)}(A_{11} - S_{111}\bar{P}_{111}) + (A_{11} - S_{111}\bar{P}_{111})^T E_{211}^{(n+1)} \\
&\quad +A_{111}^T E_{211}^{(n+1)} A_{111} \\
&= E_{111}^{(n+1)} S_{111} \bar{P}_{211} + \bar{P}_{211} S_{111} E_{111}^{(n+1)} \\
&\quad +\varepsilon(E_{211}^{(n)} S_{111} E_{111}^{(n)} + E_{111}^{(n)} S_{111} E_{211}^{(n)}) \\
&\quad -(P_{212}^{(n)} A_{21} + A_{21}^T P_{212}^{(n)T}) \\
&\quad +P_{212}^{(n)} S_{112}^T P_{111}^{(n)} + P_{111}^{(n)} S_{112} P_{212}^{(n)T} \\
&\quad +\varepsilon^2(P_{212}^{(n)} S_{122} P_{112}^{(n)T} + P_{112}^{(n)} S_{122} P_{212}^{(n)T}) \\
&\quad +\varepsilon(P_{211}^{(n)} S_{112} P_{112}^{(n)T} + P_{112}^{(n)} S_{112}^T P_{211}^{(n)}) \\
&\quad -(A_{121}^T P_{222}^{(n)} A_{121} + A_{111}^T P_{212}^{(n)} A_{121} + A_{121}^T P_{212}^{(n)T} A_{111}) \\
&\quad +P_{212}^{(n)} S_{222} P_{212}^{(n)T} + \varepsilon^2 P_{211}^{(n)} S_{211} P_{211}^{(n)} \\
&\quad +\varepsilon(P_{212}^{(n)} S_{212}^T P_{211}^{(n)} + P_{211}^{(n)} S_{212} P_{212}^{(n)T}), \qquad (14d)
\end{aligned}
$$

$$
\begin{aligned}
&E_{212}^{(n+1)}(A_{22} - S_{222}\bar{P}_{222}) + (A_{11} - S_{111}\bar{P}_{111})^T E_{212}^{(n+1)} \\
&\quad +A_{111}^T E_{212}^{(n+1)} A_{122} \\
&= -(A_{21} - S_{112}^T \bar{P}_{111} - S_{222}\bar{P}_{212}^T)^T E_{222}^{(n+1)} \\
&\quad +E_{111}^{(n+1)}(S_{112}\bar{P}_{222} + S_{111}\bar{P}_{212}) - A_{121}^T E_{222}^{(n+1)} A_{122} \\
&\quad +\varepsilon(E_{111}^{(n)} S_{112} E_{222}^{(n)} + E_{111}^{(n)} S_{111} E_{212}^{(n)} + E_{212}^{(n)} S_{222} E_{222}^{(n)}) \\
&\quad -P_{211}^{(n)} A_{12} + \varepsilon(P_{211}^{(n)} S_{112} P_{122}^{(n)} + P_{212}^{(n)} S_{112}^T P_{112}^{(n)} \\
&\quad +P_{112}^{(n)} S_{112}^T P_{212}^{(n)} + P_{112}^{(n)} S_{122} P_{222}^{(n)}) + \varepsilon^2 P_{212}^{(n)} S_{122} P_{122}^{(n)} \\
&\quad +P_{211}^{(n)} S_{111} P_{112}^{(n)} - \varepsilon A_{121}^T P_{212}^{(n)T} A_{112} - A_{111}^T P_{211}^{(n)} A_{112} \\
&\quad +\varepsilon P_{212}^{(n)} S_{212}^T P_{212}^{(n)} + P_{211}^{(n)} S_{212} P_{222}^{(n)} \\
&\quad +\varepsilon^2 P_{211}^{(n)} S_{211} P_{212}^{(n)}, \qquad (14e)
\end{aligned}
$$

$$
\begin{aligned}
&E_{112}^{(n+1)}(A_{22} - S_{222}\bar{P}_{222}) + (A_{11} - S_{111}\bar{P}_{111})^T E_{112}^{(n+1)} \\
&\quad +A_{111}^T E_{112}^{(n+1)} A_{122} \\
&= -E_{111}^{(n+1)}(A_{12} - S_{111}\bar{P}_{112} - S_{212}\bar{P}_{222}) \\
&\quad +(S_{212}^T \bar{P}_{111} + S_{222}\bar{P}_{112}^T)^T E_{222}^{(n+1)} - A_{111}^T E_{111}^{(n+1)} A_{112} \\
&\quad +\varepsilon(E_{111}^{(n)} S_{212} E_{222}^{(n)} + E_{112}^{(n)} S_{222} E_{222}^{(n)} + E_{111}^{(n)} S_{111} E_{112}^{(n)}) \\
&\quad -A_{21}^T P_{122}^{(n)} + \varepsilon(P_{211}^{(n)} S_{212} P_{122}^{(n)} + P_{111}^{(n)} S_{211} P_{212}^{(n)} \\
&\quad +P_{112}^{(n)} S_{212}^T P_{212}^{(n)} + P_{212}^{(n)} S_{212}^T P_{112}^{(n)}) + \varepsilon^2 P_{211}^{(n)} S_{211} P_{112}^{(n)} \\
&\quad +P_{212}^{(n)} S_{222} P_{122}^{(n)} - \varepsilon A_{121}^T P_{112}^{(n)T} A_{112} - A_{121}^T P_{122}^{(n)} A_{122} \\
&\quad +\varepsilon P_{112}^{(n)} S_{112}^T P_{112}^{(n)} + P_{111}^{(n)} S_{112} P_{122}^{(n)} \\
&\quad +\varepsilon^2 P_{112}^{(n)} S_{122} P_{122}^{(n)}, \ n = 0,\ 1,\ 2,\ \dots, \qquad (14f)
\end{aligned}
$$

where

$$
\begin{aligned}
&P_{i11}^{(n)} = \bar{P}_{i11} + \varepsilon E_{i11}^{(n)},\ P_{i12}^{(n)} = \bar{P}_{i12} + \varepsilon E_{i12}^{(n)}, \\
&P_{i22}^{(n)} = \bar{P}_{i22} + \varepsilon E_{i22}^{(n)},\ E_{i11}^{(0)} = E_{i12}^{(0)} = E_{i22}^{(0)} = 0,\ i = 1,\ 2.
\end{aligned}
$$

The main result of this section is given below.

Theorem 2: Under Assumptions 1-3, the iterative algorithm (14) converges to the exact solutions E_{ipq}, $i = 1,\ 2$, $pq = 11,\ 12,\ 22$ of the equation (13) with the linear rate of convergence. That is, the following relations hold.

$$
\begin{aligned}
&\|E_{ipq} - E_{ipq}^{(n)}\| = O(\varepsilon^n), \\
&n = 1,\ 2,\ \dots,\ i = 1,\ 2,\ pq = 11,\ 12,\ 22. \qquad (15)
\end{aligned}
$$

The following lemma will play an important role in establishing (15).

Lemma 2: If $dz(t) = Az(t)dt + \sum_{p=1}^{N} A_p z(t) dw_p(t)$ is exponentially mean-square stable and $Q = Q^T$, $z^T(0)Pz(0) = \int_0^\infty z^T(t)Qz(t)dt$, where P satisfies the stochastic algebraic Lyapunov equation (SALE) $A^T P + PA + \sum_{p=1}^{N} A_p^T P A_p + Q = 0$.

Proof: As a starting point it needs to show the existence of the unique and bounded solutions E_{ipq} of (13) instead of P_{ipq} of (4) in neighborhood of $\varepsilon = 0$. To prove that by the implicit function theorem, it is enough to show that the corresponding Jacobian J_E of (13) is nonsingular at $\varepsilon = 0$. The Jacobian is given by

$$
J_E = \begin{bmatrix}
\frac{\partial[\mathrm{vec}H_1]^T}{\partial \mathrm{vec}E_{111}} & \frac{\partial[\mathrm{vec}H_1]^T}{\partial \mathrm{vec}E_{112}} & \cdots & \frac{\partial[\mathrm{vec}H_1]^T}{\partial \mathrm{vec}E_{222}} \\
\frac{\partial[\mathrm{vec}H_6]^T}{\partial \mathrm{vec}E_{111}} & \frac{\partial[\mathrm{vec}H_6]^T}{\partial \mathrm{vec}E_{112}} & \cdots & \frac{\partial[\mathrm{vec}H_6]^T}{\partial \mathrm{vec}E_{222}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial[\mathrm{vec}H_2]^T}{\partial \mathrm{vec}E_{111}} & \frac{\partial[\mathrm{vec}H_2]^T}{\partial \mathrm{vec}E_{112}} & \cdots & \frac{\partial[\mathrm{vec}H_2]^T}{\partial \mathrm{vec}E_{222}}
\end{bmatrix} = J_P. \quad (16)
$$

Taking into consideration the fact that J_P is nonsingular at $\varepsilon = 0$, J_E is also nonsingular. Therefore, there exists a unique and bounded solution of the error equations (13). Secondly, the proof of (15) uses mathematical induction. When $n = 0$ for the equations (14), the first order approximations $E_{ipq}^{(1)}$ corresponding to the small parameter ε satisfy the equations (13). It follows from these equations that

$$
\|E_{ipq} - E_{ipq}^{(1)}\| = O(\varepsilon),\ i = 1,\ 2,\ pq = 11,\ 12,\ 22. \quad (17)
$$

When $n = k,\ k \geq 1$, it is assumed that $\|E_{ipq} - E_{ipq}^{(k)}\| = O(\varepsilon^k)$. Subtracting (13) from (14) and using the assumption

$\|E_{ipq} - E_{ipq}^{(k)}\| = O(\varepsilon^k)$, the following equations hold.

$$
\begin{aligned}
&(E_{iii}^{(k+1)} - E_{iii})(A_{ii} - S_{iii}\bar{P}_{iii}) \\
&\quad +(A_{ii} - S_{iii}\bar{P}_{iii})^T(E_{iii}^{(k+1)} - E_{iii}) \\
&\quad +A_{1ii}^T(E_{iii}^{(k+1)} - E_{iii})A_{1ii} = O(\varepsilon^{k+1}),\ i = 1,\ 2, \quad (18a) \\
&\quad (E_{122}^{(k+1)} - E_{122})(A_{22} - S_{222}\bar{P}_{222}) \\
&\quad +(A_{22} - S_{222}\bar{P}_{222})^T(E_{122}^{(k+1)} - E_{122}) \\
&\quad +A_{122}^T(E_{122}^{(k+1)} - E_{122})A_{122} \\
&= (E_{222}^{(k+1)} - E_{222})S_{222}\bar{P}_{222} + \bar{P}_{222}S_{222}(E_{222}^{(k+1)} - E_{222}) \\
&\quad +O(\varepsilon^{k+1}), \quad (18b) \\
&\quad (E_{211}^{(k+1)} - E_{211})(A_{11} - S_{111}\bar{P}_{111}) \\
&\quad +(A_{11} - S_{111}\bar{P}_{111})^T(E_{211}^{(k+1)} - E_{211}) \\
&\quad +A_{111}^T(E_{211}^{(k+1)} - E_{211})A_{111} \\
&= (E_{111}^{(k+1)} - E_{111})S_{111}\bar{P}_{211} + \bar{P}_{211}S_{111}(E_{111}^{(k+1)} - E_{111}) \\
&\quad +O(\varepsilon^{k+1}), \quad (18c) \\
&\quad (E_{212}^{(k+1)} - E_{212})(A_{22} - S_{222}\bar{P}_{222}) \\
&\quad +(A_{11} - S_{111}\bar{P}_{111})^T(E_{212}^{(k+1)} - E_{212}) \\
&\quad +A_{111}^T(E_{212}^{(k+1)} - E_{212})A_{122} \\
&= -(A_{21} - S_{112}^T\bar{P}_{111} - S_{222}\bar{P}_{212}^T)^T(E_{222}^{(k+1)} - E_{222}) \\
&\quad +(E_{111}^{(k+1)} - E_{111})(S_{112}\bar{P}_{222} + S_{111}\bar{P}_{212}) \\
&\quad -A_{121}^T(E_{222}^{(k+1)} - E_{222})A_{122} + O(\varepsilon^{k+1}), \quad (18d) \\
&\quad (E_{112}^{(k+1)} - E_{112})(A_{22} - S_{222}\bar{P}_{222}) \\
&\quad +(A_{11} - S_{111}\bar{P}_{111})^T(E_{112}^{(k+1)} - E_{112}) \\
&\quad +A_{111}^T(E_{112}^{(k+1)} - E_{112})A_{122} \\
&= -(E_{111}^{(k+1)} - E_{111})(A_{12} - S_{111}\bar{P}_{112} - S_{212}\bar{P}_{222}) \\
&\quad +(S_{212}^T\bar{P}_{111} + S_{222}\bar{P}_{112}^T)^T(E_{222}^{(k+1)} - E_{222}) \\
&\quad -A_{111}^T(E_{111}^{(k+1)} - E_{111})A_{112} + O(\varepsilon^{k+1}). \quad (18e)
\end{aligned}
$$

After the cancellation takes place, since D_{ii} $i = 1,\ 2$ are stable, the following results hold.

$$\|E_{ipq} - E_{ipq}^{(k+1)}\| = O(\varepsilon^{k+1}),\ i = 1,\ 2,\ pq = 11,\ 12,\ 22. \quad (19)$$

Consequently, the equation (15) holds for all $n \in \mathbf{N}$. This completes the proof of Theorem 2 concerned with the fixed point algorithm. ∎

The required iterative count associated with the Newton's method with other two fixed point algorithms and the recursive algorithm is compared. It is assumed that the required operation count is $O(k)$ by using the recursive algorithm for rough estimate. In this case, it should be noted that the required operation count of these fixed point algorithms are also $O(k)$, respectively. Then, the required operation count of the Newton's method with other two fixed point algorithms is $O(k^2 \log_2 k)$. Therefore, the proposed recursive algorithm drastically succeeds in reducing the operating count. As a result, the CPU time can be reduced.

V. HIGH-ORDER APPROXIMATE STOCHASTIC NASH STRATEGY

The design of high-order approximate stochastic Nash strategies is considered. Such strategy is obtained by using iterative solution (14).

$$
\begin{aligned}
&u_i^{(n)}(t) = F_{i\varepsilon}^{(n)}x(t) = -R_{ii}^{-1}B_{i\varepsilon}^T P_{i\varepsilon}^{(n)}x(t), \quad (20) \\
&n = 1,\ 2,\ \ldots,\ i = 1,\ 2,
\end{aligned}
$$

where

$$P_{1\varepsilon}^{(n)} = \begin{bmatrix} P_{111}^{(n)} & \varepsilon P_{112}^{(n)} \\ \varepsilon P_{112}^{(n)T} & \varepsilon P_{122}^{(n)} \end{bmatrix},\ P_{2\varepsilon}^{(n)} = \begin{bmatrix} \varepsilon P_{211}^{(n)} & \varepsilon P_{212}^{(n)} \\ \varepsilon P_{212}^{(n)T} & P_{222}^{(n)} \end{bmatrix}.$$

The degradation of the cost functional via new high-order approximate stochastic Nash strategies (20) is given as follows.

Theorem 3: Under Assumptions 1-3, the use of the high-order approximate stochastic Nash strategies (20) results in (21)

$$
\begin{aligned}
&J_i(u_1^{(n)},\ u_2^{(n)},\ x(0)) = J_i(u_1^*,\ u_2^*,\ x(0)) + O(\varepsilon^{n+2}) \\
&n = 0,\ 1,\ \ldots,\ i = 1,\ 2. \quad (21)
\end{aligned}
$$

Proof: When $u_i^{(n)}(t)$ is used, the equilibrium values of the cost functional is given by (22).

$$J_i(u_1^{(n)},\ u_2^{(n)},\ x(0)) = x^T(0)Z_{i\varepsilon}x(0), \quad (22)$$

where $Z_{i\varepsilon}$ is a positive semidefinite solution of the following SALE

$$
\begin{aligned}
&Z_{i\varepsilon}\left(A_\varepsilon - S_{1\varepsilon}P_{1\varepsilon}^{(n)} - S_{2\varepsilon}P_{2\varepsilon}^{(n)}\right) \\
&\quad + \left(A_\varepsilon - S_{1\varepsilon}P_{1\varepsilon}^{(n)} - S_{2\varepsilon}P_{2\varepsilon}^{(n)}\right)^T Z_{i\varepsilon} + A_{1\varepsilon}^T Z_{i\varepsilon} A_{1\varepsilon} \\
&\quad +P_{i\varepsilon}^{(n)}S_{i\varepsilon}P_{i\varepsilon}^{(n)} + Q_{i\varepsilon} = 0,\ i,\ j = 1,\ 2,\ i \neq j. \quad (23)
\end{aligned}
$$

Subtracting (23) from (4), $V_\varepsilon = Z_{i\varepsilon} - P_{i\varepsilon}$ satisfies the following SALE

$$
\begin{aligned}
&V_{i\varepsilon}\left(A_\varepsilon - S_{1\varepsilon}P_{1\varepsilon}^{(n)} - S_{2\varepsilon}P_{2\varepsilon}^{(n)}\right) \\
&\quad + \left(A_\varepsilon - S_{1\varepsilon}P_{1\varepsilon}^{(n)} - S_{2\varepsilon}P_{2\varepsilon}^{(n)}\right)^T V_{i\varepsilon} + A_{1\varepsilon}^T V_{i\varepsilon} A_{1\varepsilon} \\
&\quad +P_{i\varepsilon}S_{j\varepsilon}\left(P_{j\varepsilon} - P_{j\varepsilon}^{(n)}\right) + \left(P_{j\varepsilon} - P_{j\varepsilon}^{(n)}\right)S_{j\varepsilon}P_{i\varepsilon} \\
&\quad + \left(P_{i\varepsilon}^{(n)} - P_{i\varepsilon}\right)S_{i\varepsilon}\left(P_{i\varepsilon}^{(n)} - P_{i\varepsilon}\right) = 0. \quad (24)
\end{aligned}
$$

By using the result of (15) and $P_{i\varepsilon}S_{j\varepsilon} = O(\varepsilon)$, it is easy to verify that

$$
\begin{aligned}
&V_{i\varepsilon}\left(A_\varepsilon - S_{1\varepsilon}P_{1\varepsilon}^{(n)} - S_{2\varepsilon}P_{2\varepsilon}^{(n)}\right) \\
&\quad + \left(A_\varepsilon - S_{1\varepsilon}P_{1\varepsilon}^{(n)} - S_{2\varepsilon}P_{2\varepsilon}^{(n)}\right)^T V_{i\varepsilon} \\
&\quad +A_{1\varepsilon}^T V_{i\varepsilon} A_{1\varepsilon} + O(\varepsilon^{n+2}) = 0. \quad (25)
\end{aligned}
$$

Therefore, $V_{i\varepsilon} = O(\varepsilon^{n+2})$ because of Lemma 2. Hence, since $V_{i\varepsilon} = O(\varepsilon^{n+2})$,

$$
\begin{aligned}
&x(0)^T V_{i\varepsilon}x(0) = x(0)^T Z_{i\varepsilon}x(0) - x^T(0)P_{i\varepsilon}x(0) \\
&= J_i(u_1^{(n)},\ u_2^{(n)},\ x(0)) - J_i(u_1^*,\ u_2^*,\ x(0)) = O(\varepsilon^{n+2}) \quad (26)
\end{aligned}
$$

results in (21). ∎

In the rest of this section, an important implication is given. If the parameter ε is unknown, then the following corollary is easily seen in view of Theorem 3.

Corollary 1: Consider ε-independent approximate stochastic Nash strategies

$$\bar{u}_i^*(t) = -R_{ii}^{-1} B_i^T \bar{P}_i x(t),\ i = 1,\ 2, \tag{27}$$

where

$$B_1 := \begin{bmatrix} B_{11} \\ 0 \end{bmatrix},\ B_2 := \begin{bmatrix} 0 \\ B_{22} \end{bmatrix}.$$

Under Assumptions 1-3, the use of ε-independent approximate stochastic Nash strategies (27) results in (28)

$$J_i(\bar{u}_1^*,\ \bar{u}_2^*,\ x(0)) = J_i(u_1^*,\ u_2^*,\ x(0)) + O(\varepsilon^2),\quad i=1,\ 2. \tag{28}$$

Proof: Since the result of Corollary 1 can be proved by using a technique similar to that used in Theorem 3 under the fact that $P_{i\varepsilon}^* - \bar{P}_i = O(\varepsilon)$, the proof is omitted. ∎

The following result establishes the mean-square stable for the closed-loop systems with the proposed strategies (20).

Theorem 4: Under Assumptions 1-3, there exists a small constant σ and the positive scalar parameters $\alpha > 0$ and $\beta > 0$ such that for all $\varepsilon \in (0,\ \sigma)$, $\| \exp[(A_\varepsilon + \sum_{p=1}^2 B_{p\varepsilon} F_{p\varepsilon}^{(n)})t] \| \le \alpha e^{-\beta t}$. Moreover, if this condition $\alpha^2/\beta \|A_{1\varepsilon}\|^2 \le \omega < 1$ is met, the closed-loop stochastic system is exponentially mean-square stable.

Proof: First, it is easy to verify that

$$A_\varepsilon + \sum_{p=1}^2 B_{p\varepsilon} F_{p\varepsilon}^{(n)} = \textbf{block diag}\begin{pmatrix} D_{11} & D_{22} \end{pmatrix} + O(\varepsilon). \tag{29}$$

Hence, using the stability assumption of D_{ii}, it can be shown that there exists a small constant σ and the positive scalar parameters $\alpha > 0$ and $\beta > 0$ such that for all $\varepsilon \in (0,\ \sigma)$, $\| \exp[(A_\varepsilon + \sum_{p=1}^2 B_{p\varepsilon} F_{p\varepsilon}^{(n)})t] \| \le \alpha e^{-\beta t}$. Let us consider the closed-loop stochastic system (30).

$$dx(t) = \left[A_\varepsilon + \sum_{p=1}^2 B_{p\varepsilon} F_{p\varepsilon}^{(n)} \right] x(t)dt + A_{1\varepsilon} x(t) dw(t). \tag{30}$$

The representation of the solution of equation (30) is given as

$$\begin{aligned} x(t) &= \exp\left[\left[A_\varepsilon + \sum_{p=1}^2 B_{p\varepsilon} F_{p\varepsilon}^{(n)} \right](t-s) \right] x(0) \\ &\quad + \int_s^t \exp\left[\left[A_\varepsilon + \sum_{p=1}^2 B_{p\varepsilon} F_{p\varepsilon}^{(n)} \right](t-\tau) \right] \\ &\quad \times A_{1\varepsilon} x(\tau) dw(\tau). \end{aligned} \tag{31}$$

Using inequality (31) and considering the independence of the Wiener processes $w(t)$ yields

$$\begin{aligned} &E\|x(t)\|^2 \\ &\le 2 \left\| \exp\left[\left[A_\varepsilon + \sum_{p=1}^2 B_{p\varepsilon} F_{p\varepsilon}^{(n)} \right](t-s) \right] \right\|^2 E\|x(0)\|^2 \\ &\quad + 2 \int_s^t \left\| \exp\left[\left[A_\varepsilon + \sum_{p=1}^2 B_{p\varepsilon} F_{p\varepsilon}^{(n)} \right](t-\tau) \right] \right\|^2 \\ &\quad \times \|A_{1\varepsilon}\|^2 E\|x(\tau)\|^2 d\tau. \end{aligned} \tag{32}$$

Thus, the conditions $\| \exp[(A_\varepsilon + \sum_{p=1}^2 B_{p\varepsilon} F_{p\varepsilon}^{(n)})t] \| \le \alpha e^{-\beta t}$, $\exists\ \alpha,\ \beta > 0$ and $\alpha^2/\beta \|A_{1\varepsilon}\|^2 \le \omega$ imply that

$$\begin{aligned} &e^{2\beta(t-s)} E\|x(t)\|^2 \\ &\le 2\alpha^2 E\|x(0)\|^2 + 2\beta\omega \int_s^t e^{2\beta(\tau-s)} E\|x(\tau)\|^2 d\tau. \end{aligned} \tag{33}$$

From the Bellman-Gronwall inequality [11], it follows that

$$E\|x(t)\|^2 \le 2\alpha^2 E\|x(0)\|^2 e^{2\beta(\omega-1)(t-s)}. \tag{34}$$

Since ω has been selected such that $\omega < 1$, then equation (30) is exponentially mean-square stable. ∎

It should be noted that the use of ε-independent approximate stochastic Nash strategies (27) has the similar result of Theorem 4.

VI. NUMERICAL EXAMPLE

In order to demonstrate the efficiency of the proposed algorithm, a numerical example is given. The system matrices have been chosen randomly in the interval $[0,\ 1]$ with four-dimensional coefficient matrix.

The small parameter is chosen as $\varepsilon = 0.01$. It is verified that the solution of the CSARE (4) converges to the exact solution with accuracy of $\mathcal{G} < 10^{-9}$ after five iterations, where the function $\mathcal{G}(\varepsilon)$ is defined as follows: $\mathcal{G}(\varepsilon) = \sum_{p=1}^2 \|\mathcal{G}_p(\varepsilon,\ P_{1\varepsilon},\ P_{2\varepsilon}\|$. In order to verify the exactitude of the solution, the remainder per iteration by substituting $P_{i\varepsilon}$ into the (4) is computed for various parameter ε. It can be seen from Table I that the recursive algorithm (14) generates the required optimal solution for small value of ε. Moreover, using different values of ε in the same example, it is easy to observe that the proposed algorithm has linear convergence. Therefore, the resulting algorithm of this paper is very reliable in the sense that the proposed algorithm has the linear convergence. Furthermore, the resulting algorithm of this paper is very useful in the sense that the required computation work space is the same as the reduced-order systems. In other words, even if the stochastic weakly-coupled large-scale systems (1) are composed of two four-dimensional subsystems, the required workspace is four.

Table II presents the results of the CPU time with regard to the comparison between the Newton's method [10] and the proposed recursive algorithm. The CPU time represents the average based on the computations of ten runs. It can be observed from Table II that as compared to the Newton's method [10], the recursive algorithm (14) requires considerably less CPU time. This is because the existing result

TABLE I

ERROR PER ITERATIONS.

n	$\|\mathcal{G}(1.0e-02)\|$	$\|\mathcal{G}(1.0e-03)\|$	$\|\mathcal{G}(1.0e-04)\|$	$\|\mathcal{G}(1.0e-05)\|$
0	$1.6579e-003$	$1.6593e-005$	$1.6595e-007$	$1.6594e-009$
1	$8.1016e-005$	$8.1583e-008$	$8.1118e-011$	$3.8485e-013$
2	$1.8251e-006$	$1.7730e-010$		
3	$6.4043e-008$			
4	$1.4268e-009$			
5	$3.7959e-010$			

TABLE II

CPU TIME [SEC]

ε	Newton's Method [10]	Recursive Algorithm
$1.0e-02$	$6.4360e-001$	$4.9700e-002$
$1.0e-03$	$2.2660e-001$	$1.2400e-002$

TABLE III

DEGRADATION OF COST.

ε	$J_1(u_1^*(t),\ u_2^*(t))$	$J_2(u_1^*(t),\ u_2^*(t))$	$J_1(\bar{u}_1^*(t),\ \bar{u}_2^*(t))$	$J_2(\bar{u}_1^*(t),\ \bar{u}_2^*(t))$	ψ_1	ψ_2
10^{-2}	4.5318	2.0430	4.5326	2.0431	7.7041	$5.0126e-1$
10^{-3}	4.4023	1.8951	4.4024	1.8951	6.9374	$2.5624e-1$
10^{-4}	4.3898	1.8806	4.3898	1.8806	6.8633	$3.3259e-1$
10^{-5}	4.3886	1.8792	4.3886	1.8792	6.8607	$3.4009e-1$

[10] consists of the Newton's method and another two fixed point algorithms. Therefore, the reduction of the CPU can be attained by using the recursive algorithm.

Finally, the cost degradation of (28) is verified. The values of the optimal cost performance and ε-independent approximate stochastic Nash strategies (27) for various ε are given in Table III, where $\psi_i := |J_i(\bar{u}_1^*(t),\ \bar{u}_2^*(t)) - J_i(u_1^*(t),\ u_2^*(t))|/\varepsilon^2,\ i = 1,\ 2$. It is easy to verify that for each parameter ε, $|J_i(\bar{u}_1^*,\ \bar{u}_2^*) - J_i(u_1^*,\ u_2^*)| = O(\varepsilon^2)$ because of $\phi_i < \infty$. Therefore, the new result for the loss of performance which is indicated by (28) is correct.

Since it seems that the proposed algorithm and the existing one of [10] both run very quickly, it may be hard to see why the high-order algorithm is needed. In this case, it should be noted that the dimension of the simulation data is quite small. If the large-scale dimension is treated, it can be seen that the proposed algorithm still attains the fast convergence.

VII. CONCLUSIONS

Infinite-horizon stochastic Nash games have been discussed. First, recursive algorithm for solving the CSARE that arose in the stochastic Nash games for weakly coupled large-scale systems has been studied. By using this algorithm, it has been shown that both linear convergence and reduced-order computations can be attained. Thus, the proposed algorithm is expected to be very useful and reliable for a sufficiently small value of ε. As another important feature, it has been shown that a high-order approximate strategy attained better cost performance. In fact, the cost degradation for using the proposed approximate strategy has been proved. Additionally, the mean-square stable of the stochastic Nash strategies has been proved. Finally, numerical example has yielded excellent results using which linear convergence has been verified and the proposed algorithm has succeeded in reducing the CPU time.

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