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Simultaneous Design of Reliable $\mathcal{H}_\infty$ Filter and Fault Detector for Linear Continuous-Time Systems with Sensor Outage Faults

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Abstract— **The paper studies the problem of simultaneous design of reliable filter and fault detector for a class of linear continuous-time systems with bounded disturbances and nonzero constant reference inputs. An $\mathcal{H}_\infty$ filter and two detection weighting matrices are designed simultaneously. The filter is designed for both fault free and faulty cases, and by manipulating the steady-state values of the filter states and the measured outputs with the detection weighting matrices, a residual is then generated, through which the sensor outage faults can be detected effectively. A convergent iterative algorithm based on linear matrix inequality (LMI) is given to obtain the solutions. A numerical example is given to illustrate the effectiveness of the proposed methods.**

I. INTRODUCTION

During the last decades, the $\mathcal{H}_\infty$ filtering approach, has received considerable attention recently due to its wide applicability when robustness is imposed, where the main objective is to minimize the $\mathcal{H}_\infty$ norm from the process noise to the estimation error [1]-[3], and in [4]-[6], the parameter dependent Lyapunov method is adopted which reduces conservatism in some extent.

Note that all the above filtering approaches are based on the assumption that the sensors can provide uninterrupted signal measurement. In practice, however, contingent failures are possible for all sensors in a system, which may result in a large degree of filter performance degradation and, more importantly, possible hazard. In [7]-[9], reliable filters have been designed considering both the normal and sensor faulty cases, however, the sensor faults have not been detected in the process of filtering.

In this paper, in addition to designing a reliable $\mathcal{H}_\infty$ filter against sensor outage faults, a detection scheme is also designed simultaneously to detect sensor outage faults. Different from classical detection methods as stated in [10]-[14], the residual in this work is generated through manipulating the steady-state values of the filter states and the measured outputs with two weighting matrices, which is used to detect sensor outage faults. Based on an analysis of the simultaneous reliable filtering and fault detection problem, some performance indexes are derived. These indexes reflect the design constraints on the transfer functions from noises and constant reference inputs to the state estimation error and the residual signals for both normal and faulty cases. Further, the GKYP lemma proposed recently in [15], is also used in this paper to give solutions to some of the indexes. At last, an iterative LMI approach is given to solve the simultaneous reliable $\mathcal{H}_\infty$ filtering and fault detection problem.

Notation: For a matrix A, $A^T, A^*, A^\perp$ denote its transpose, complex conjugate transpose and orthogonal complement, respectively. The Hermitian part of a square matrix M is denoted by $\mathrm{He}(M) := M + M^*$. The symbol $\mathbf{H}_n$ stands for the set of $n \times n$ Hermitian matrices. $\sigma_{max}(G)$ and $\sigma_{min}(G)$ denote maximum and minimum singular values of the transfer matrix G, respectively.

II. PROBLEM FORMULATION

A. System model

Consider a stable linear time-invariant system with known reference input described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t) + B_r r_0 \\ y(t) &= Cx(t) + Dw(t) \\ z(t) &= Lx(t) \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^{n_w}$ is the bounded disturbance input satisfying $w(t)^T w(t) \leq \bar{w}^2$, $r_0 \in \mathbb{R}^p$ is a known constant reference input, $y(t) \in \mathbb{R}^m$ denotes the measured output, $z(t) \in \mathbb{R}^q$ is the vector to be measured. All matrices are of compatible dimensions. We assume that matrix L and B_r are known and that the time-invariant parameters gathered in the matrix $\bar{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ are unknown but belong to a given convex bounded polyhedral domain D_c. That is each uncertain matrix in this domain may be written as an unknown convex combination of N_p given extreme matrices $\bar{M}_1, \bar{M}_2, \ldots, \bar{M}_{N_p}$ such that

$D_c := \{\bar{M}(\lambda) : \bar{M}(\lambda) = \sum_{l=1}^{N_p} \lambda_l \bar{M}_l, \lambda_l \geq 0, \sum_{l=1}^{N_p} \lambda_l = 1\}$

The filter is of the form:

$$\begin{aligned} \dot{\hat{x}}(t) &= A_f \hat{x}(t) + B_f y(t) \\ \hat{z}(t) &= C_f \hat{x}(t) \end{aligned} \tag{2}$$

where the vector $\hat{x}(t)$ is the filter state vector, A_f, B_f, and C_f are real matrices of appropriate dimensions to be

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determined. The order of the filter n_f is restricted to be equal to the system n.
The dynamics of (1) and (2) can be rewritten as the following augmented system:

$$\dot{\xi}(t) = \bar{A}\xi(t) + \bar{B}w(t) + \bar{B}_r \quad (3)$$
$$e(t) = \bar{C}\xi(t) \quad (4)$$

where $e(t) = z(t) - \hat{z}(t)$ is the estimation error, $\xi(t) = \begin{bmatrix} x(t)^T & \hat{x}(t)^T \end{bmatrix}^T$, $\bar{B}_r = \begin{bmatrix} B_r r_0 \\ 0 \end{bmatrix}$, and

$$\begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & 0 \end{bmatrix} = \left[\begin{array}{cc|c} A & 0 & B \\ B_f C & A_f & B_f D \\ \hline L & -C_f & 0 \end{array}\right]$$

Remark 1: The reference input r_0 described in (1) is a known constant, which is general in practice [16].

B. Fault model

In this paper, the following type sensor fault model is adopted.
Definition 1 (Sensor outage fault): when sensor outage faults occur, the sensor signals of systems are given by

$$y_{si}(t) = F_i y(t), i = 1, \dots, N_s \quad (5)$$

where F_i's are diagonal matrices defined as

$$F_i = diag \begin{bmatrix} F_{i1} & F_{i2} & \dots & F_{im} \end{bmatrix} \quad (6)$$

where $F_{ik} = 1$ if the kth sensor is fault free, and $F_{ik} = 0$ if the kth sensor is of outage.

Consider the ith faulty model, system (1) becomes

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t) + B_r r_0 \\ y(t) &= F_i Cx(t) + F_i Dw(t) \\ z(t) &= Lx(t) \end{aligned} \quad (7)$$

and (3) becomes

$$\dot{\xi}(t) = \bar{A}_{F_i}\xi(t) + \bar{B}_{F_i}w(t) + \bar{B}_r \quad (8)$$
$$e(t) = \bar{C}_{F_i}\xi(t) \quad (9)$$

where $\bar{B}_r = \begin{bmatrix} B_r r_0 \\ 0 \end{bmatrix}$, and

$$\begin{bmatrix} \bar{A}_{F_i} & \bar{B}_{F_i} \\ \bar{C}_{F_i} & 0 \end{bmatrix} = \left[\begin{array}{cc|c} A & 0 & B \\ B_f F_i C & A_f & B_f F_i D \\ \hline L & -C_f & 0 \end{array}\right]$$

Denote $G_{ew}(j\omega)$, $G_{ew_i}(j\omega)$ as the transfer functions from noise inputs $w(t)$ to estimation errors $e(t)$ for fault free case (3)-(4) and the ith faulty case (8)-(9), respectively. The $\mathcal{H}_\infty$ filtering problem for systems with sensor outage faults is to find a guaranteed estimation performance index $\gamma > 0$ and $\gamma_f > 0$ such that

$$\sup \sigma_{max}(G_{ew}(j\omega)) < \gamma \quad (10)$$

for fault free case, and

$$\sup \sigma_{max}(G_{ew_i}(j\omega)) < \gamma_f \quad (11)$$

for the ith faulty case, where $G_{ew}(j\omega) = \bar{C}(j\omega I - \bar{A})^{-1}\bar{B}, G_{ew_i}(j\omega) = \bar{C}_{F_i}(j\omega I - \bar{A}_{F_i})^{-1}\bar{B}_{F_i}$.

Then, the reliable $\mathcal{H}_\infty$ filtering problem can be formulated as to design a filter (2) such that the augmented error system (3)-(4) and (8)-(9) are both stable and satisfy performance indexes (10)-(11).

In addition to designing a reliable filter, another task of this work is to design a detector to detect the sensor outage faults whenever they occur. To attain the detection task, the following preliminaries are essential.

C. Preliminaries for fault detection

Solve differential equations (3) and (8) respectively, we have

$$\lim_{t\to\infty} \xi(t) = -\bar{A}^{-1}\bar{B}_r + \int_{t_0}^{\infty} e^{\bar{A}(t-\tau)}\bar{B}w(\tau)d\tau \quad (12)$$

for fault free case, and

$$\lim_{t\to\infty} \xi(t) = -\bar{A}_{F_i}^{-1}\bar{B}_{rF_i} + \int_{t_0}^{\infty} e^{\bar{A}_{F_i}(t-\tau)}\bar{B}_{F_i}w(\tau)d\tau \quad (13)$$

for the ith faulty case.

Design two weighting matrices $V_1 \in \mathbb{R}^{1\times m}$, $V_2 \in \mathbb{R}^{1\times n}$, and define the residual $r(t)$ as

$$r(t) = V_1 y(t) + V_2 \hat{x}(t) \quad (14)$$

which is used to detect the sensor outage faults.

Consider both the fault free and faulty cases, (14) becomes

$$r(t) = C_v \xi(t) + D_w w(t) \quad (15)$$

for the fault free case, where $C_v = \begin{bmatrix} V_1 C & V_2 \end{bmatrix}, D_w = V_1 D$, and

$$r(t) = C_{vF_i}\xi(t) + D_{wF_i}w(t) \quad (16)$$

for the ith faulty case, where $C_{vF_i} = \begin{bmatrix} V_1 F_i C & V_2 \end{bmatrix}, D_{wF_i} = V_1 F_i D$.

Notice that for the fault free system model

$$\begin{aligned} |\lim_{t\to\infty} r(t)| &\le |C_v \bar{A}^{-1}\bar{B}_r| \\ &+ \lim_{t\to\infty} |C_v \int_{t_0}^{t} e^{\bar{A}(t-\tau)}\bar{B}w(\tau)d\tau + D_w| \\ &\le |C_v \bar{A}^{-1}\bar{B}_r| + \|G_{rw}\|_{peak}\bar{w} \end{aligned} \quad (17)$$

and for the ith faulty case

$$\begin{aligned} |\lim_{t\to\infty} r(t)| &\ge |C_{vF_i}\bar{A}_{F_i}^{-1}\bar{B}_r| \\ &- \lim_{t\to\infty} |C_{vF_i} \int_{t_0}^{t} e^{\bar{A}_{F_i}(t-\tau)}\bar{B}_{F_i}w(\tau)d\tau + D_{wF_i}| \\ &\ge |C_{vF_i}\bar{A}_{F_i}^{-1}\bar{B}_r| - \|G_{rw_i}\|_{peak}\bar{w} \end{aligned} \quad (18)$$

where $\|G_{rw}\|_{peak}$ and $\|G_{rw_i}\|_{peak}$ of (17) and (18) are peak-to-peak gains of the transfer matrices from disturbance $w(t)$ to residual $r(t)$ for both the fault free and faulty cases [17] which should be minimized to attenuate the effects of disturbances, and

$$G_{rw}(j\omega) = C_v(j\omega I - \bar{A})^{-1}\bar{B} + D_w,$$
$$G_{rw_i}(j\omega) = C_{vF_i}(j\omega I - \bar{A}_{F_i})^{-1}\bar{B}_{F_i} + D_{wF_i}$$

The bounds of the peak-to-peak gains are formulated as

$$\|G_{rw}(j\omega)\|_{peak} < \zeta \quad (19)$$

$$\|G_{rw_i}(j\omega)\|_{peak} < \zeta_f \quad (20)$$

for both fault free and faulty cases.

To discriminate the fault free and faulty system models, the following conditions should be satisfied

$$|C_v\bar{A}^{-1}\bar{B}_r| + \|G_{rw}\|_{peak}\bar{w} + \|G_{rw_i}\|_{peak}\bar{w} < |C_{vF_i}\bar{A}_{F_i}^{-1}\bar{B}_r|, \ i = 1, \ldots, N_s \quad (21)$$

In order to satisfy condition (21), the following performance indexes are to be satisfied

$$\sigma_{max}(G_n(j\omega)) < \beta, \ \text{for } \omega = 0 \quad (22)$$

$$\sigma_{min}(G_{rF_i}(j\omega)) > \beta_f, \ \text{for } \omega = 0, \ i = 1, \ldots, N_s \quad (23)$$

$$\|G_{rw}(j\omega)\|_{peak} < \zeta, \quad (24)$$

$$\|G_{rw_i}(j\omega)\|_{peak} < \zeta_f, \ i = 1, \ldots, N_s \quad (25)$$

where $G_n(j\omega) = C_v(j\omega I - \bar{A})^{-1}\bar{B}_r, G_{rF_i}(j\omega) = C_{vF_i}(j\omega I - \bar{A}_{F_i})^{-1}\bar{B}_r$.

D. Problem formulation

Summarize the statements stated in the last subsection, the simultaneous design of reliable $\mathcal{H}_\infty$ filter and fault detector can be expressed as to design a reliable filter (2) and weighting matrices V_1, V_2 through solving the following optimization problem: Given proper scalars $\beta, \zeta, \zeta_f, \gamma, \gamma_f$ according to practical requirements, solve

$$\begin{aligned} &\max \ \beta_f \\ &\quad s.t. \ (10)-(11), (22)-(25) \end{aligned} \quad (26)$$

In the following section, LMI conditions are presented for conditions (10)-(11), (22)-(25), and the design task is illustrated in details.

III. Simultaneous design of reliable filter and fault detector

A. Conditions for fault free case

Firstly, consider the fault free system model, combining (3)-(4) and (15), we have

$$\begin{aligned} \dot{\xi}(t) &= \bar{A}\xi(t) + \bar{B}w(t) + \bar{B}_r \\ e(t) &= \bar{C}\xi(t) \\ r(t) &= C_v\xi(t) + D_w w(t) \end{aligned} \quad (27)$$

a) Conditions for performance indexes (22)

Firstly, the following Lemma 1 is given which is essential for the main theorems of this paper.

Lemma 1: Given the same matrices $\bar{A}$, $\bar{B}_r$, C_v, D_w as stated in (27), the following statements are equivalent:

i) There exist matrix variables P_1, Q_1, A_f, B_f, C_f, D_f, V_1, V_2, $X = \begin{bmatrix} X_{11} & X_{12} \\ \star & X_{22} \end{bmatrix}$ and positive scalar β such that

$$\begin{bmatrix} -Q_1 & P_1 & 0 \\ \star & C_v^*C_v & C_v^*D_w \\ \star & \star & D_w^*D_w - \beta^2 \end{bmatrix} < \text{He} \begin{bmatrix} -I \\ \bar{A}^* \\ \bar{B}^* \end{bmatrix} X \begin{bmatrix} 0 \\ -I \\ 0 \end{bmatrix}^* \quad (28)$$

holds.

ii) There exist matrix variables P_{a1}, Q_{a1}, A_{fe}, B_{fe}, C_{fe}, D_{fe}, V_1, V_{2e}, $X_a = \begin{bmatrix} Y & -N \\ -N & N \end{bmatrix}$ and positive scalar β such that

$$\begin{bmatrix} -Q_{a1} & P_{a1} & 0 \\ \star & C_{av}^*C_{av} & C_{av}^*D_w \\ \star & \star & D_w^*D_w - \beta^2 \end{bmatrix} < \text{He} \begin{bmatrix} -I \\ \bar{A}_a^* \\ \bar{B}_a^* \end{bmatrix} X_a \begin{bmatrix} 0 \\ -I \\ 0 \end{bmatrix}^* \quad (29)$$

holds, where $\bar{A}_a = \begin{bmatrix} A & 0 \\ B_{fe}C & A_{fe} \end{bmatrix}$, $\bar{B}_a = \begin{bmatrix} A \\ B_{fe}C \end{bmatrix}$, $C_{av} = \begin{bmatrix} V_1C & V_{2e} \end{bmatrix}$. $A_{fe} = (X_{12}^*)^{-1}X_{22}A_fX_{22}^{-1}X_{12}^*, B_{fe} = -(X_{12}^*)^{-1}X_{22}B_f, C_{av} = \begin{bmatrix} V_1C & V_{2e} \end{bmatrix}, V_{2e} = -V_2X_{22}^{-1}X_{12}^*$.

Theorem 1: Consider the fault-free system model (27), let real matrices $\bar{A} \in \mathbb{R}^{2n\times 2n}$, $\bar{B}_r \in \mathbb{R}^{2n\times 1}$, $C_v \in \mathbb{R}^{1\times 2n}$, a symmetric matrix $\Pi_1 = \begin{bmatrix} 1 & 0 \\ 0 & -\beta^2 \end{bmatrix}$ be given. Then, the inequality condition

$$\sigma_{max}(G_n(j\omega)) < \beta, \ \text{for } \omega = 0 \quad (30)$$

holds if there exist matrix variables $Y, N, \mathcal{A}, \mathcal{B}$, Hermitian matrices $\bar{P}_{1l} = \begin{bmatrix} P_{1l} & P_{2l} \\ \star & P_{3l} \end{bmatrix}$, $\bar{Q}_{1l} = \begin{bmatrix} Q_{1l} & Q_{2l} \\ \star & Q_{3l} \end{bmatrix} \in \mathbf{H}_n$ satisfying $\bar{Q}_{1l} > 0$, and

$$\begin{bmatrix} -Q_{1l} & -Q_{2l} & P_{1l} - Y & P_{2l} + N & 0 & 0 \\ \star & -Q_{3l} & P_{2l}^* + N & P_{3l} - N & 0 & 0 \\ \star & \star & \Phi_{1l} & \Phi_{2l} & YB_rr_0 & C_l^*V_1^* \\ \star & \star & \star & \mathcal{A} + \mathcal{A}^* & -NB_rr_0 & V_2^* \\ \star & \star & \star & \star & -\beta^2 & D^*V_1^* \\ \star & \star & \star & \star & \star & -I \end{bmatrix} < 0, \quad l = 1, \ldots, N_p \quad (31)$$

where $\Phi_{1l} = YA_l - \mathcal{B}C_l + (YA_l - \mathcal{B}C_l)^*, \mathcal{A} = NA_f, \Phi_{2l} = -\mathcal{A} + (-NA_l + \mathcal{B}C_l)^*, \mathcal{B} = NB_f$.

Proof: From Lemma 2 in Appendix, it can be concluded that condition (45) is equivalent to (30). Let $R = \begin{bmatrix} 0 & -I & 0 \end{bmatrix}$, applying Lemma 4 in Appendix, it can be seen that performance index (30) is satisfied if

$$\begin{bmatrix} -\bar{Q}_1 & \bar{P}_1 & 0 \\ \star & C_v^*C_v & C_v^*D_w \\ \star & \star & D_w^*D_w - \beta^2 \end{bmatrix} < \text{He}(\begin{bmatrix} -I \\ \bar{A}^* \\ \bar{B}^* \end{bmatrix} X \begin{bmatrix} 0 \\ -I \\ 0 \end{bmatrix}^*) \quad (32)$$

holds, from Lemma 1, X can be chosen as $X = \begin{bmatrix} Y & -N \\ -N & N \end{bmatrix}$ without introducing any conservatism. After some matrix manipulation and using Schur complement, (32) becomes

$$\begin{bmatrix} -\bar{Q}_1 & \bar{P}_1 - X & 0 & 0 \\ \star & \bar{A}^*X + X\bar{A} & X\bar{B} & C_v^* \\ \star & \star & -\beta^2 & D_w^* \\ \star & \star & \star & -I \end{bmatrix} < 0 \quad (33)$$

with $\bar{P}_1 := \begin{bmatrix} P_1 & P_2 \\ \star & P_3 \end{bmatrix}$, $\bar{Q}_1 := \begin{bmatrix} Q_1 & Q_2 \\ \star & Q_3 \end{bmatrix}$. Multiplying each inequality in (31) by the uncertain parameter λ_l and then evaluating the sum from $l = 1, \dots, N_p$ produces (33) with

$$P_1 = \sum_{l=1}^{N_p} \lambda_l P_{1l}, P_2 = \sum_{l=1}^{N_p} \lambda_l P_{2l}, P_3 = \sum_{l=1}^{N_p} \lambda_l P_{3l},$$
$$Q_1 = \sum_{l=1}^{N_p} \lambda_l Q_{1l}, Q_2 = \sum_{l=1}^{N_p} \lambda_l Q_{2l}, Q_3 = \sum_{l=1}^{N_p} \lambda_l Q_{3l}$$

Then we have that inequalities in (31) provide sufficient conditions for performance index (30), which completes the proof. ■

B. Conditions for faulty cases

Secondly, consider the ith faulty model, combining (8)-(9) and (16), we have

$$\begin{aligned} \dot{\xi}(t) &= \bar{A}_{F_i}\xi(t) + \bar{B}_{F_i}w(t) + \bar{B}_r \\ e(t) &= \bar{C}_{F_i}\xi(t) \\ r(t) &= C_{vF_i}\xi(t) + D_{wF_i}w(t) \end{aligned} \tag{34}$$

a) Conditions for performance index (23)
Consider system model (34), the following Theorem 2 provides inequality conditions for performance index (23).
Theorem 2: Consider the ith faulty system model (34), let real matrices $\bar{A}_{F_i} \in \mathbb{R}^{2n\times 2n}$, $\bar{B}_{F_i} \in \mathbb{R}^{2n\times 1}$, $C_{vF_i} \in \mathbb{R}^{1\times 2n}$, a symmetric matrix $\Pi_2 = \begin{bmatrix} -1 & 0 \\ 0 & \beta_f^2 \end{bmatrix}$ be given. Then, the inequality condition

$$\sigma_{min}(G_{rF_i}(j\omega)) > \beta_f, \text{ for } \omega = 0 \tag{35}$$

holds, if there exist matrix variables $Y, N, \mathcal{A}, \mathcal{B}$, Hermitian matrices $\bar{P}_{1l}^i = \begin{bmatrix} P_{1l}^i & P_{2l}^i \\ \star & P_{3l}^i \end{bmatrix}$, $\bar{Q}_{1l}^i = \begin{bmatrix} Q_{1l}^i & Q_{2l}^i \\ \star & Q_{3l}^i \end{bmatrix} \in \mathbf{H}_n$ satisfying $\bar{Q}_{1l}^i > 0$, and

$$\begin{bmatrix} -Q_{1l}^i & -Q_{2l}^i & P_{1l}^i - Y & P_{2l}^i + N & -Y\ell \\ \star & -Q_{3l}^i & P_{2l}^{i^*} + N & P_{3l}^i - N & N\ell \\ \star & \star & \Upsilon_1^i & \Upsilon_2^i & \Upsilon_4^i \\ \star & \star & \star & \Upsilon_3^i & \Upsilon_5^i \\ \star & \star & \star & \star & \Upsilon_6^i \end{bmatrix} < 0, l = 1 \dots, N_p \tag{36}$$

$$\begin{aligned} \Upsilon_1^i =& YA_l - \mathcal{B}F_iC_l + (YA_l - \mathcal{B}F_iC_l)^* - C_l^*F_iV_1^*V_{10}F_iC_l \\ &- C_l^*F_iV_{10}^*V_1F_iC_l + C_l^*F_iV_{10}^*V_{10}F_iC_l, \\ \Upsilon_2^i =& -\mathcal{A} + (-NA_l + \mathcal{B}F_iC_l)^* - C_l^*F_iV_1^*V_{20} \\ &- C_l^*F_iV_{10}^*V_2 + C_l^*F_iV_{10}^*V_{20}, \\ \Upsilon_3^i =& \mathcal{A} + \mathcal{A}^* - V_2^*V_{20} - V_{20}^*V_2 + V_{20}^*V_{20}, \\ \Upsilon_4^i =& YB_rr_0 + (\ell^*YA_l - \ell^*\mathcal{B}F_iC_l)^* \\ \Upsilon_5^i =& -NB_rr_0 - \mathcal{A}^*\ell \\ \Upsilon_6^i =& \ell^*YB_rr_0 + (\ell^*YB_rr_0)^* + \beta_f^2 \end{aligned}$$

where $\mathcal{A} = NA_f$, $\mathcal{B} = NB_f$, and $\ell = \begin{bmatrix} \ell_1 & \dots & \ell_n \end{bmatrix}^T \in \mathbb{R}^{n\times 1}$ is a vector that should be determined beforehand.
Proof: Similar to Theorem 1, let $R = \begin{bmatrix} 0 & -I & \begin{bmatrix} -\ell \\ 0 \end{bmatrix} \end{bmatrix}$, it can be concluded that performance index (35) is satisfied if the following inequality is feasible

$$\Omega - \Theta^*\Theta < 0 \tag{37}$$

where $\Omega = \begin{bmatrix} -\bar{Q}_1^i & \bar{P}_1^i - X & -X\begin{bmatrix} \ell \\ 0 \end{bmatrix} \\ \star & X\bar{A}_{F_i} + \bar{A}_{F_i}^*X & \phi_1 \\ \star & \star & \phi_2 \end{bmatrix}$, $\phi_1 = X\bar{B}_r + \bar{A}_{F_i}^*X\begin{bmatrix} \ell \\ 0 \end{bmatrix}$, $\phi_2 = \bar{B}_r^*X\begin{bmatrix} \ell \\ 0 \end{bmatrix} + (\bar{B}_r^*X\begin{bmatrix} \ell \\ 0 \end{bmatrix})^* + \beta_f^2$, $\Theta = \begin{bmatrix} 0 & V_1F_iC & V_2 & V_1(I - F_i)f_i \end{bmatrix}$.

As is known to all, there exists $\Theta_0 = \begin{bmatrix} 0 & V_{10}F_iC & V_{20} & 0 \end{bmatrix}$ such that $(\Theta - \Theta_0)^*(\Theta - \Theta_0) \geq 0$ holds, it can be concluded that if

$$\Omega - \Theta^*\Theta + (\Theta - \Theta_0)^*(\Theta - \Theta_0) < 0 \tag{38}$$

holds, (37) readily holds, and on the other hand if (37) holds, there always exists $\Theta_0 = \Theta$ such that (38) becomes (37), so we have that (38) is equivalent to (37).

Similar to Theorem 1, here X can be chosen as $X = \begin{bmatrix} Y & -N \\ -N & N \end{bmatrix}$ without introducing any conservatism. Since inequalities in (36) are all linear dependent on $P_{1l}^i, P_{2l}^i, P_{3l}^i$, $Q_{1l}^i, Q_{2l}^i, Q_{3l}^i$, A_l, B_l, C_l, again, following the same lines of that for Theorem 1, we have if each inequality condition in (36) holds, (38) then holds, which completes the proof. ■
Remark 2: Vector ℓ in inequality (36) should be determined beforehand which can be obtained through heuristic method. An algorithm will be given later in this section to determine the initial values V_{10}, V_{20} of V_1, V_2. After ℓ, V_{10}, V_{20} be determined, inequality (36) becomes LMI.
b) Conditions for performance indexes (24)-(25) and (10)-(11)
Lemma 5: Consider system (34), the peak-to-peak gain of $G_{rw_i}(j\omega)$ is bounded by

$$\|G_{rw_i}(j\omega)\|_{peak} < \zeta_f \tag{39}$$

if there exist matrix variables $0 < N < Y$, $\mathcal{A}, \mathcal{B}$, $\lambda_f > 0, \mu_f$ and ζ_f such that the following inequalities hold

$$\begin{bmatrix} \phi_{3l}^i & \phi_{4l}^i & YB_l - \mathcal{B}F_iD_l \\ \star & \mathcal{A} + \mathcal{A}^T + \lambda_fN & -NB_l + \mathcal{B}F_iD_l \\ \star & \star & -\mu_fI \end{bmatrix} < 0, \quad l = 1, \dots, N_p \tag{40}$$

$$\begin{bmatrix} \lambda_fY & -\lambda_fN & 0 & C_l^TF_iV_1^T \\ \star & \lambda_fN & 0 & V_2^T \\ \star & \star & (\frac{\zeta_f}{\overline{w}} - \mu_f)I & D_l^TF_iV_1^T \\ \star & \star & \star & \frac{\zeta_f}{\overline{w}}I \end{bmatrix} > 0, \quad l = 1, \dots, N_p \tag{41}$$

where $\phi_{3l}^i = YA_l - \mathcal{B}F_iC_l + (YA_l - \mathcal{B}F_iC_l)^T + \lambda_fY, \phi_{4l}^i = -\mathcal{A} + (-NA_l + \mathcal{B}F_iC_l)^T - \lambda_fN, \mathcal{A} = NA_f, \mathcal{B} = NB_f$
Proof: Considering the inequality conditions for the peak-to-peak gain of a transfer matrix as stated in [17], let the lyapunov variable matrix X be chosen as $X = \begin{bmatrix} Y & -N \\ -N & N \end{bmatrix}$, the conclusion is immediate. ■

Lemma 6: Consider system (34), the inequality condition

$$\sigma_{max}(G_{ew_i}(j\omega)) < \gamma_f,$$

holds, if there exist variables $0 < N < Y$, $\mathcal{A}, \mathcal{B}$, satisfying

$$\begin{bmatrix} Y & -N \\ \star & N \end{bmatrix} > 0$$

$$\begin{bmatrix} \phi_{5l}^i & \phi_{6l}^i & YB_l - \mathcal{B}F_iD_l & L^T \\ \star & \mathcal{A}+\mathcal{A}^T & -NB_l + \mathcal{B}F_iD_l & -C_f^T \\ \star & \star & -\gamma_f I & 0 \\ \star & \star & \star & -\gamma_f I \end{bmatrix} < 0,$$

$$l = 1, \ldots, N_p \qquad (42)$$

where $\phi_{5l}^i = YA_l - \mathcal{B}F_iC_l + (YA_l - \mathcal{B}F_iC_l)^T, \mathcal{A} = NA_f, \phi_{6l}^i = -\mathcal{A} + (-NA_l + \mathcal{B}F_iC_l)^T, \mathcal{B} = NB_f$.
Proof: Using the Bounded Real Lemma (BRL) and restricting $X = \begin{bmatrix} Y & -N \\ -N & N \end{bmatrix}$, it is immediate. ■

Remark 3: Let $F_i = I$ Lemma 5 and Lemma 6 viz., let $F_i = I$ of LMIs (40)-(41), and (42), then the LMI conditions in Lemma 5 and Lemma 6 also provide sufficient conditions for performance indexes (24) and (10), respectively.

C. Solutions

Till now, inequality conditions for performance indexes (10)-(11) and (22)-(25) have been formulated in Theorems 1-2 and Lemmas 5-6, respectively. Summarily, we have the following theorem.

Theorem 3: Consider system model (1), there exist a filter (2) and weighting matrices V_1, V_2 such that the fault free augmented system model (27) and the faulty augmented model (34) satisfy performance indexes (10)-(11) and (22)-(25) if inequality conditions (31), (36), and (40)-(42) for $i = 0, 1, \ldots, N_s, l = 1, \ldots, N_p$ hold, where $F_0 = I$.

Proof: Combining Theorems 1-2,and Lemmas 5-6, it is immediate. ■

The following Algorithm 1 is proposed which gives an integrated design process for the appropriate solutions of the filter parameters A_f, B_f, C_f and weights V_1, V_2.

Algorithm 1 Let ϵ_0 be a given large enough constant specifying a stop criterion of this algorithm.

- Step 1) Minimize $a_1\gamma + a_2\zeta + a_3\alpha + a_4\beta$ with weights $a_1, a_2, a_3, a_4 \in \mathbb{R}$ subject to LMI constraints (31), (40)-(41), and (42) with $F_i = I$. The optimal solution is denoted as $V_{1_{opt}}^0$, $V_{2_{opt}}^0$, and $\gamma_{opt}, \zeta_{opt}, \alpha_{opt}, \beta_{opt}$.
- Step 2) Choose $\gamma > \gamma_{opt}, \zeta > \zeta_{opt}, \alpha > \alpha_{opt}, \beta > \beta_{opt}$, $V_1^1 = V_{1_{opt}}^0$, $V_2^1 = V_{2_{opt}}^0$. Given small enough scalars $\gamma_f, \zeta_f, \alpha_f$, maximize β_f subject to LMI constraints (31), (36), and (40)-(42) for $i = 0, 1, \ldots, N_s, l = 1, \ldots, N_p$ with $F_0 = I$. Let $V_1^v = V_{1_{opt}}^{v-1}$, $V_2^v = V_{2_{opt}}^{v-1}$, where $V_{1_{opt}}^{v-1}$ and $V_{2_{opt}}^{v-1}$ are the solutions of the $(v-1)$th optimization.
  If $\beta^{v_0} < \epsilon_0$ for some $V_{1_{opt}}^{v_0}, V_{2_{opt}}^{v_0}$, denote $V_1^{v_0+1} = V_{1_{opt}}^{v_0}$, $V_2^{v_0+1} = V_{2_{opt}}^{v_0}$ and repeat the above optimization, else continue.
- Step 3) When $\beta_f^v \geq \epsilon_0$ for any v in Step 2), stop.

After weighting matrices V_1, V_2 are determined, the residual $r(t)$ is obtained as

$$r(t) = V_1 y(t) + V_2 \hat{x}(t)$$

Consider the fault free case, the steady-state value of $r(t)$ satisfies

$$|r(t)| \leq \sigma_{max}(G_n(j0)) + \|G_{rw}\|_{peak}\bar{w}$$

where $|r(t)|$ denotes the absolute value of $r(t)$. Define the threshold r_{th} as

$$r_{th} := \sup |r(t)| = \sigma_{max}(G_n(j0)) + \|G_{rw}(j\omega)\|_{peak}\bar{w}$$

the sensor outage faults can be detected according to the following logic rule

$$\begin{cases} |r(t)|_{steady} \leq r_{th} & \text{no alarm} \\ |r(t)|_{steady} > r_{th} & \text{alarm} \end{cases} \qquad (43)$$

where $|r(t)|_{steady}$ denotes the steady-state value of the residual output.

IV. Numerical example

This section gives a numerical example to illustrate the effectiveness of our approach. Considering the following system model

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & -2+\rho \end{bmatrix} x(t) + \begin{bmatrix} 0.1 \\ 0 \\ 0.2 \end{bmatrix} w(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} r_0$$

$$y(t) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} w(t)$$

$$z(t) = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} x(t) \qquad (44)$$

where $|\rho| \leq 0.2$. Assume that the reference input $r_0 = 0.4$, the disturbance $\|w(t)\| \leq 0.3$, and $\|w(t)\| := \sqrt{w(t)^T w(t)}$.

The initial values of the weighting matrices V_{10}, V_{20} are obtained through Step 1 in Algorithm 1 as $V_{10} = \begin{bmatrix} -1.0742 & 0.4370 \end{bmatrix}$, $V_{20} = \begin{bmatrix} 0.9251 & 0.2656 & 1.9013 \end{bmatrix}$. Applying Algorithm 1, finally, the filter parameters and the weighting matrices are obtained as

$$A_f = \begin{bmatrix} -0.9202 & -1.0884 & 5.6471 \\ -0.1792 & -0.8514 & -0.8780 \\ -0.4229 & -0.0449 & -2.8528 \end{bmatrix},$$

$$B_f = \begin{bmatrix} 0.8732 & 0.6728 \\ -0.2500 & -0.3228 \\ 0.1913 & 0.4360 \end{bmatrix},$$

$$C_f = \begin{bmatrix} 0.5424 & 0.4010 & 1.8906 \end{bmatrix},$$

$$V_1 = \begin{bmatrix} -1.1997 & 1.7089 \end{bmatrix},$$

$$V_2 = \begin{bmatrix} -0.0866 & -0.7312 & 0.7336 \end{bmatrix}$$

with the performance index $\gamma = 0.6, \gamma_f = 0.8$, and $\beta = 0.3, \beta_f = 0.4050$. Matrix ℓ is chosen beforehand as $\ell = \begin{bmatrix} -0.6 & 0.2 & 0.1 \end{bmatrix}^T$.

To illustrate the simulation results, assume that the disturbance $w(t) = 0.3sin(t)$. When sensor 1 is of outage at $t = 50s$, the residual output is shown in Fig. 2(a), and if sensor 2 is of outage instead of sensor 1, the residual output is shown in Fig. 2(b).

Through the threshold design approach proposed in Section III, the threshold is determined as $r_{th} = 0.0680$ which is denoted by the dashed lines in Fig. 2(a, b). From Fig. 2(a, b), it can easily be formulated that either sensor 1 or senor 2 is of outage, it can be detected according to logic rule (43).

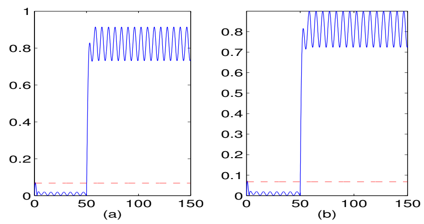

Fig. 1. Residual outputs for different cases.

V. Conclusions

In this paper, the problem of simultaneous reliable $\mathcal{H}_\infty$ filtering and fault detection problem for linear continuous-time systems with bounded disturbances and nonzero constant reference inputs has been investigated. The considered system models are modeled via multiple modes, namely, fault free case and faulty cases. The numerical example has illustrated the effectiveness of the proposed approach.

VI. Appendix

Lemma 2: (Generalized KYP Lemma [15]) Given system matrices (A, B, C, D), and a symmetric matrix Π, the following statements are equivalent:
i) The finite frequency inequality

$$\begin{bmatrix} G(j\omega)^* & I \end{bmatrix} \Pi \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \text{ for all } |\omega| \le \varpi \tag{45}$$

where $G(j\omega) = C(j\omega I - A)^{-1}B + D$.
ii) There exist Hermitian matrices $P, Q \in \mathbf{H}_n$ satisfying $Q > 0$, and

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} -Q & P \\ P & \varpi^2 Q \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} < 0 \tag{46}$$

Lemma 3: (*Projection Lemma* [18]) Let Γ, Λ, Θ be given. There exists a matrix F satisfying $\Gamma F \Lambda + (\Gamma F \Lambda)^T + \Theta < 0$ if and only if the following two conditions hold

$$\Gamma^\perp \Theta \Gamma^{\perp^T} < 0, \quad \Lambda^{T^\perp} \Theta \Lambda^{T^{\perp^T}} < 0$$

The following lemma provides an alternative condition to (46) by introducing a multiplier R through the projection lemma, which is similar to that of [19]. Firstly, define $J \in \mathbb{R}^{(2n+n_z)}$, $\bar{H} \in \mathbb{R}^{(2n+n_z)\times(n_w+n_z)}$, and $\bar{L} \in \mathbb{R}^{(2n+n_z)\times n}$ as

$$J := \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}, \bar{H} := \begin{bmatrix} 0 & 0 \\ C^* & 0 \\ D^* & I \end{bmatrix}, \bar{L} := \begin{bmatrix} -I \\ A^* \\ B^* \end{bmatrix}$$

Lemma 4: Let Hermitian matrix variables $P, Q \in \mathbf{H}_n$ and $Q > 0, R \in \mathbb{R}^{n\times(2n+n_z)}$. Let N_R be the null space of R. The following statements are equivalent:
i) The condition in (46) holds and

$$N_R^*(J \begin{bmatrix} -Q & P \\ P & \varpi^2 Q \end{bmatrix} J^* + \bar{H}\Pi\bar{H}^*)N_R < 0 \tag{47}$$

ii) There exists $X \in \mathbb{R}^{n\times n}$ such that

$$J \begin{bmatrix} -Q & P \\ P & \varpi^2 Q \end{bmatrix} J^* + \bar{H}\Pi\bar{H}^* < \mathrm{He}(\bar{L}XR) \tag{48}$$

Proof. Notice that the null space of $\bar{L}$ is $\begin{bmatrix} A^* & I & 0 \\ B^* & 0 & I \end{bmatrix}$, and using Lemma 3, we have that ii) is equivalent to i). □

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