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On control of a class of MIMO sparse plants

Mauricio Cea and Mario E. Salgado

***Abstract*— This paper deals with achievable performance in the control of multivariable systems, with a particular degree of sparsity. The systems to be considered are linear, stable, time-invariant and discrete-time, and it is assumed that they can be reasonably described by a diagonal plus one interconnection structure. The performance loss when ignoring the off-diagonal term is quantified and also a comparison with a classical feedforward strategy is made.**

I. INTRODUCTION

In practice, the control of multi-input multi-output (MIMO) systems is mostly based on diagonal models for the plant. However in many situations, diagonal models do not capture essential plant dynamics features, such as nonminimum-phase zeros (NMP) with non-canonical directions; also, diagonal models do not account for significant cross-channel interactions. Performance degradation via detuning is then a natural consequence of the oversimplification. Thus, a key question is how the control performance can be improved without resourcing to full MIMO control design.

A first, gradual increase in the plant model complexity can be obtained by considering a diagonal transfer function model plus one off-diagonal element. In the sequel we will refer to this structure as a sparse-1 model.

The contribution of this paper relates to achievable performance bounds for sparse-1 models, in quadratic norm, connections to classical feedforward schemes and performance gain when we transit from diagonal to sparse-1 models.

II. PRELIMINARY DEFINITIONS AND RESULTS

A. Sparse-1 models

On this paper we will refer to sparse-1 model as a diagonal model plus one additional off-diagonal element. If the plant has p inputs and p outputs, the sparse-1 transfer function has $(p+1)$ nonzero entries. If we start with a full MIMO model, the first question is how do we choose the most relevant $(p+1)$ scalar transfer functions. One possible tool to achieve that is the Participation Matrix (PM) [1], which can be used to truncate or to approximate the full MIMO transfer function for a member of the sparse-1 class.

B. Matrix transfer functions

Throughout this paper we will use bold face to denote matrices. Thus $\mathbf{X}$ is a matrix with the (i,j) element denoted by either X_{ij} or $X^{(i,j)}$, and $\mathbf{X} = [X^{(i,j)}]$. Then, a full MIMO transfer function is $\mathbf{G}(z) = [G_{ij}(z)] \in \mathbb{C}^{n\times n}$, and the diagonal transfer function $\mathbf{G_D}(z)$, given by $\mathbf{G_D}(z) = [G_D^{(i,i)}(z) \neq 0]$ with $i = 1,...,n$ and 0 elsewhere. The sparse-1 model, through appropriate permutations of inputs and outputs, can always be described by a transfer function $\mathbf{G_s}(z)$, given by $\mathbf{G_s}(z) = [G_s^{(i,i)}(z) \neq 0]$ with $i = 1,...,n$, $G_s^{(2,1)}(z) \neq 0$ and 0 elsewhere.

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The authors are with the Department of Electronic Engineering, Universidad Técnica Federico Santa María, Valparaíso, Chile. mcea@elo.utfsm.cl, msb@elo.utfsm.cl

C. Inversion, interactors and Youla

Inversion is the basic paradigm in control design. One key step in this construction process is to extract the invertible factor in the model; this is done using interactors [2]. In this paper, given that we are only dealing with stable models, these interactors are chosen to be stable unitary transfer functions. Thus any stable transfer function $\mathbf{H}(z)$, which is nonzero for $|z| = 1$, can be expressed as

$$\mathbf{H}(z) = \mathbf{E_H}(z)\tilde{\mathbf{H}}(z) \tag{1}$$

where $\tilde{\mathbf{H}}(z)$ is stable, minimum phase and biproper, and $\mathbf{E_H}(z^{-1})^T\mathbf{E_H}(z) = \mathbf{I}$, with $\mathbf{E_H}(1) = \mathbf{I}$. Note that $\mathbf{E_H}(z)^{-1}$ is unstable, improper and extracts all zeros of $\mathbf{H}(z)$ lying outside the unit disk; this set includes finite and infinite zeros. For simplicity on this paper we deal with sparse-1 models that have only diagonal unitary interactors. This factorization can be used in conjunction with the Youla parametrization of all stabilizing controllers to synthesize a good (in some sense) inverse. Using this approach we set the synthesis problem as the minimization of the cost function [3],[2]

$$J(\mathbf{H},\mathbf{Q}) = \left|\left|\frac{\mathbf{I}-\mathbf{H}(z)\mathbf{Q}(z)}{z-1}\right|\right|_2^2 = \left|\left|\frac{\mathbf{S}(z)}{z-1}\right|\right|_2^2 \tag{2}$$

where $\mathbf{H}$ generically represents the chosen (stable) plant model, $\mathbf{S}(z)$ corresponds to the sensitivity function in the control loop, and $\mathbf{Q}(z)$ is the Youla parameter [4].

D. $\mathcal{H}_2$ synthesis

The minimization of (2) can be done by expressing the cost function as

$$J(\mathbf{H},\mathbf{Q}) = \left|\left|\underbrace{\frac{\mathbf{E_H}(z)^{-1}-\mathbf{I}}{z-1}}_{\mathbf{M}(z)} + \underbrace{\frac{\mathbf{I}-\tilde{\mathbf{H}}(z)\mathbf{Q}(z)}{z-1}}_{\mathbf{N}(z)}\right|\right|_2^2 \tag{3}$$

where we observe that $\mathbf{M}(z) \in \mathcal{H}_2^\perp$ and $\mathbf{N}(z) \in \mathcal{H}_2$. Then they are orthogonal. Thus

$$\mathbf{Q}^{opt}(z) = \tilde{\mathbf{H}}^{-1}(z) \tag{4}$$

$$J(\mathbf{H},\mathbf{Q}^{opt}) = \left|\left|\mathbf{M}(z)\right|\right|_2^2 \tag{5}$$

This minimal cost is then a function of finite and infinite NMP zeros, and their associated directions [2].

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III. CONTROL SYNTHESIS FOR SPARSE-1 MODELS

The solution for the $\mathcal{H}_2$ synthesis for the sparse-1 case is $\mathbf{Q_s}^{opt}(z) = \tilde{\mathbf{G}}_\mathbf{s}(z)^{-1}$.

$$J(\mathbf{G_s}, \mathbf{Q_s^{opt}}) = \sum_{i=1}^{n} \ell_i + \sum_{i=1}^{n}\sum_{k=1}^{m_i} \frac{|c_k^i|^2 - 1}{|1 - c_k^i|^2} \tag{6}$$

The controller $\mathbf{C_s}(z)$ can then be computed from

$$\mathbf{C_s}(z) = \mathbf{Q_s(z)}(\mathbf{I}-\mathbf{G_s(z)}\mathbf{Q_s(z)})^{-1} = \tilde{\mathbf{G}}_\mathbf{s}\mathbf{(z)}^{-1}(\mathbf{I}-\mathbf{E_s(z)})^{-1} \tag{7}$$

Given that $\mathbf{E_s}(z)$ is diagonal, then $\mathbf{C_s}(z)$ inherits the block structure of $\mathbf{G_s}(z)$. Hence

$$C_s^{(2,1)}(z) = \frac{-E_s^{(1,1)}G_{21}(z)}{(1 - E_s^{(1,1)}(z))G_{11}(z)G_{22}(z)} \tag{8}$$

$$C_s^{(i,i)}(z) = \frac{E_s^{(i,i)}(z)}{G_s^{(i,i)}(z)(1 - E_s^{(i,i)}(z))} \tag{9}$$

IV. PERFORMANCE GAIN

Assume that an optimal diagonal controller is computed, based upon the diagonal model $\mathbf{G_D}(z)$. If we characterize the controller through the corresponding optimal Youla parameter $Q_D^{opt}(z)$, then the resulting sensitivity is

$$\mathbf{S_D}^{opt}(z) = \mathbf{I} - \mathbf{Q_D^{opt}}(z)\mathbf{G_D}(z) = \mathbf{I} - \mathbf{E_D}(z) \tag{10}$$

Given the assumption of having the same interactor for the diagonal and sparse-1 model, the optimal synthesis yields the same optimal sensitivity, that is $\mathbf{S_D^{opt}}(z) = \mathbf{S_s^{opt}}(z)$; however the optimal Youla parameters are different.

Assume now that $\mathbf{Q_D^{opt}}(z)$ is used to control the sparse-1 model. Then the achieved sensitivity is given by [4]

$$\mathbf{S}(z) = \mathbf{S_D^{opt}}(z)(\mathbf{I} + \mathbf{Q_D^{opt}}(z)(\mathbf{G_s}(z) - \mathbf{G_D}(z)))^{-1} \tag{11}$$

We then have the following lemma

Lemma 1: Given the sensitivity function in (11), then

$$\left|\left|\frac{\mathbf{S}(z)}{z-1}\right|\right|_2^2 = \left|\left|\frac{\mathbf{S_D^{opt}}(z)}{z-1}\right|\right|_2^2 + \left|\left|\frac{1 - E_D^{(2,2)}(z)}{z-1}\,\frac{G_{21}(z)}{G_{22}(z)}\right|\right|_2^2 \tag{12}$$

Proof: Direct upon using the definition of the 2-norm and the fact that $\mathbf{Q_D^{opt}}(z) = \mathbf{G_D}(z)^{-1}\mathbf{E_D}(z)$.

Note that the performance degradation depends on the relative magnitude of the off-diagonal term $G_{21}(z)$, with respect to the corresponding diagonal element, $G_{22}(z)$, and also on the location of the NMP zeros in $G_{22}(z)$.

A. Example

This example illustrates the deleterious impact of ignoring the off-diagonal term. Consider a plant with a sparse-1 model $\mathbf{G_s}(z)$ and its unitary interactor $\mathbf{E_s}(z)$

$$\mathbf{G_s}(z) = \begin{bmatrix} \frac{0.2(z+1.5)}{z^2(z-0.5)} & 0 & 0 \\ \frac{2(z-0.8)}{z^4(z-0.2)} & \frac{z-0.2}{z^2} & 0 \\ 0 & 0 & \frac{0.32(z-1.5)}{(z-0.2)(z-0.8)} \end{bmatrix} \tag{13}$$

The interactor, $E_s^{(1,1)}(z) = (z+1.5)(z^2(1.5z+1))^{-1}$, $E_s^{(2,2)}(z) = z^{-1}$, $E_s^{(3,3)}(z) = (z-1.5)(z(-1.5z+1))^{-1}$ and 0 elsewhere. Thus

$$\left|\left|\frac{\mathbf{S_s^{opt}}(z)}{z-1}\right|\right| = \left|\left|\frac{\mathbf{S_D^{opt}}(z)}{z-1}\right|\right| = 9.2 \tag{14}$$

However, when $\mathbf{Q_D^{opt}}(z)$ is used in conjunction with the sparse-1 model the achieved sensitivity (11) satisfies

$$J(\mathbf{G_s}, \mathbf{Q_D^{opt}}) = \left|\left|\frac{\mathbf{S}(z)}{z-1}\right|\right|_2^2 = 14.0 \tag{15}$$

V. CLASSICAL FEEDFORWARD ARCHITECTURE

When using sparse-1 models, the classical control design theory suggests that the cross-coupling contributed by the off-diagonal term , be dealt with using the idea of disturbance feedforward. We now explore that strategy in two steps:

- We first synthesize a diagonal controller based upon the plant diagonal model.
- Design a feedforward controller to compensate the cross-coupling.

The diagonal controller is based upon the transfer function $\mathbf{G_D}(z)$. This controller is given by $C_D(z) = [C_D^{(i,i)}(z)]$, where $C_D^{(i,i)}(z)$ is the same that (9). We have used the fact that $\mathbf{G_D}(z)$ and $\mathbf{G_s}(z)$ have the same interactor.

To completely compensate the effect of cross-coupling, we choose the feedforward controller $C_{ff}(z)$ as

$$C_{ff}(z) = -G_{21}(z)G_{22}^{-1}(z) \tag{16}$$

This solution is stable and proper due to the assumption regarding diagonal directions. This choice leads to the same off-diagonal element (8). This result shows that the $\mathcal{H}_2$ optimization for the sparse-1 model yields the same controller as that obtained by combining $\mathcal{H}_2$ optimization for the diagonal model and a feedforward mechanism.

VI. CONCLUSIONS

This paper reports preliminary results regarding achievable performance bounds in the control of sparse-1 models. The performance gain for going from diagonal to sparse-1 model has been computed. Finally, it is also shown that the same results can be achieved via classical feedforward approach, under a particular assumption.

VII. ACKNOWLEDGMENTS

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