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A Procedure for Robust Stability Analysis of Discrete-Time Systems via Polyhedral Lyapunov Functions

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Abstract— **In this paper we study the robustness analysis problem for linear discrete-time systems subject to parametric time-varying uncertainties making use of polyhedral Lyapunov functions. We propose a novel procedure to construct, in the general n-dimensional case, a polyhedral Lyapunov function to prove the robust stability of a given system.**

I. Introduction

In this paper we focus on the robust stability analysis problem for linear discrete-time systems subject to parametric time-varying uncertainties. Typically, this problem is tackled by means of quadratic Lyapunov functions. As a matter of fact, this approach has been shown to be conservative with respect to approaches using other types of Lyapunov functions [8].

The use of polyhedral Lyapunov functions for robust stability analysis was first proposed in [5]. In [6] it is proved that they are universal for the robustness analysis problem involving linear systems subject to parametric uncertainties.

One problem concerning the use of polyhedral functions in the robust stability context, consists in the development of an efficient numerical approach to find, for a given uncertain system, an optimal polyhedral Lyapunov function (see for instance [3], [4]). In this paper, we propose a novel procedure to construct, in the general n-dimensional case, a polyhedral Lyapunov function for the class of linear discrete-time systems subject to parametric uncertainties. A necessary and sufficient condition for the existence of a polyhedral Lyapunov function is provided. Such condition requires that a certain optimization problem admit a feasible solution. A numerical example is included to illustrate the effectiveness of the proposed approach.

II. Preliminaries

In this paper we deal with the stability of a linear discrete-time system subject to uncertain parameters

$$x(k+1) = A(p)x(k)\,, \tag{1}$$

where $A(\cdot) : \mathcal{R} \subset \mathbb{R}^q \to \mathbb{R}^{n\times n}$, and $\mathcal{R}$ is a box, i. e. $\mathcal{R} := [\underline{p}_1, \overline{p}_1]\times[\underline{p}_2, \overline{p}_2]\times\cdots\times[\underline{p}_q, \overline{p}_q]$. In the sequel we assume that: *i)* The vector-valued function $p(\cdot)$ is any function $p(\cdot) : N \to \mathcal{R}$; *ii)* The matrix-valued function $A(\cdot)$ depends affinely on the parameter vector p, that is

$$A(p) = \sum_{\substack{i_1,\ldots,i_q\\=0}}^{1} A_{i_1,\ldots,i_q} p_1^{i_1}\cdots p_q^{i_q}\,. \tag{2}$$

Definition 1 (Robust stability): System (1) is said to be robustly stable if for any function $p(\cdot) : N \to \mathcal{R}$, the resulting linear time-varying system

$$x(k+1) = A\big(p(k)\big)x(k)$$

is exponentially stable.

We focus on the problem of determining some conditions guaranteeing the robust stability of system (1) making use of the class of (symmetrical) polyhedral Lyapunov functions, which are piecewise linear functions of the following form

$$V(x) = \|Q^T x\|_\infty\,, \tag{3}$$

where $Q \in \mathbb{R}^{n\times m}$ is a full row rank matrix.

Definition 2 (Polyhedral stability, [4]): System (1) is said to be polyhedrally stable *iff* there exist a polyhedral Lyapunov function in the form (3) such that the one step Lyapunov difference along the solutions of system (1)

$$D^+V(x) = V\left(A\left(p\right)x\right) - V\left(x\right)$$

is negative definite for all $p \in \mathcal{R}$.

Remark 1: As shown in [4], [5], the class of polyhedral Lyapunov functions is universal for system (1), i. e. the existence of a Lyapunov function which proves robust stability of the uncertain system implies the existence of a Lyapunov function belonging to the class which does the same job.

A. Notions on polytopes

If we deal with a finite set, say $K = \{x^{(1)}, \ldots, x^{(l)}\} \subset \mathbb{R}^n$, the *convex hull* of K turns out to be a *polytope*, whose *dimension* ([9], p. 5), is given by the dimension of the affine hull of K, i. e.

$$\operatorname{rank}\left[\begin{array}{cccc} x^{(2)} - x^{(1)} & x^{(3)} - x^{(1)} & \ldots & x^{(l)} - x^{(1)} \end{array}\right].$$

In this paper we will focus on polytopes symmetrical with respect to the origin of $\mathbb{R}^n$. To this regard note that, given any symmetrical polytope $\mathcal{P} \subset \mathbb{R}^n$, there always exists a full row rank matrix $Q \in \mathbb{R}^{n\times m}$, $m \geq n$, such that the polytope $\mathcal{P}$ can be alternatively defined as (see [7], p. 6)

$$\mathcal{P} = \wp(Q) := \{x \in \mathbb{R}^n : \|Q^T x\|_\infty \leq 1\}\,. \tag{4}$$

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Therefore a given symmetric polytope $\mathcal{P}$ admits two different equivalent descriptions. The algorithm in [1], implemented in the Matlab routine *convhulln*, enables to find the description matrix Q of a polytope starting from its vertices.

In the following, given a symmetric polytope $\wp(Q)$, we indicate with $x_Q^{(i)}$ with $i = 1 \dots 2l$ the vertices of $\wp(Q)$ and we suppose that $x_Q^{(i)} = -x_Q^{(i+l)}$ for $i = 1 \dots l$.

III. Main Result

The following result provides a necessary and sufficient condition for polyhedral stability of system (1).

Theorem 1: System (1) is polyhedrally stable *iff* there exists a polytope $\wp(Q)$ of dimension n such that the following condition holds for all $i = 1, \dots, 2l$,

$$\max_{p \in \mathrm{vert}(\mathcal{R})} \max_j \tilde{q}_j^T A(p) x_Q^{(i)} < 1 \,, \tag{5}$$

where $\tilde{q}_j$ denotes the j-th column of $\tilde{Q} = \begin{pmatrix} Q & -Q \end{pmatrix}$, and $\mathrm{vert}(\mathcal{R})$ denotes the set of all vertices of $\mathcal{R}$.

Proof: We will show that the one step difference of the polyhedral Lyapunov function $V(x) = \|Q^T x\|_\infty$ along the solutions of system (1) is negative definite *iff* condition (5) holds. To this end, the one step Lyapunov difference can be expressed as

$$\begin{aligned} D^+ V(x) &= \|Q^T A(p) x\|_\infty - \|Q^T x\|_\infty = \\ &= \max_j \tilde{q}_j^T A(p) x - \max_j \tilde{q}_j x \,. \end{aligned} \tag{6}$$

Thus equation (6) is negative definite *iff*

$$\max_j \tilde{q}_j^T A(p) x < \max_j \tilde{q}_j x \qquad \forall x \in \mathbb{R}^n \quad \forall p \in \mathcal{R} \tag{7}$$

Taking into account that for each vector $x \in \mathbb{R}^n$ there exist a scalar λ and a point x_Q belonging to the boundary $\partial\wp(Q)$ of the polytope $\wp(Q)$ such that $x_Q = \lambda x$, equation (7) can be rewritten as

$$\max_j \tilde{q}_j^T A(p) x < \max_j \tilde{q}_j x = 1 \qquad \forall x \in \partial\wp(Q) \quad \forall p \in \mathcal{R}$$

from the definition of boundary of a polytope. It is straightforward to recognize that the previous condition is equivalent to the following

$$\max_j \tilde{q}_j^T A(p) x_Q^{(i)} < 1 \qquad \forall p \in \mathcal{R} \,. \tag{8}$$

To conclude the proof, note that inequality (8) holds for all $p \in \mathcal{R}$ *iff* its maximum on $\mathcal{R}$ is negative. Thus we have

$$\max_{p \in \mathrm{vert}(\mathcal{R})} \max_j \tilde{q}_j^T A(p) x_Q^{(i)} < 1 \tag{9}$$

where we have used the fact that a multiaffine function defined on a box $\mathcal{R}$ attains its maximum at one of the vertices of $\mathcal{R}$ (see [2]). ■

In order to find a polyhedral Lyapunov function satisfying the conditions of Theorem 1, the following procedure can be adopted.

Procedure 1 (Implementation of Theorem 1):

1) Fix an initial number $2l \geq 2n$ of symmetric points $x_Q^{(i)}$ on a hypersphere with radius 1. Let indicate with $K_0 = \{x_Q^{(i)}\}_{i=1,\dots,2l}$ the set of such points.
2) Find a set of symmetric points K solving the problem
$$\min_K \max_{p \in \mathrm{vert}(\mathcal{R})} f(K,p) \quad s.t. \quad \mathrm{rank}(Q) = n \tag{10}$$
with initial condition K_0, where
$$f(K,p) = \max_j \tilde{q}_j^T A(p) x_Q^{(i)} - 1 \,.$$
3) Let $M = \min_K (\max_p f(K,p))$. If $M < 0$ then set $K_{opt} = \arg M$, and go to step 4, else set $l = l+1$ and
$$K_0 = K \cup \left\{ x_Q^{(l+1)}, -x_Q^{(l+1)} \right\}, \; x_Q^{(l+1)} \in \mathbb{R}^n$$
and go to step 2.
4) The polyhedral Lyapunov function that proves the polyhedral stability of system (1) is $V(x) = \|Q^T x\|_\infty$ where Q describes the polytope of vertices K_{opt}.

A. Numerical Example

Consider the linear uncertain system

$$\dot{x} = (A_1 + A_2 p)\, x \,, \quad p \in [-\gamma, \gamma] \,, \tag{11}$$

with

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -0.4 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0.4 & 0 \end{pmatrix}.$$

It is easy to show that system (11) is quadratically stable for $|\gamma| \leq 0.92$. On the other hand, using the approach based on polyhedral functions, it is possible to prove the robust stability of (11) up to $\gamma = 0.999$ with the polytope $\wp(Q)$ of 12 vertices ($l = 6$).

IV. Conclusions

In this paper we have considered the robustness analysis problem for a linear uncertain discrete-time system subject to parametric time-varying uncertainties. To tackle this problem we have made use of polyhedral Lyapunov functions. A novel procedure, which enables to construct a polyhedral Lyapunov function proving robust stability of a given uncertain system, has been provided, and an example has been presented.

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