2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008



# Global robust stabilization of nonlinear strict feedforward systems with input unmodeled dynamics

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*Abstract*— **In this paper, we study the global robust stabilization problem of strict feedforward systems subject to input unmodeled dynamics. We present a recursive design method for a nested saturation controller which globally stabilizes the closed-loop system in the presence of input unmodeled dynamics. One of the specific difficulties of the problem is that the Jacobian linearization of our system at the origin may not be stabilizable. We overcome this difficulty by employing a special version of the small gain theorem to address the stability of the closed-loop system while employing the asymptotic small gain theorem to establish the global attractiveness property of the closed-loop system. An example is given to show that a redesign of the controller is necessary to guarantee the global robust asymptotic stability when the input unmodeled dynamics is present.**

## I. Introduction

In this paper, we study the global robust stabilization problem of strict feedforward systems described by

$$\begin{aligned} \dot{x}_i &= g_i(x_{i-1}^{p_{i-1}}, ..., x_1^{p_1}, v, d),\ i = n, ..., 2 \\ \dot{x}_1 &= g_1(v, d) \end{aligned} \tag{1}$$

subject to the following input unmodeled dynamics

$$\dot{\xi}_1 = q(\xi_1, u, d), v = p(\xi_1, u, d) \tag{2}$$

where $p_1, ..., p_{n-1}$ are odd positive integers, $x_i \in \mathbb{R}, i = 1, ..., n$, $\xi_1 \in \mathbb{R}^{n_{\xi_1}}$, $u, v \in \mathbb{R}$, $d : [0, \infty) \to \mathcal{D} \subset \mathbb{R}^{n_d}$ with $\mathcal{D}$ a compact set having a known bound, is piecewise continuous representing a static time-varying disturbance, and $g_i$, $i = 1, ..., n$, $p, q$ are locally *Lipschitz*, and are continuously differentiable and vanish at $(0, ..., 0, d)$ for all $d \in \mathcal{D}$.

The robust stabilization problem of nonlinear systems subject to input unmodeled dynamics has been studied for over fifteen years, see, e.g., [1], [6], [5], [7], [8], [9] and the references therein. Among them, [1], [7], [9] studied various special cases of system (1) with $p_1 = ... = p_{n-1} = 1$. For example, it is assumed in [1] that, for $i = 2, ..., n$, $g_i(x_{i-1}, ..., x_1, v, d) = x_{i-1} + \tilde{g}_i(x_{i-1}, ..., x_1, v, d)$ where $\tilde{g}_i(x_{i-1}, ..., x_1, v, 0) = o(x_{i-1}, ..., x_1, v)$. A common assumption of these papers is the stabilizability of the Jacobian linearization of system (1) at $(0, ..., 0, d)$. However, this assumption is not satisfied by system (1) when some of the $p_i$'s are greater than one. As a result, the approaches in [1], [7], [9] do not work for our problem. In particular, the *Lyapunov linearization* technique cannot be used to establish the local stability of the closed-loop system as what was done in [1]. In this paper, we will adopt the small gain approach to handle the global robust stabilization problem of system (1) subject to the input unmodeled dynamics (2), and to design a nested saturation controller recursively to guarantee the global robust asymptotic stability in the presence of the input unmodeled dynamics. More specifically, we will employ two versions of the small gain theorem adapted from [12] to establish the local stability and global attractiveness of the closed-loop system at the origin respectively.

It is noted that when the input unmodeled dynamics (2) is not present, the problem reduces to the global robust stabilization problem of system (1) viewing $v$ as the input. This special case is also treated in [10], [13], [14] under various assumptions. The approaches in [10], [13], [14] are *Lyapunov* based. In contrast, ours is a small gain approach which leads to the well-known nested saturation controller. It will be seen later that even for this special case, the results in [10], [13], [14] do not contain ours because the functions $g_i$'s in this paper only need to satisfy Assumption 4.1 to be given in Section IV, while in [10], [13], [14], the functions $g_i$'s are subject to some other assumptions. For example, in [14], it is assumed that, for $i = 2, ..., n$, $\dot{x}_i = x_{i-1}^{p_{i-1}} + \hat{g}_i(x, v, d)$ where $|\hat{g}_i(x, v, d)| \leq a_i(1 + |x_i|)(x_1^{p_1+1} + ... + x_{i-1}^{p_{i-1}+1} + v^2) \times \chi_i(x_1, ..., x_{i-1}, v)$ with $a_i \geq 0$ being an unknown constant and $\chi_i(x_1, ..., x_{i-1}, v) \geq 0$ being a known function. In the case when $\hat{g}_i(x, v, d)$ is a polynomial in $x_1, ..., x_{i-1}, v$, the above assumption implies that the degree of each $x_j$ ($j = 1, ..., i-2$) and $v$ has to be greater than $p_j$ and 1 respectively. However, we allow the degree of each $x_j$ ($j = 1, ..., i-2$) and $v$ to be equal to $p_j$ and 1 respectively. As an illustration of this point, a simple example that cannot be handled by the approaches in [10], [13], [14] will be given in Section V.

## II. Preliminary

Throughout the paper, we will use $(x_1, x_2)$ with $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$ to denote the vector $(x_1^T, x_2^T)^T \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, and let $\mathcal{L}_\infty^m$ be the set of all piecewise continuous functions $w : [0, \infty) \to \mathbb{R}^m$ with a finite supremum norm $\|w\|_\infty = \sup_{t\geq 0} \|w(t)\|$, and let $\|w\|_a = \limsup_{t\to\infty} \|w(t)\|$ denote the asymptotic $\mathcal{L}_\infty$ norm of $w$, where $\|\cdot\|$ is the standard Euclidean norm. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is called a gain function if it is continuous, nondecreasing, and satisfies $\gamma(0) = 0$. Let $Id$ denote the gain function $\gamma(s) = s$, and let $O(\|v\|^p) : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ be a function such that $\lim_{\|v\|\to 0} \frac{\|O(\|v\|^p)\|}{\|v\|^p}$ is a finite constant for some positive

The work described in this paper was substantially supported by a grant from the Research Grants Council of the Hong Kong Special Administration Region (Project no.: CUHK412305)

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978-1-4244-2079-7/08/$25.00 ©2008 AACC. 

integer $p$. For a function $g(v)$, the notation $g(v) = o(v)$ means $\lim_{\|v\|\to 0} \frac{\|g(v)\|}{\|v\|} = 0$.

In the following, we first review some terminologies introduced in [4], [12].

Consider the following nonlinear system

$$\dot{x} = f(x, w, d),\ y = h(x, w, d) \tag{3}$$

where $x \in \mathbb{R}^n$ is the plant state, $y \in \mathbb{R}^p$ the output, $w \in \mathbb{R}^m$ the piecewise continuous input, $f(x, w, d)$ and $h(x, w, d)$ are locally *Lipschitz* functions satisfying $f(0, 0, d) = 0$ and $h(0, 0, d) = 0$ for all $d \in \mathcal{D}$, and $d : [0, \infty) \to \mathcal{D}$ is a piecewise continuous function with its range $\mathcal{D}$ a compact subset of $\mathbb{R}^{n_d}$. Let $x(t)$ denote the solution of system (3) with initial state $x(0)$, input $w$ and $d$.

*Definition 2.1:* [4] The output $y$ of system (3) is said to satisfy a robust $\mathcal{L}_\infty$ stability bound (RLB) with restrictions $X_s, \Delta$ on $x(0), w$ and gains $\gamma^0, \gamma$ respectively, if there exist open set $X_s$ of the origin of $\mathbb{R}^n$, positive real number $\Delta$, gain functions $\gamma^0, \gamma$, all independent of $d$, such that, for each $x(0) \in X_s$, $d \in \mathcal{D}$, $\|w\|_\infty < \Delta$, the solution of (3) exists for all $t \geq 0$ and

$$\|y\|_\infty \leq \max\{\gamma^0(\|x(0)\|), \gamma(\|w\|_\infty)\} \tag{4}$$

*Definition 2.2:* [12] The output $y$ of system (3) is said to satisfy a robust asymptotic bound (RAB) with restriction $X_a$ on $x(0)$, restriction $\Delta$ on $w$ and gain $\gamma$, if there exist open set $X_a$ of the origin of $\mathbb{R}^n$, non-negative real number $\Delta$, gain function $\gamma$, all independent of $d$, such that, for each $x(0) \in X_a$, $d \in \mathcal{D}$ and piecewise continuous $w$ satisfying $\|w\|_a \leq \Delta$, the solution of (3) exists for all $t \geq 0$ and

$$\|y\|_a \leq \gamma(\|w\|_a) \tag{5}$$

*Remark 2.1:* In both Definitions 2.1 and 2.2, the word *robust* is used to emphasize that the inequalities (4) and (5) hold regardless of the presence of the disturbance $d$ in (3). For convenience, in the following, we will simply use LB and AB to mean RLB and RAB respectively. Moreover, if the state $x$ of system (3) satisfies LB or AB, then we will say system (3) satisfies LB or AB. If the output $y$ of system (3) satisfies LB with restriction on $x(0)$, restriction $\Delta$ on $w$ and gain $\gamma$, and satisfies AB with no restriction on $x(0)$, restriction $\Delta$ on $w$ and gain $\gamma$, then we will say $y$ satisfies LB with restriction and AB with no restriction on $x(0)$, both with restriction $\Delta$ on $w$ and gain $\gamma$.

## III. A TECHNICAL LEMMA

Like [1], [12], our approach will utilize saturation functions characterized as follows.

*Definition 3.1:* A locally *Lipschitz* function $\sigma(\cdot) : \mathbb{R} \to [-\lambda, \lambda]$ is said to be a saturation function with saturation level $\lambda > 0$, if for all $s \in \mathbb{R}$,

(1) $\sigma(s) = s$ when $|s| \leq \frac{\lambda}{2}$;

(2) $\frac{\lambda}{2} \leq sgn(s)\sigma(s) \leq \min\{|s|, \lambda\}$ when $|s| \geq \frac{\lambda}{2}$.

In the following, we consider the system

$$\dot{z} = \theta(d)u + F(\xi, u, d), \dot{\xi} = G(\xi, u, d) \tag{6}$$

where $F : \mathbb{R}^{n_\xi} \times \mathbb{R} \times \mathcal{D} \to \mathbb{R}, G : \mathbb{R}^{n_\xi} \times \mathbb{R} \times \mathcal{D} \to \mathbb{R}^{n_\xi}$ are locally *Lipschitz* functions vanishing at $(0, 0, d)$ for all $d \in \mathcal{D}$, and $\theta : \mathcal{D} \to \mathbb{R}$ is continuous, nonzero and does not change its sign.

Under the control

$$u = -\sigma(k(z + H(d)\xi)^p - \bar{u}) \tag{7}$$

where $\sigma$ is a saturation function with level $\lambda > 0$, $k$ is a nonzero real number with the same sign as $\theta(d)$, $H(d)$ is a $1 \times n_\xi$ matrix depending on $d$ satisfying $\|H(d)\| \leq \nu$ for all $d \in \mathcal{D}$ and some positive constant $\nu$, and $p$ is an odd positive integer, system (6) takes the form

$$\begin{aligned}\dot{z} &= -\theta(d)\sigma(k(z + H(d)\xi)^p - \bar{u}) \\ &\quad + F(\xi, -\sigma(k(z + H(d)\xi)^p - \bar{u}), d) \\ \dot{\xi} &= G(\xi, -\sigma(k(z + H(d)\xi)^p - \bar{u}), d)\end{aligned} \tag{8}$$

which can be viewed as the interconnection

$$v_1 = y_2, \quad v_2 = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = y_1 \tag{9}$$

of the following two subsystems

$$\Sigma_1 : \begin{array}{l} \dot{\xi} = G(\xi, -\sigma(kv_1), d), \\ y_1 = \begin{bmatrix} y_{11} \\ y_{12} \end{bmatrix} = \begin{bmatrix} H(d)\xi \\ F(\xi, -\sigma(kv_1), d)/\tilde{k} \end{bmatrix}, \end{array}$$

$$\Sigma_2 : \begin{array}{l} \dot{z} = -\tilde{\sigma}(\tilde{k}[(z + v_{21})^p - \frac{\bar{u}}{k}]) + \tilde{k}v_{22}, \\ y_2 = (z + v_{21})^p - \frac{\bar{u}}{k} \end{array}$$

where $\tilde{\sigma}(s) = \theta(d)\sigma(s/\theta(d))$ is a saturation function with level $\tilde{\lambda} = |\theta(d)|\lambda$, and $\tilde{k} = \theta(d)k > 0$.

*Lemma 3.1:* Consider system (6). Assume $\xi$ subsystem satisfies LB with restrictions $\Xi, \Delta$, on $\xi(0), u$ and gains $\gamma_1^0, \gamma$ respectively, and satisfies AB with no restriction on $\xi(0)$, restriction $\Delta$ on $u$ and gain $\gamma$. Then under the control (7), the following results hold:

1) With $\lambda < \Delta$, the output $y_{1i}$, $i = 1, 2$, of $\Sigma_1$ subsystem satisfies LB with restrictions $\Xi, \bar{\Delta}_1$ on $\xi(0), v_1$ and gains $\bar{\gamma}_1^0, \bar{\gamma}_{1i}$ respectively, and satisfies AB with no restriction on $\xi(0)$, restriction $\bar{\Delta}_1$ on $v_1$ and gain $\bar{\gamma}_{1i}$.
2) If there exist sufficiently small $\lambda, |k|$ such that the following small gain condition holds:

$$\max\{2 \cdot 6^p(\bar{\gamma}_{11}(s))^p, 2 \cdot 6^p\bar{\gamma}_{12}(s)\} < s, s > 0, \tag{10}$$

then the state $z, \xi$ of the closed-loop system (8) satisfy LB with restriction and AB with no restriction on $(z(0), \xi(0))$, both with restriction $\frac{\lambda}{3}$ on $\bar{u}$ and gains $3(\frac{s}{|k|})^{\frac{1}{p}}, \gamma(2 \cdot 6^p s)$ respectively.

∎

*Remark 3.1:* Lemma 3.1 is a generalization of the corresponding one in [4] in three aspects. First, the control (7) takes a more general form; second, $\xi$ subsystem is allowed to satisfy LB and AB with a nonlinear gain from $u$ instead of the linear gain; and finally, the higher order assumption $F(\xi, u, 0) = o(\xi, u)$ has been removed.

*Remark 3.2:* The following inequalities are used in the proof of Lemma 3.1:

$$\begin{aligned} 2 \cdot 6^p(\bar{\gamma}_{11}(s))^p < s &\Leftrightarrow \bar{\gamma}_{11}(2 \cdot 6^p s^p) < s \\ &\Leftrightarrow \bar{\gamma}_{11}(s) < 2^{-\frac{1}{p}}6^{-1}s^{\frac{1}{p}}, s > 0, \\ 2 \cdot 6^p\bar{\gamma}_{12}(s) < s \ &\Leftrightarrow\ \bar{\gamma}_{12}(2 \cdot 6^p s) < s,\ s > 0. \end{aligned} \tag{11}$$

*Proof: Part 1):* The assumption on $\xi$ subsystem and $|\sigma(kv_1)| \le \min\{|k||v_1|, \lambda\}$ with $\lambda < \Delta$ implies that, for all $\xi(0) \in \mathbb{R}^{n_\xi}$, $d \in \mathcal{D}$, piecewise continuous $v_1$, $\xi(t)$ exists for all $t \ge 0$, and satisfies

$$\|\xi\|_\infty \le \max\{\gamma_1^0(\|\xi(0)\|), \gamma(\min\{|k|\|v_1\|_\infty, \lambda\})\} \quad (12)$$

for all $\xi(0) \in \Xi$, $d \in \mathcal{D}$, $v_1 \in \mathcal{L}_\infty^1$, and

$$\|\xi\|_a \le \gamma(\min\{|k|\|v_1\|_a, \lambda\}) \quad (13)$$

for all $\xi(0) \in \mathbb{R}^{n_\xi}$, $d \in \mathcal{D}$ and piecewise continuous $v_1$.

Noting $|y_{11}| = |H(d)\xi| \le \|H(d)\|\|\xi\| \le \nu\|\xi\|$ yields

$$\|y_{11}\|_\infty \le \max\{\nu\gamma_1^0(\|\xi(0)\|), \bar{\gamma}_{11}(\|v_1\|_\infty)\} \quad (14)$$

for all $\xi(0) \in \Xi$, $d \in \mathcal{D}$, $v_1 \in \mathcal{L}_\infty^1$, where $\bar{\gamma}_{11}(s) = \nu\gamma(\min\{|k|s, \lambda\})$ and

$$\|y_{11}\|_a \le \nu\gamma(\min\{|k|\|v_1\|_a, \lambda\}) = \bar{\gamma}_{11}(\|v_1\|_a) \quad (15)$$

for all $\xi(0) \in \mathbb{R}^{n_\xi}$, $d \in \mathcal{D}$ and piecewise continuous $v_1$.

Next consider $y_{12}$. Since $F(\xi, u, d)$ is locally Lipschitz and $F(0, 0, d) = 0$ for all $d \in \mathcal{D}$, there exists a gain function $\gamma^o(s)$ such that $|F(\xi, u, d)| \le \gamma^o(\|(\xi, u)\|)$ for any $\xi \in \mathbb{R}^{n_\xi}, u \in \mathbb{R}$ and $d \in \mathcal{D}$. Then, using (12) and (13) gives

$$\|y_{12}\|_\infty \le \max\{\gamma^o(2\gamma_1^0(\|\xi(0)\|))/\tilde{k}, \bar{\gamma}_{12}(\|v_1\|_\infty)\} \quad (16)$$

for all $\xi(0) \in \Xi$, $d \in \mathcal{D}$, $v_1 \in \mathcal{L}_\infty^1$, where $\bar{\gamma}_{12}(s) = \max\{\gamma^o(2\gamma(\min\{|k|s, \lambda\})), \gamma^o(2\min\{|k|s, \lambda\})\}/\tilde{k}$, and

$$\|y_{12}\|_a \le \bar{\gamma}_{12}(\|v_1\|_a) \quad (17)$$

for all $\xi(0) \in \mathbb{R}^{n_\xi}$, $d \in \mathcal{D}$ and piecewise continuous $v_1$. Defining $\bar{\Delta}_1 = \infty$ and $\bar{\gamma}_1^0(s) = \max\{\nu\gamma_1^0(s), \gamma^o \circ 2\gamma_1^0(s)/\tilde{k}\}$ completes Part 1).

*Part 2):* Let us first apply Propositions 2.1 and 2.2 of [3] to show that the output $(y_1, y_2)$ of $\Sigma_1$ and $\Sigma_2$ under the interconnection (9) satisfies LB with restriction and AB with no restriction on $(z(0), \xi(0))$, both with restriction on $\bar{u}$.

*Step 1.* Show that, for $\Sigma_2$ system viewing $v_{21}, v_{22}, \bar{u}$ as inputs, there exists a gain function $\gamma_2^0$ such that, $y_2$ satisfies LB with no restriction on $z(0)$ and gain $\bar{\gamma}_2^0$, restrictions $\bar{\Delta}_{21}, \bar{\Delta}_{22}, \bar{\Delta}_{\bar{u}}$ on $v_{21}, v_{22}, \bar{u}$ and gains $\bar{\gamma}_{21}, \bar{\gamma}_{22}, \bar{\gamma}_2^{\bar{u}}$ respectively, and satisfies AB with no restriction on $z(0)$, restrictions $\bar{\Delta}_{21}, \bar{\Delta}_{22}, \bar{\Delta}_{\bar{u}}$ on $v_{21}, v_{22}, \bar{u}$ and gains $\bar{\gamma}_{21}, \bar{\gamma}_{22}, \bar{\gamma}_2^{\bar{u}}$ respectively, where $\bar{\Delta}_{21} = \bar{\Delta}_{\bar{u}} = \infty, \bar{\Delta}_{22} = \frac{\lambda}{3|k|}$ and $\bar{\gamma}_2^0(s) = 2^{p+1}(\gamma_2^0(s))^p, \bar{\gamma}_{21}(s) = 2 \cdot 6^p s^p, \bar{\gamma}_{22}(s) = 2 \cdot 6^p s, \bar{\gamma}_2^{\bar{u}}(s) = \frac{2 \cdot 6^p}{|k|} s$.

Let $V(z) = \frac{1}{2}z^2$. Then its time derivative along the trajectory of $\Sigma_2$ subsystem satisfies

$$\dot{V} = -(\tilde{\sigma}(\tilde{k}[(z + v_{21})^p - \frac{\bar{u}}{k}]) - \tilde{k}v_{22})z$$

Now consider the following three cases:
(1) $\tilde{k}|(z + v_{21})^p - \frac{\bar{u}}{k}| \le \frac{\tilde{\lambda}}{2}$: We have $\dot{V} = -\tilde{k}((z + v_{21})^p - \frac{\bar{u}}{k} - v_{22})z$. Thus,

$$\begin{aligned}&|z| > 3\max\{|v_{21}|, |\frac{\bar{u}}{k}|^{\frac{1}{p}}, |v_{22}|^{\frac{1}{p}}\} \ge |v_{21}| + |\frac{\bar{u}}{k}|^{\frac{1}{p}} + |v_{22}|^{\frac{1}{p}} \\ &\Rightarrow |z| > |v_{21}| + |\tfrac{\bar{u}}{k} + v_{22}|^{\frac{1}{p}} \\ &\left.\begin{aligned}&\Rightarrow\ |z + v_{21}|^p > |\tfrac{\bar{u}}{k} + v_{22}| \\ &\quad |z| > 3|v_{21}|\end{aligned}\right\} \Rightarrow \dot{V} < 0\end{aligned} \quad (18)$$

(2) $\tilde{k}|(z + v_{21})^p - \frac{\bar{u}}{k}| > \frac{\tilde{\lambda}}{2}$ and $z > 0$: We have

$$\begin{aligned}z = |z| &> 2\max\{|v_{21}|, |\tfrac{\bar{u}}{k}|^{\frac{1}{p}}\} \ge -v_{21} + (\tfrac{\bar{u}}{k})^{\frac{1}{p}} \\ &\Rightarrow \tilde{k}|(z + v_{21})^p - \tfrac{\bar{u}}{k}| = \tilde{k}((z + v_{21})^p - \tfrac{\bar{u}}{k}) > \tfrac{\tilde{\lambda}}{2} \\ &\Rightarrow \tilde{\sigma}(\tilde{k}[(z + v_{21})^p - \tfrac{\bar{u}}{k}]) > \tfrac{\tilde{\lambda}}{2} \\ &\Rightarrow \dot{V} < -z(\tfrac{\tilde{\lambda}}{2} - \tilde{k}|v_{22}|) < 0\end{aligned} \quad (19)$$

for all $|v_{22}| < \frac{\tilde{\lambda}}{2\tilde{k}} = \frac{\lambda}{2|k|}$.
(3) $\tilde{k}|(z + v_{21})^p - \frac{\bar{u}}{k}| > \frac{\tilde{\lambda}}{2}$ and $z < 0$: We have

$$\begin{aligned}-z = |z| &> 2\max\{|v_{21}|, |\tfrac{\bar{u}}{k}|^{\frac{1}{p}}\} \ge v_{21} - (\tfrac{\bar{u}}{k})^{\frac{1}{p}} \\ &\Rightarrow \tilde{k}|(z + v_{21})^p - \tfrac{\bar{u}}{k}| = -\tilde{k}((z + v_{21})^p - \tfrac{\bar{u}}{k}) > \tfrac{\tilde{\lambda}}{2} \\ &\Rightarrow \tilde{\sigma}(\tilde{k}[(z + v_{21})^p - \tfrac{\bar{u}}{k}]) < -\tfrac{\tilde{\lambda}}{2} \\ &\Rightarrow \dot{V} < -z(-\tfrac{\tilde{\lambda}}{2} + \tilde{k}|v_{22}|) < 0\end{aligned} \quad (20)$$

for all $|v_{22}| < \frac{\tilde{\lambda}}{2\tilde{k}} = \frac{\lambda}{2|k|}$.

Noting (18) to (20), we claim that, there exists a gain function $\gamma_2^0$ such that, for all $z(0) \in \mathbb{R}$, $d \in \mathcal{D}$, piecewise continuous $\bar{u}, v_{21}$ and $v_{22}$ satisfying $\|v_{22}\|_a \le \frac{\lambda}{3|k|}$, $z(t)$ exists for $t \ge 0$, and satisfies

$$\begin{aligned}\|z\|_\infty \le \max\{&\gamma_2^0(|z(0)|), \\ &3\|v_{21}\|_\infty, 3(\|v_{22}\|_\infty)^{\frac{1}{p}}, 3(\tfrac{\|\bar{u}\|_\infty}{|k|})^{\frac{1}{p}}\}\end{aligned} \quad (21)$$

for all $z(0) \in \mathbb{R}, d \in \mathcal{D}, \bar{u}, v_{21} \in \mathcal{L}_\infty^1, \|v_{22}\|_\infty < \frac{\lambda}{3|k|}$, and

$$\|z\|_a \le \max\{3\|v_{21}\|_a, 3(\|v_{22}\|_a)^{\frac{1}{p}}, 3(\tfrac{\|\bar{u}\|_a}{|k|})^{\frac{1}{p}}\} \quad (22)$$

for all $z(0) \in \mathbb{R}, d \in \mathcal{D}$, piecewise continuous $\bar{u}, v_{21}$ and $v_{22}$ satisfying $\|v_{22}\|_a \le \frac{\lambda}{3|k|}$. In fact, the proof of (21) can be derived by Lemma 3.3 in [12] and the proof of (22) can be derived from the derivation of (A.16) of [1].

Then it follows from (21), (22) and $y_2 = (z + v_{21})^p - \frac{\bar{u}}{k}$ that, $y_2$ satisfies

$$\begin{aligned}\|y_2\|_\infty \le \max\{&\bar{\gamma}_2^0(|z(0)|), \\ &\bar{\gamma}_{21}(\|v_{21}\|_\infty), \bar{\gamma}_{22}(\|v_{22}\|_\infty), \bar{\gamma}_2^{\bar{u}}(\|\bar{u}\|_\infty)\}\end{aligned} \quad (23)$$

for all $z(0) \in \mathbb{R}, d \in \mathcal{D}, \bar{u}, v_{21} \in \mathcal{L}_\infty^1, \|v_{22}\|_\infty < \frac{\lambda}{3|k|}$, and

$$\|y_2\|_a \le \max\{\bar{\gamma}_{21}(\|v_{21}\|_a), \bar{\gamma}_{22}(\|v_{22}\|_a), \bar{\gamma}_2^{\bar{u}}(\|\bar{u}\|_a\}) \quad (24)$$

for all $z(0) \in \mathbb{R}, d \in \mathcal{D}$, piecewise continuous $\bar{u}, v_{21}$ and $v_{22}$ satisfying $\|v_{22}\|_a \le \frac{\lambda}{3|k|}$.

*Step 2.* Choose $\lambda, k$ appropriately to satisfy the small gain conditions of Propositions 2.1 and 2.2 of [3].

Clearly, the small gain condition $\bar{\gamma}_{2i} \circ \bar{\gamma}_{1i}(s) < s$ for $s > 0$ and $i = 1, 2$ of Proposition 2.1 of [3] is guaranteed according to (10) with sufficiently small $\lambda, |k|$.

From (14), (16), (23), (11), and by Proposition 2.1 of [3],

$$\begin{aligned}\|y_{11}\|_\infty \le \max\{&\bar{\gamma}_1^0(\|\xi(0)\|), (\bar{\gamma}_1^0(\|\xi(0)\|))^{\frac{1}{p}}, \\ &2^{-\frac{1}{p}}6^{-1}(\bar{\gamma}_2^0(|z(0)|))^{\frac{1}{p}}, (\tfrac{\|\bar{u}\|_\infty}{|k|})^{\frac{1}{p}}\},\end{aligned} \quad (25)$$

$$\begin{aligned}\|y_{12}\|_\infty \le \max\{&\bar{\gamma}_1^0(\|\xi(0)\|), (\bar{\gamma}_1^0(\|\xi(0)\|))^p, \\ &2^{-1}6^{-p}\bar{\gamma}_2^0(|z(0)|), \tfrac{\|\bar{u}\|_\infty}{|k|}\},\end{aligned} \quad (26)$$

$$\begin{aligned}\|y_2\|_\infty \le \max\{&2 \cdot 6^p\bar{\gamma}_1^0(\|\xi(0)\|), \\ &2 \cdot 6^p(\bar{\gamma}_1^0(\|\xi(0)\|))^p, \bar{\gamma}_2^0(|z(0)|), 2 \cdot 6^p\tfrac{\|\bar{u}\|_\infty}{|k|}\},\end{aligned} \quad (27)$$

for all $z(0) \in Z = \{z \in \mathbb{R} : \bar{\gamma}_2^0(|z|) < \frac{2\cdot 6^p \lambda}{3|k|}\}$, $\xi(0) \in \hat{\Xi} = \{\xi \in \Xi : \max\{\bar{\gamma}_1^0(\|\xi\|), (\bar{\gamma}_1^0(\|\xi\|))^p\} < \frac{\lambda}{3|k|}\}$, $d \in \mathcal{D}$ and $\|\bar{u}\|_\infty < \frac{\lambda}{3}$.

Next consider Proposition 2.2 of [3]. First note that the solution of the interconnected system exists for all $t \geq 0$ using the same argument as that in Lemma 3.5 of [12]. Then $\bar{\Delta}_1 = \infty$ implies that the first condition of Proposition 2.2 of [3] is satisfied. To check the second condition, note that $\lim_{s\to\infty} \bar{\gamma}_{11}(s) = \bar{\gamma}_{11}(\frac{\lambda}{|k|}) < \infty$, $\lim_{s\to\infty} \bar{\gamma}_{12}(s) = \bar{\gamma}_{12}(\frac{\lambda}{|k|}) < \infty$ and $\bar{\Delta}_{21} = \infty$, $\bar{\Delta}_{22} = \frac{\lambda}{3|k|}$. Then we only need to check $\lim_{s\to\infty} \bar{\gamma}_{12}(s) \leq \frac{\lambda}{3|k|}$. From (10), we have

$$\lim_{s\to\infty} \bar{\gamma}_{12}(s) = \bar{\gamma}_{12}(\tfrac{\lambda}{|k|}) < \tfrac{\lambda}{2\cdot 6^p|k|} < \tfrac{\lambda}{3|k|}$$

Finally, note that the small gain condition, i.e., $\bar{\gamma}_{2i} \circ \bar{\gamma}_{1i}(s) < s$, for all $s > 0$ and $i = 1, 2$, is also guaranteed due to (10) with sufficiently small $\lambda, |k|$.

From (15), (17), (24), (11), and by Proposition 2.2 of [3],

$$\|y_{11}\|_a \leq (\tfrac{\|\bar{u}\|_a}{|k|})^{\frac{1}{p}}, \ \|y_{12}\|_a \leq \tfrac{\|\bar{u}\|_a}{|k|} \tag{28}$$

$$\|y_2\|_a \leq 2\cdot 6^p \tfrac{\|\bar{u}\|_a}{|k|} \tag{29}$$

for all $z(0) \in \mathbb{R}$, $\xi(0) \in \mathbb{R}^{n_\xi}$, $d \in \mathcal{D}$ and piecewise continuous $\bar{u}$.

Let $\tilde{\gamma}^0(s) = \max\{\gamma_1^0(s), \gamma_2^0(s), 3\bar{\gamma}_1^0(s), 3(\bar{\gamma}_1^0(s))^{\frac{1}{p}}, 2^{-1-\frac{1}{p}} (\bar{\gamma}_2^0(s))^{\frac{1}{p}}, \gamma(k\bar{\gamma}_2^0(s)), \gamma(2k6^p(\bar{\gamma}_1^0(s))^p), \gamma(2k6^p\bar{\gamma}_1^0(s))\}$. For all $(z(0), \xi(0)) \in Z \times \hat{\Xi}$, $d \in \mathcal{D}$ and $\|\bar{u}\|_\infty < \frac{\lambda}{3}$, (26) implies $\|y_{12}\|_\infty < \frac{\lambda}{3|k|}$. Using (25), (26), (21) and (27), (12), yields

$$\|z\|_\infty \leq \max\{\tilde{\gamma}^0(\|(z(0), \xi(0))\|), 3(\tfrac{\|\bar{u}\|_\infty}{|k|})^{\frac{1}{p}}\}$$
$$\|\xi\|_\infty \leq \max\{\tilde{\gamma}^0(\|(z(0), \xi(0))\|), \gamma(2\cdot 6^p\|\bar{u}\|_\infty)\}$$

for all $(z(0), \xi(0)) \in Z \times \hat{\Xi}$, $d \in \mathcal{D}$ and $\|\bar{u}\|_\infty < \frac{\lambda}{3}$.

Then for all $(z(0), \xi(0)) \in \mathbb{R} \times \mathbb{R}^{n_\xi}$, $d \in \mathcal{D}$ and piecewise continuous $\bar{u}$ satisfying $\|\bar{u}\|_a \leq \frac{\lambda}{3}$, (28) implies $\|y_{12}\|_a \leq \frac{\lambda}{3|k|}$. Using (28), (22) and (29), (13), yields

$$\|z\|_a \leq 3(\frac{\|\bar{u}\|_a}{|k|})^{\frac{1}{p}}, \ \|\xi\|_a \leq \gamma(2\cdot 6^p\|\bar{u}\|_a)$$

for all $(z(0), \xi(0)) \in \mathbb{R} \times \mathbb{R}^{n_\xi}$, $d \in \mathcal{D}$ and piecewise continuous $\bar{u}$ satisfying $\|\bar{u}\|_a \leq \frac{\lambda}{3}$.

*Remark 3.3:* Let $x = z + H(d)\xi = z + y_{11}$. Noting $|x| \leq 2\max\{|z|, |y_{11}|\}$, $u = -\sigma(k(z + H(d)\xi)^p - \bar{u}) = -\sigma(kv_1) = -\sigma(ky_2)$ and equations (25) to (29), yields that $x, u$ satisfy LB with restriction and AB with no restriction on $(z(0), \xi(0))$, both with restriction $\frac{\lambda}{3}$ on $\bar{u}$ and gains $6(\frac{s}{|k|})^{\frac{1}{p}}, 2\cdot 6^p \cdot Id$ respectively. Similarly, assume $y = h(\xi, u, d)$ is an output of $\xi$ subsystem, and satisfies LB with restriction and AB with no restriction on $\xi(0)$, both with restriction $\Delta$ on $u$ and gain $\tilde{\gamma}$. Then $y$ satisfies LB with restriction and AB with no restriction on $(z(0), \xi(0))$, both with restriction $\frac{\lambda}{3}$ on $\bar{u}$ and gain $\tilde{\gamma}(2\cdot 6^p s)$. These facts will be used in the proof of Theorem 4.2.

## IV. Nested Saturation Controller Design

We first design a nested saturation controller for system (1) and then show how to redesign the controller when the input unmodeled dynamics (2) is present.

Let us make the following assumption.

*Assumption 4.1:* Assume $p_1, ..., p_{n-1}$ are odd positive integers satisfying $p_1 \leq p_2 \leq ... \leq p_{n-1}$ and $c_1(d) = \frac{\partial g_1}{\partial v}|_{(0,d)}$, $c_i(d) = \frac{\partial g_i}{\partial x_{i-1}^{p_{i-1}}}|_{(0,...,0,d)}$, $i = 2, ..., n$, are nonzero and do not change their signs for all $d \in \mathcal{D}$.

As a result of this assumption, system (1) can be rewritten in the following form:

$$\begin{aligned} \dot{x}_i &= c_i x_{i-1}^{p_{i-1}} + g_i^r(x_{i-1}^{p_{i-1}}, ..., x_1^{p_1}, v, d), \ i = n, ..., 2 \\ \dot{x}_1 &= c_1 v + g_1^r(v, d) \end{aligned} \tag{30}$$

where $g_i^r$, $i = 1, ..., n$, are suitably defined functions vanishing at $(0, ..., 0, d)$, and for simplicity, we have dropped the argument $d$ in $c_i$, $i = 1, ..., n$.

*Theorem 4.1:* Consider system (30). Under Assumption 4.1, there exist $\lambda_i > 0$ and nonzero $k_i$ with the same sign as $\theta_i$ where $\theta_1 = c_1$ and $\theta_i = c_i/k_{i-1}$, $i = 2, ..., n$, such that under the control

$$v = -\sigma_1(k_1 x_1^{p_1} + ... + \sigma_n(k_n x_n^{p_n} - u_n)) \tag{31}$$

where, for $i = 1, ..., n$, $\sigma_i$ is a saturation function with level $\lambda_i$ and $p_n \geq p_{n-1}$ is an odd positive integer, the closed-loop system satisfies LB with restriction and AB with no restriction on $(x_n(0), ..., x_1(0))$, both with restriction on $u_n$. In particular, when $u_n = 0$, the closed-loop system at $(0, ..., 0, d)$ is globally asymptotically stable. ∎

*Proof:* The proof is a special case of the proof of Theorem 4.2 with $v = u$ and $n_{\xi_1} = 0$, i.e., when the input unmodeled dynamics (2) is not present.

As will be shown in Section V, the control law (31) can be destabilizing when the input unmodeled dynamics (2) is present. So we have to *redesign*. Let us make the following assumption on the input unmodeled dynamics (2).

*Assumption 4.2:* $\xi_1$ subsystem satisfies an a-$\mathcal{L}_\infty$ stability bound (a-LB, see [12]) with no restriction on $\xi_1(0)$, restriction $\Delta_1$ on $u$ and gain $\bar{N}_1 \cdot Id$, and moreover, the linearization of $\xi_1$ subsystem at $(0, 0, d)$, $A_1(d) = \frac{\partial q}{\partial \xi_1}|_{(0,0,d)}, B_1(d) = \frac{\partial q}{\partial u}|_{(0,0,d)}, D_1(d) = \frac{\partial p}{\partial \xi_1}|_{(0,0,d)}, e_1(d) = \frac{\partial p}{\partial u}|_{(0,0,d)}$ is such that $e_1(d) - D_1(d)A_1^{-1}(d)B_1(d)$ is nonzero and does not change its sign for all $d \in \mathcal{D}$.

To simplify the notation, we drop the argument $d$ in the above defined matrices and numbers. Then, system (1) subject to (2) can be written as follows:

$$\begin{aligned} \dot{x}_i &= c_i x_{i-1}^{p_{i-1}} + f_i(x_{i-1}^{p_{i-1}}, ..., x_1^{p_1}, \xi_1, u, d), \ i = n, ..., 2 \\ \dot{x}_1 &= c_1 D_1 \xi_1 + c_1 e_1 u + f_1(\xi_1, u, d) \\ \dot{\xi}_1 &= A_1\xi_1 + B_1 u + f_0(\xi_1, u, d) \end{aligned} \tag{32}$$

where $f_0(\xi_1, u, d) = q(\xi_1, u, d) - A_1\xi_1 - B_1 u$, $f_1(\xi_1, u, d) = g_1^r(p(\xi_1, u, d), d) + c_1(p(\xi_1, u, d) - D_1\xi_1 - e_1 u)$, and $f_i(x_{i-1}^{p_{i-1}}, ..., x_1^{p_1}, \xi_1, u, d) = g_i^r(x_{i-1}^{p_{i-1}}, ..., x_1^{p_1}, p(\xi_1, u, d), d)$, $i = 2, ..., n$.

*Theorem 4.2:* Consider system (32). Under Assumptions 4.1-4.2, there exist $\lambda_i > 0$ and nonzero $k_i$ with the same sign as $\theta_i$ where $\theta_1 = c_1(e_1 - D_1 A_1^{-1} B_1)$ and $\theta_i = c_i/k_{i-1},\ i = 2, ..., n$, such that under the control

$$u = -\sigma_1(k_1 x_1^{p_1} + ... + \sigma_n(k_n x_n^{p_n} - u_n)) \quad (33)$$

where for $i = 1, ..., n$, $\sigma_i$ is a saturation function with level $\lambda_i$, and $p_n \geq p_{n-1}$ is an odd positive integer, the closed-loop system satisfies LB with restriction and AB with no restriction on $(x_n(0), ..., x_1(0), \xi_1(0))$, both with restriction on $u_n$. In particular, when $u_n = 0$, the closed-loop system at $(0, ..., 0, d)$ is globally asymptotically stable. ∎

*Proof:* The proof is omitted due to the space limit.

*Remark 4.1:* Like [1], our design does not require the detailed knowledge of the input unmodeled dynamics either, and the *redesigned* control can guarantee the global robust stability against a class of input unmodeled dynamics characterized by the bounds on $\mathcal{D}$, $c_i,\ i = 1, ..., n$ and $e_1 - D_1 A_1^{-1} B_1$ respectively, and the upper bounds on $\bar{N}_1$, $\Delta_1$, $\|D_1 A_1^{-1}\|$, and $f_0(\xi_1, u, d) - c_1 D_1 A_1^{-1} f_1(\xi_1, u, d)$ respectively.

## V. An Example

Consider the system

$$\begin{aligned} \dot{x}_2 &= x_1^3 + av \\ \dot{x}_1 &= v \end{aligned} \quad (34)$$

where $a$ is a positive real number. Clearly, system (34) satisfies Assumption 4.1 although it does not satisfy the assumptions needed in [10], [13], [14]. When $a = 2^{-5}$, we can design the following nested saturation control law:

$$v = -\sigma_1(0.24 x_1^3 + \sigma_2(6.10 \times 10^{-5} x_2^3)) \quad (35)$$

where $\sigma_1, \sigma_2$ are saturation functions with level 10 and 2.5 respectively.

However, for system (34), in the presence of the input unmodeled dynamics

$$v(s) = \frac{s-1}{s+1} u(s) \quad (36)$$

and under the control

$$u = -\sigma_1(k_1 x_1^3 + \sigma_2(k_2 x_2^3))$$

where $\sigma_1, \sigma_2$ are saturation functions with arbitrary level $\lambda_1, \lambda_2$ respectively, and $k_1, k_2$ are arbitrary positive real numbers, the resulting closed-loop system is not only unstable at the origin, but also has unbounded solutions.

Note that the state space equation of (36) is

$$\dot{\xi}_1 = -\xi_1 + u, v = -2\xi_1 + u$$

then the resulting closed-loop system takes the following form

$$\begin{aligned} \dot{x}_2 &= x_1^3 + a(-2\xi_1 - \sigma_1(k_1 x_1^3 + \sigma_2(k_2 x_2^3))) \\ \dot{x}_1 &= -2\xi_1 - \sigma_1(k_1 x_1^3 + \sigma_2(k_2 x_2^3)) \\ \dot{\xi}_1 &= -\xi_1 - \sigma_1(k_1 x_1^3 + \sigma_2(k_2 x_2^3)) \end{aligned} \quad (37)$$

Suppose $x_1(0), x_2(0)$ are any positive real numbers such that $\sigma_1(\cdot)$ is saturated, i.e., $\sigma_1(k_1 x_1(0)^3 + \sigma_2(k_2 x_2(0)^3)) = \lambda_1 > 0$. Let $\xi_1(0) = -\lambda_1$. Then, we have $\xi_1(t) = -\lambda_1$ for all $t \geq 0$ and $x_1(t), x_2(t)$ are strictly increasing and diverge to infinity as $t \to \infty$, because $\dot{x}_1(t) = \lambda_1 > 0, \dot{x}_2(t) = x_1(t)^3 + a\lambda_1 > 0$ for all $t \geq 0$.

To show the instability of the origin of (37), let $\phi = (\phi_2, \phi_1)$ where $\phi_2 = x_2 - ax_1$ and $\phi_1 = x_1 - 2\xi_1$. Then, system (37) can be written as follows:

$$\begin{aligned} \dot{\phi} &= f(\phi, \xi_1) = \begin{bmatrix} \bar{f}_2(\phi, \xi_1) \\ \bar{f}_1(\phi, \xi_1) \end{bmatrix} \\ \dot{\xi}_1 &= -\xi_1 + g(\phi, \xi_1) \end{aligned} \quad (38)$$

where $\bar{f}_2(\phi, \xi_1) = (\phi_1 + 2\xi_1)^3$ and $\bar{f}_1(\phi, \xi_1) = \sigma_1(k_1(\phi_1 + 2\xi_1)^3 + \sigma_2(k_2(a\phi_1 + \phi_2 + 2a\xi_1)^3))$, and $g(\phi, \xi_1) = -\sigma_1(k_1(\phi_1 + 2\xi_1)^3 + \sigma_2(k_2(a\phi_1 + \phi_2 + 2a\xi_1)^3))$.

By the property of the saturation function and the Local Center Manifold Theorem [11] (see also [2]), there exists a local center manifold $\xi_1 = h(\phi)$ for sufficiently small $\phi$, where $h$ is $\mathcal{C}^2$ and $h(0) = 0, h'(0) = 0$. Let

$$\chi(\phi) = -k_1 \phi_1^3 - k_2(a\phi_1 + \phi_2)^3$$

then we have, for sufficiently small $\phi$, that

$$\frac{\partial \chi(\phi)}{\partial \phi} f(\phi, \chi(\phi)) + \chi(\phi) - g(\phi, \chi(\phi)) = O(\|\phi\|^5)$$

Then by Theorem 3 in Chap. 1 of [2], $h(\phi) = \chi(\phi) + O(\|\phi\|^5)$ for sufficiently small $\phi$. In turn, by Theorem 2 in Chap. 1 of [2], we obtain that, the equation which determines the stability of (38) is

$$\begin{aligned} \dot{\phi} &= f(\phi, \chi(\phi) + O(\|\phi\|^5)) \\ &= \begin{bmatrix} \phi_1^3 \\ k_1 \phi_1^3 + k_2(a\phi_1 + \phi_2)^3 \end{bmatrix} + O(\|\phi\|^5) \end{aligned} \quad (39)$$

We further perform the coordinate transform $\varphi_1 = \phi_2, \varphi_2 = \phi_1 - k_1 \phi_2$ on (39) and yield

$$\begin{bmatrix} \dot{\varphi}_2 \\ \dot{\varphi}_1 \end{bmatrix} = \begin{bmatrix} k_2[a\varphi_2 + (ak_1 + 1)\varphi_1]^3 \\ (\varphi_2 + k_1 \varphi_1)^3 \end{bmatrix} + O(\|\varphi\|^5) \quad (40)$$

where $\varphi = (\varphi_2, \varphi_1)$. If we can show the instability of the origin of system (40) for arbitrary positive $a, \lambda_1, \lambda_2, k_1, k_2$, then the origin of (38), and thus the origin of (37), is unstable for arbitrary positive $a, \lambda_1, \lambda_2, k_1, k_2$.

Let $\varphi(t)$ denote the solution of (40) starting from $\varphi(0)$. We will show the instability of the origin of (40) by definition, that is, given some $\varepsilon > 0$, there exist some $\varphi(0)$ (can be arbitrarily small) and a finite $T > 0$ such that $\|\varphi(T)\| \geq \varepsilon$. To show this, note that if $\varphi_i > 0,\ i = 1, 2$, then

$$\begin{aligned} k_2[a\varphi_2 + (ak_1 + 1)\varphi_1]^3 &\geq q_1(\varphi_2 + \varphi_1)^3 \\ &\geq q_1(\varphi_2^2 + \varphi_1^2)^{\frac{3}{2}} = q_1\|\varphi\|^3 \end{aligned}$$

$$(\varphi_2 + k_1\varphi_1)^3 \geq q_2(\varphi_2 + \varphi_1)^3 \geq q_2(\varphi_2^2 + \varphi_1^2)^{\frac{3}{2}} = q_2\|\varphi\|^3$$

where $q_1 = k_2(\min\{a, ak_1 + 1\})^3, q_2 = (\min\{k_1, 1\})^3$. By the definition of $O(\|\varphi\|^5)$, given any $0 < q_3 < \frac{1}{2}\min\{q_1, q_2\}$, there exists $q_4 > 0$ such that $\|O(\|\varphi\|^5)\| \leq$

$q_3\|\varphi\|^3$ for all $\|\varphi\| \le q_4$. Let $\varepsilon = \frac{1}{2}q_4$. Then, for any positive $\varphi_1, \varphi_2$ satisfying $\|\varphi\| \le 2\varepsilon$, we have

$$\dot{\varphi}_1 \ge (q_1 - q_3)\|\varphi\|^3 > 0,\ \dot{\varphi}_2 \ge (q_2 - q_3)\|\varphi\|^3 > 0,$$

which implies that, the solutions $\varphi_1(t), \varphi_2(t)$ starting from any positive $\varphi_1(0), \varphi_2(0)$ satisfying $\|\varphi(0)\| < \varepsilon$, are strictly increasing and there must exist a finite $T > 0$ such that $\|\varphi(T)\| \ge \varepsilon$. Hence, the origin of system (40) is unstable for arbitrary positive $a, \lambda_1, \lambda_2, k_1, k_2$.

As a result, we have to redesign the nested saturation control law by Theorem 4.2. In the presence of (36), system (34) becomes

$$\begin{aligned} \dot{x}_2 &= x_1^3 + a(-2\xi_1 + u) \\ \dot{x}_1 &= -2\xi_1 + u \\ \dot{\xi}_1 &= -\xi_1 + u \end{aligned} \tag{41}$$

It can be verified that, system (41) satisfies the assumptions of Theorem 4.2. When $a = 2^{-5}$, we can design the following nested saturation control law:

$$u = -\sigma_1(-6.5 \times 10^{-4}x_1^3 + \sigma_2(-3.5 \times 10^{-16}x_2^3)) \tag{42}$$

where $\sigma_1, \sigma_2$ are saturation functions with level $1.2247$ and $0.0047$ respectively.

For illustration, Fig. 1 and Fig. 2 show the simulation results with the initial condition $(\xi_1(0), x_1(0), x_2(0)) = (0.1, 0.2, -5)$ for system (34) under the control (35), and for system (34) subject to the input unmodeled dynamics (36) under the control (42), respectively.

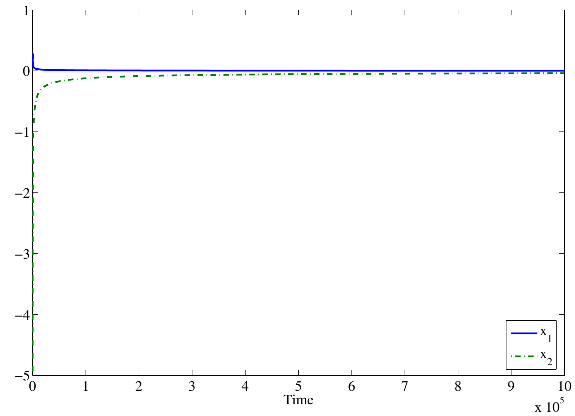


Fig. 1. Simulation results without input unmodeled dynamics

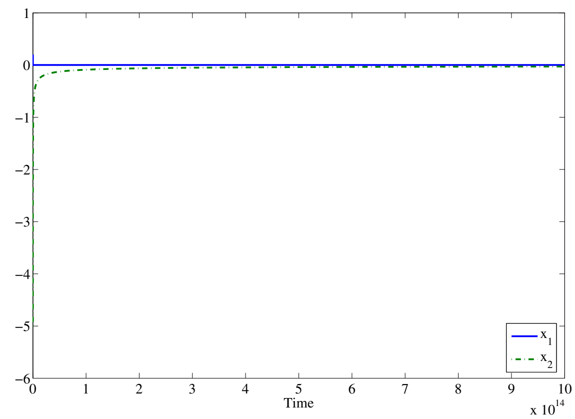


Fig. 2. Simulation results with input unmodeled dynamics

## VI. Conclusion

In this paper, we have addressed the global robust stabilization problem for strict feedforward system (1) subject to some type of input unmodeled dynamics (2). A specific difficulty in dealing with this problem is that the Jacobian linearization of (1) is not stabilizable. We have overcome this difficulty by employing two versions of the small gain theorem adapted from [12] to establish the local stability and global attractiveness of the closed-loop system at the origin respectively.

It is noted that, even in the special case where the input unmodeled dynamics (2) is not present, our result cannot be covered by the existing results in [10], [13], [14] because in this paper the functions $g_i$'s do not have to satisfy some structural constraints needed in [10], [13], [14]. In particular, the functions $g_i$'s are allowed to be linear in its arguments.

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