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Small-Gain Conditions and Lyapunov Functions Applicable Equally to iISS and ISS Systems without Uniformity Assumption

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Abstract— **This paper addresses the problem of verifying stability of the interconnection of integral input-to-state stable (iISS) systems. This paper strengthens a Lyapunov approach to the problem to the point where the necessity of a small-gain condition holds without technical assumptions such as uniformity and analyticity required by previous work. This paper unifies the treatment of iISS and input-to-state stable (ISS) systems, which has not been achieved by any previous technique of constructing Lyapunov functions. It is shown that iISS systems and ISS systems can share a common form of Lyapunov functions to characterize stability of their interconnection with respect to external signals. Global asymptotic stability is also discussed.**

I. Introduction

Input-to-state stability(ISS) is one of the useful classes of dissipative properties of dynamical systems[10]. For instance, the cascade of any two ISS systems is ISS. This idea has been generalized into the ISS small-gain theorem dealing with interconnections of ISS systems[9], [14], [12], [3]. Integral input-to-state stability (iISS) is another important dissipative property which is not necessarily ISS[11], [2]. Recently, a stability criterion covering iISS systems has been developed in [4], [5]. It explicitly constructs Lyapunov functions establishing stability of interconnected systems with external signals. Global asymptotic stability is a special case where no external signals affect systems. The possibility of establishing stability for the interconnection of iISS systems by means of gain conditions is followed up by a nullcline approach[1] in the absence of external signals. Generalizing the result of [1] to the case of external stability with respect to external signals is by no means easy. As a matter of fact, the relationship between the nullcline approach and the Lyapunov constructive approach has not been investigated yet.

This paper continues to pursue nonlinear small-gain techniques for iISS systems. An emphasis is placed on less restrictive construction of Lyapunov functions conforming to the necessity of small-gain technique. This paper accomplishes

A1) deriving *less restrictive Lyapunov functions* from dissipation inequalities of individual subsystems with *non-uniformly contractive loop gain*

A2) *merging small-gain conditions for iISS and ISS systems* into a single condition *without analyticity assumption*

A3) attaining the *necessity* of the small-gain condition *without uniformity assumption*

A4) enabling iISS and ISS systems to be established by *a common form of Lyapunov functions*

A5) providing Lyapunov functions in the situations considered by the nullcline approach to global asymptotic stability in the absence of external signals

Supply rate functions considered in this paper are more general than those covered in [8], [4], [5], [6], [7]. No result has been available for the construction of Lyapunov functions of interconnected iISS systems when nonlinear loop gain is non-uniformly contractive, which allows the loop gain to approach unity asymptotically as magnitude of signals tends to zero and infinity. Some of the results in [5] and their earlier version [4] deal with iISS systems. However, the small-gain criterion proposed there for iISS systems looks more complicated and more restrictive than the small-gain condition for ISS systems. It is should be mentioned that analyticity and smoothness assumptions are introduced in [6] to accomplish simplification of the criterion for iISS systems for the first time. This paper completely merges the small-gain criterion for iISS systems into the small-gain condition of popular ISS-type without introducing any analyticity and smoothness assumptions. The necessity of the nonlinear small-gain condition has been investigated in [6]. However, the uniformity of the small-gain condition is assumed there. This paper removes this assumption. Moreover, this paper resolves the unnatural situation that the aforementioned previous studies employ nonidentical Lyapunov functions for the ISS case and the iISS case. This paper develops a single unified formula for explicitly constructing a Lyapunov function applicable to iISS systems and ISS systems equally. For an interpretation of materials in connection with studies for ISS systems, see [7].

By $\gamma \in \mathscr{P}_0$ this paper means that γ is a continuous function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\gamma(0) = 0$. The set of $\gamma \in \mathscr{P}_0$ satisfying $\gamma(s) > 0$ for all $s \in \mathbb{R}_+ \setminus \{0\}$ is denoted by $\gamma \in \mathscr{P}$. The symbol **id** stands for the identity function. For a function $h \in \mathscr{P}$, we write $h \in \mathscr{O}(>L)$ with a non-negative real number L if there exists a positive real number $K > L$ such that $\limsup_{s\to 0+} h(s)/s^K < \infty$. For $f, g : \mathbb{R}_+ \to \mathbb{R}_+$, we use the simple notation $\lim f(s) = \lim g(s)$ to describe $\{\lim f(s) = \infty \wedge \lim g(s) = \infty\} \vee \{\infty > \lim f(s) = \lim g(s)\}$. Note that the ∞ case is included. In a similar manner, $\lim f(s) \geq \lim g(s)$

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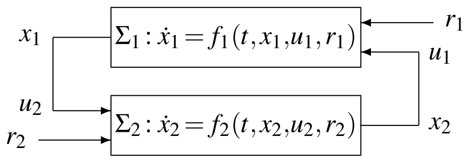

Fig. 1. Interconnected system Σ

denotes $\{\lim f(s) = \infty \ \vee \ \infty > \lim f(s) \geq \lim g(s)\}$. The symbols $\vee$ and $\wedge$ denote logical sum and logical product, respectively. Negation is $\neg$.

II. Illustrative Examples

This section illustrates some of the features listed in the previous section using two examples. Consider the interconnected system shown in Fig.1, and suppose that the systems Σ_1 and Σ_2 satisfy the following dissipation inequalities:

$$\Sigma_i: \ \dot{S}_i(|x_i|) \leq -\alpha_i(|x_i|) + \sigma_i(|x_{3-i}|) + \sigma_{ri}(|r_i|), \ i = 1,2 \quad (1)$$

where the supply rates are given by

$$\alpha_1(s) = \frac{\beta s^2}{s^2+\beta}, \ \sigma_1(s) = s^2, \alpha_2(s) = s^4, \ \sigma_2(s) = \left(\frac{\gamma\beta s^2}{s^2+\beta}\right)^2 \quad (2)$$

for some $\gamma > 0$, $\sigma_{ri} \in \mathscr{P}_0$ and some storage function $S_i \in \mathscr{K}_\infty$, $i = 1,2$. The real number β is supposed to be positive. In the limit case of $\beta = \infty$, the above functions become

$$\alpha_1(s) = s^2, \ \sigma_1(s) = s^2, \quad \alpha_2(s) = s^4, \ \sigma_2(s) = \gamma^2 s^4$$

For each $\beta \in (0, \infty]$, there exist c_1, $c_2 > 1$ such that

$$c_1\sigma_1 \circ \alpha_2^{-1} \circ c_2\sigma_2(s) \leq \alpha_1(s), \quad \forall s \in \mathbb{R}_+ \quad (3)$$

holds if and only if $\gamma < 1$. This paper will show that, for arbitrary pair $\sigma_{r1}, \sigma_{r2} \in \mathscr{P}_0$, the function

$$V_{cl}(x) = \int_0^{S_1(|x_1|)} \lambda_1(s) ds + \int_0^{S_2(|x_2|)} \lambda_2(s) ds \quad (4)$$

establishes iISS of the interconnected system with respect to input (r_1, r_2) and state (x_1, x_2) if $\gamma < 1$. The integrands $\lambda_1, \lambda_2 : \mathbb{R}_+ \to \mathbb{R}_+$ derived by this paper are

$$\lambda_1(s) = \frac{d(c_2-1)}{c_2}\left(\frac{\beta(S_1^{-1}(s))^2}{c_1((S_1^{-1}(s))^2+\beta)}\right)^{K+1} \quad (5)$$

$$\lambda_2(s) = (S_2^{-1}(s))^{2K} \quad (6)$$

$$c_1 = \frac{1}{2}\left(1 + \frac{1}{\sqrt{c_2}\gamma}\right), \ c_2 = \frac{1}{4}\left(1+\frac{1}{\gamma}\right)^2, \ d = \sqrt{\gamma c_1 \sqrt{c_2}} \quad (7)$$

$$q = \frac{\gamma c_1 \sqrt{c_2}}{1 - \gamma c_1 \sqrt{c_2}}\left(\frac{1}{(c_1-1)(c_2-1)} - 1\right) \quad (8)$$

$$K = \begin{cases} 0 & , \ \text{if } (c_1-1)(c_2-1) > 1 \\ \max\{1, q\}, & \text{otherwise} \end{cases} \quad (9)$$

Note that, in the case of $\beta < \infty$, the first system Σ_1 is not necessarily ISS with resect to input x_2 and state x_1, and it is only iISS, while Σ_1 is ISS if $\beta = \infty$. This paper demonstrates that iISS systems and ISS systems can be dealt with equally by a common small-gain condition (3) and a common formula for constructing a Lyapunov function of the closed loop. In fact, the function (4) defined with (5)-(9) functions serves as an iISS Lyapunov function for each $\beta \in (0, \infty]$ whenever $\gamma < 1$ holds. In the case of $\beta = \infty$, it is also an ISS Lyapunov function. This paper will also show that there is always a counterexample when $\gamma \geq 1$ holds in each case of $\beta \in (0, \infty]$. Let differential equation models of Σ_1 and Σ_2 be given by

$$\dot{x}_1 = \frac{8}{3}\left\{\frac{-\beta|x_1|^{\frac{1}{2}}}{|x_1|^2+\beta} + \left(\frac{\beta^3(\tilde{\sigma}_1(|u_1|)+\tilde{\sigma}_{r1}(|r_1|))}{(|x_1|^2+\beta)^3}\right)^{\frac{1}{4}}\right\} x_1 \quad (10)$$

$$x_1 \in \mathbb{R}^2, \ \tilde{\sigma}_1(s) = \begin{cases} 100s^4 & , \quad 0 \leq s < 0.1 \\ s^2 & , \quad 0.1 \leq s \end{cases}$$

$$\dot{x}_2 = \frac{8}{3}\left\{-|x_2|^{\frac{5}{2}} + |x_2|^{\frac{3}{2}}(\tilde{\sigma}_2(|u_2|)+\tilde{\sigma}_{r2}(|r_2|))^{\frac{1}{4}}\right\} x_2 \quad (11)$$

$$x_2 \in \mathbb{R}^2, \ \tilde{\sigma}_2(s) = \begin{cases} \left(\frac{0.01+\beta}{0.01\gamma\beta}\right)^2 \left(\frac{\gamma\beta s^2}{s^2+\beta}\right)^4, & 0 \leq s < 0.1 \\ \left(\frac{\gamma\beta s^2}{s^2+\beta}\right)^2 & , \ 0.1 \leq s \end{cases}$$

$$\tilde{\sigma}_{ri}(s) = \begin{cases} 100s^4 & , \quad 0 \leq s < 0.1 \\ s^2 & , \quad 0.1 \leq s \end{cases} \quad , \ i = 1,2$$

For $S_i(|x_i|) = |x_i|^{3/2}$, $i = 1,2$, these systems satisfy (1) with (2). In the $\beta = \infty$ case, the $\dot{x}_1$-equation and $\tilde{\sigma}_2$ become

$$\dot{x}_1 = \frac{8}{3}\left\{-|x_1|^{\frac{1}{2}} + (\tilde{\sigma}_1(|u_1|)+\tilde{\sigma}_{r1}(|r_1|))^{\frac{1}{4}}\right\} x_1, \ x_1 \in \mathbb{R}^2$$

$$\tilde{\sigma}_2(s) = \begin{cases} 10^4\gamma^2 s^8, & 0 \leq s < 0.1 \\ \gamma^2 s^4 & , \ 0.1 \leq s \end{cases}$$

and therefore fulfill (1). Numerical simulation tells that the interconnection (10) and (11) is globally asymptotically stable at the origin if and only if $\gamma < 1$, which confirms a main theoretical result of this paper. Global asymptotic stability is necessary for iISS and ISS. This paper explicitly gives a pair $\{\Sigma_1, \Sigma_2\}$ achieving the instability of interconnection whenever $\gamma \geq 1$ holds for each $\beta \in (0, \infty]$.

The condition (3) is a special case of

$$\sigma_1 \circ \alpha_2^{-1} \circ \sigma_2(s) < \alpha_1(s), \quad \forall s \in (0, \infty) \quad (12)$$

Note that this condition does not guarantee the existence of c_1, $c_2 > 1$ achieving (3). To illustrate this, consider supply rates in (1) given by the following functions:

$$\alpha_1(s) = \frac{\beta s^2}{s^2+\beta}, \quad \sigma_1(s) = \frac{\beta s^2}{s^2+2\beta}, \quad \sigma_{r1}(s) = 0 \quad (13)$$

$$\alpha_2(s) = \left(\frac{\beta s^2}{s^2+2\beta}\right)^2, \ \sigma_2(s) = \left(\frac{\gamma\beta s^2}{\gamma s^2+\beta}\right)^2, \ \sigma_{r2}(s) = 0 \quad (14)$$

For each $\beta \in (0, \infty]$, these functions fulfill (12) if and only if $\gamma < 1$. However, the inequality (12) is never achieved in a uniform manner. The gap between both the sides of (12) approaches zero as s tends to infinity for all $\gamma < 1$. The condition (3) can be regarded as

$$(\mathbf{id} + \omega_1) \circ \sigma_1 \circ \alpha_2^{-1} \circ (\mathbf{id} + \omega_2) \circ \sigma_2(s) \leq \alpha_1(s), \ \forall s \in \mathbb{R}_+ \quad (15)$$

with $(\mathbf{id} + \omega_i)(s) = c_i s$, $i = 1,2$. The generalized condition (12) allows ω_i(s)'s to be nonlinear. In fact, the inequality

(15) is satisfied for (13) and (14) by the following functions:

$$\omega_i(s) = \tau_i(s) - s, \quad \omega_i \in \mathscr{P}_0, \quad i = 1,2 \tag{16}$$
$$\tau_1(s) = \alpha_1 \circ \chi_1^{-1}(s) > s, \quad \forall s \in (0,\beta)$$
$$\tau_2(s) = [\chi_2 \circ \sigma_2^{-1}(s)]^2 > s, \ \forall s \in (0,\beta^2)$$
$$\tau_1(s) = s, \ \forall s \geq \beta, \quad \tau_2(s) = s, \ \forall s \geq \beta^2$$
$$\chi_i(s) = \alpha_1(s)\sqrt{\frac{s^2+\beta}{s^2+\frac{\beta}{1+\zeta_i(\gamma-1)}}}, \qquad \zeta_1 = 0.99, \quad \zeta_2 = 1$$

According to a main result of this paper, there exists a Lyapunov function verifying global asymptotic stability of the origin if $\gamma < 1$. The Lyapunov function can be obtained in the form of (4) for all $\beta \in (0,\infty]$. The integrands $\lambda_1(s)$ and $\lambda_2(s)$ similar to (5)-(9) are computed by replacing (2) and c_i with (13)-(14) and τ_i. Even for the supply rates (13)-(14) resulting in the non-uniform loop gain, we can find a counterexample whenever $\gamma \geq 1$. A pair $\{\Sigma_1, \Sigma_2\}$ whose interconnection is not globally asymptotically stable can be always explicitly constructed for each $\beta \in (0,\infty]$.

III. Description of Interconnected Systems

Consider the nonlinear interconnected system Σ shown in Fig.1. The subsystems Σ_1 and Σ_2 are connected with each other through $u_1 = x_2$ and $u_2 = x_1$. The state vector of Σ is $x = [x_1^T, x_2^T]^T \in \mathbb{R}^n$. The signals r_1 and r_2 form a vector $r = [r_1^T, r_2^T]^T \in \mathbb{R}^k$. The following sets are considered.

Definition 1: Given $\alpha_i \in \mathscr{P}$, $\sigma_i \in \mathscr{K}$ and $\sigma_{ri} \in \mathscr{P}_0$ for $i = 1,2$, let $\mathscr{S}_{Vi}(n_i, \alpha_i, \sigma_i, \sigma_{ri})$, $i = 1,2$ denote the pair of sets containing systems Σ_i in the form of

$$\dot{x}_i = f_i(t, x_i, u_i, r_i), \ x_i \in \mathbb{R}^{n_i}, u_i \in \mathbb{R}^{m_i}, r_i \in \mathbb{R}^{k_i}, t \in \mathbb{R}_+ \tag{17}$$
$$f_i(t,0,0,0) = 0, \qquad \forall t \in \mathbb{R}_+ \tag{18}$$
$$f_i \text{ is locally Lipschitz in } (x_i, u_i, r_i) \text{ and piecewise continuous in } t \tag{19}$$

for which there exist positive definite and radially unbounded $\mathbf{C}^1$ functions V_i: $\mathbb{R}_+ \times \mathbb{R}^{n_i} \to \mathbb{R}$ such that

$$\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1 \leq -\alpha_1(V_1(t,x_1)) + \sigma_1(V_2(t,x_2)) + \sigma_{r1}(|r_1|) \tag{20}$$
$$\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2} f_2 \leq -\alpha_2(V_2(t,x_2)) + \sigma_2(V_1(t,x_1)) + \sigma_{r2}(|r_2|) \tag{21}$$

hold for all $x_i \in \mathbb{R}^{n_i}$, $r_i \in \mathbb{R}^{k_i}$ and $t \in \mathbb{R}_+$, $i = 1,2$.

The integers m_i's are supposed to satisfy $m_1 = n_2$ and $m_2 = n_1$ so that the interconnection of Σ_1 and Σ_2 makes sense. The Lipschitzness imposed on f_i guarantees the existence of a unique maximal solution of Σ for locally essentially bounded $r_i(t)$. If the exogenous signal r_i is absent, the set of systems is denoted by $\mathscr{S}_{Vi}(n_i, \alpha_i, \sigma_i)$.

The inequalities (20) and (21) are often referred to as dissipation inequalities, and their right hand sides are called supply rates. The function V_i fulfilling the above definition is called a $\mathbf{C}^1$ iISS Lyapunov function[11], [2]. If $\alpha_i \in \mathscr{K}_\infty$ holds, the function V_i is also a $\mathbf{C}^1$ ISS Lyapunov function[13]. Standard definition of iISS and ISS of systems is given in terms of trajectories, which is equivalent to the existence of a $\mathbf{C}^1$ iISS and ISS Lyapunov functions, respectively[2], [13].

Definition 2: Given $\alpha_i \in \mathscr{P}$, $\sigma_i \in \mathscr{K}$, $\sigma_{ri} \in \mathscr{P}_0$ and $\underline{\alpha}_i, \bar{\alpha}_i \in \mathscr{K}_\infty$ for $i = 1,2$, let $\mathscr{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri}, \underline{\alpha}_i, \bar{\alpha}_i)$ denote the set of systems Σ_i of the form (17), (18) and (19) which admit the existence of a $\mathbf{C}^1$ function V_i: $\mathbb{R}_+ \times \mathbb{R}^{n_i} \to \mathbb{R}$ satisfying

$$\underline{\alpha}_i(|x_i|) \leq V_i(t, x_i) \leq \bar{\alpha}_i(|x_i|) \tag{22}$$
$$\frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial x_i} f_i \leq -\alpha_i(|x_i|) + \sigma_i(|u_i|) + \sigma_{ri}(|r_i|) \tag{23}$$

for all $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, $r_i \in \mathbb{R}^{k_i}$ and $t \in \mathbb{R}_+$, $i = 1,2$.

Definition 3: Let $\mathscr{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri})$ denote the set of all $\mathscr{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri}, \underline{\alpha}_i, \bar{\alpha}_i)$'s defined with arbitrary $\underline{\alpha}_i, \bar{\alpha}_i \in \mathscr{K}_\infty$.

As we will see in the sequel, for the set $\mathscr{S}_{Vi}(n_i, \alpha_i, \sigma_i, \sigma_{ri})$ which does not need (22), stability criteria become simpler than those for $\mathscr{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri})$ and $\mathscr{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri}, \underline{\alpha}_i, \bar{\alpha}_i)$. The definition of the latter involves two functions $|\cdot|$ and $V_i(\cdot)$ to measure the magnitude of feedback signals. The third set $\mathscr{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri})$ naturally generalizes the notion of $\mathscr{L}^p$ gain. The set $\mathscr{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri}, \underline{\alpha}_i, \bar{\alpha}_i)$ has information $\underline{\alpha}_i$, $\bar{\alpha}_i$ on the discrepancy between $|\cdot|$ and $V_i(\cdot)$, which is essential to the analysis of global asymptotic stability.

IV. Small-Gain Stability Criteria

This section describes the big picture of results. The following provides a necessary and sufficient condition for a stability property of a set of interconnected iISS systems.

Theorem 1: Let n_i be a positive integer for each $i = 1,2$. Assume that $\alpha_1, \alpha_2, \sigma_1, \sigma_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are $\mathbf{C}^1$ and satisfy

$$\alpha_1, \alpha_2 \in \mathscr{O}(>1), \ \sigma_1, \sigma_2 \in \mathscr{O}(>0) \tag{24}$$
$$\alpha_1, \alpha_2 \in \mathscr{K}, \tag{25}$$
$$\lim_{s\to\infty} \alpha_{3-j}(s) \geq \lim_{s\to\infty} \sigma_{3-j}(s) \tag{26}$$

and one of the following conditions

(B1) $\lim_{s\to\infty} \alpha_{3-j}(s) = \lim_{s\to\infty} \sigma_{3-j}(s)$

(B2) $\lim_{s\to\infty} \dfrac{\sigma_j \circ \alpha_{3-j}^{-1} \circ \sigma_{3-j}(s)}{\alpha_j(s)} \neq 1$

(B3) $\exists s \in (0,\infty)$ s.t. $\dfrac{\sigma_j \circ \alpha_{3-j}^{-1} \circ \sigma_{3-j}(s)}{\alpha_j(s)} \geq 1$

for "an" integer $j \in \{1,2\}$. Then, the equilibrium $x = 0$ of the interconnected system Σ without external signals is uniformly globally asymptotically stable(UGAS) for all pairs $\Sigma_i \in \mathscr{S}_{Vi}(n_i, \alpha_i, \sigma_i)$, $i = 1,2$ if and only if

$$\sigma_j \circ \alpha_{3-j}^{-1} \circ \sigma_{3-j}(s) < \alpha_j(s), \quad \forall s \in (0,\infty) \tag{27}$$

holds for "that" j. Furthermore, a Lyapunov function of Σ characterizing the UGAS is given as

$$V_{cl}(t,x) = \int_0^{V_1(t,x_1)} \lambda_1(s)ds + \int_0^{V_2(t,x_2)} \lambda_2(s)ds \tag{28}$$

for some non-decreasing continuous functions $\lambda_1, \lambda_2 : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\lambda_1(s) > 0, \ \lambda_2(s) > 0, \quad s \in (0,\infty) \tag{29}$$

It is emphasized that j in (27) is the same as in (26), and one of *(B1)-(B3)*. It is also stressed that *(B1)-(B3)* are not simultaneous constraints. Only one of them is required to hold. Theorem 1 includes *(B3)* to demonstrate that (27) is proved to be necessary in the set which does not require (27).

When (27) is violated, there always exists a pair of Σ_i, $i=1,2$ such that their interconnection is not UGAS at the origin. The properties (25) and (26) are assumed before presenting (27) for only simplicity of expressions. It can be verified that the existence of such functions is also necessary for UGAS. Let the inequality (27) be referred to as a small-gain condition. It is worth noting that the sufficiency of the small-gain condition for UGAS requires neither the smoothness nor (24) of α_i and σ_i. The UGAS in Theorem 1 is derived from

$$\exists \alpha_{cl} \in \mathscr{P} \quad \text{s.t.} \quad \dot{V}_{cl}(t,x) \leq -\alpha_{cl}(|x|), \quad \forall x \in \mathbb{R}^n \qquad (30)$$

satisfied along the trajectories of Σ with $r_i(t) \equiv 0$, $i=1,2$.

One can obtain iISS of a set of interconnected systems if amplification factors ω_i, $i=1,2$, are introduced to the small-gain condition. A stronger property, ISS, is a special case.

Theorem 2: Assume that functions α_1, α_2, σ_1, σ_2, σ_{r1}, σ_{r2}, $\underline{\alpha}_1$, $\underline{\alpha}_2$, $\bar{\alpha}_1$, $\bar{\alpha}_2 : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy (25),

$$\lim_{s\to\infty} \alpha_{3-j}(s) = \infty \;\vee\; \lim_{s\to\infty} \alpha_{3-j}(s) > \lim_{s\to\infty} \sigma_{3-j}(s) \qquad (31)$$

and one of the following conditions

(C1) $\lim_{s\to\infty} \alpha_1(s) = \infty \quad \wedge \quad \lim_{s\to\infty} \alpha_2(s) = \infty$

(C2) $\lim_{s\to\infty} \alpha_{3-j}(s) = \infty \quad \wedge \quad \lim_{s\to\infty} \sigma_{3-j}(s) < \infty$

(C3) $\lim_{s\to\infty} \sigma_j(s) < \infty \quad \wedge \quad \lim_{s\to\infty} \sigma_{3-j}(s) < \infty$

for "an" integer $j \in \{1,2\}$. Then, the interconnected system Σ is iISS with respect to input r and state x for all pairs $\mathscr{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri}, \underline{\alpha}_i, \bar{\alpha}_i)$ with any positive integer n_i, $i=1,2$ if there exist $\omega_i \in \mathscr{K}_\infty$, $i=1,2$ such that

$$(\mathbf{id}+\omega_j)\circ\sigma_j\circ\underline{\alpha}_{3-j}^{-1}\circ\bar{\alpha}_{3-j}\circ\alpha_{3-j}^{-1}\circ(\mathbf{id}+\omega_{3-j})\circ\sigma_{3-j}(s) \leq \alpha_j\circ\bar{\alpha}_j^{-1}\circ\underline{\alpha}_j(s), \; \forall s \in \mathbb{R}_+ \quad (32)$$

holds for "that" j. Furthermore, an iISS Lyapunov function of Σ is given as (28) for some non-decreasing continuous functions $\lambda_1, \lambda_2 : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (29). In the case of *(C1)*, the function V_{cl} is also an ISS Lyapunov function.

It is stressed that j in (32) is the same as in (31), and one of *(C1)-(C3)*.

Theorem 3: Let n_i be a positive integer for $i=1,2$. Assume that $\alpha_i, \sigma_i, \sigma_{ri} : \mathbb{R}_+ \to \mathbb{R}_+$, $i=1,2$ satisfy (24), (25), *(C1)* and

$$\sigma_{ri} \in \mathscr{K}_\infty, \quad \sigma_{ri} \in \mathscr{O}(>0), \quad i=1,2 \qquad (33)$$

Then, the interconnected system Σ is ISS with respect to input r and state x for all pairs $\mathscr{S}_{Vi}(n_i, \alpha_i, \sigma_i, \sigma_{ri})$, $i=1,2$ if and only if there exist $\omega_i \in \mathscr{K}_\infty$, $i=1,2$ such that

$$(\mathbf{id}+\omega_1)\circ\sigma_1\circ\alpha_2^{-1}\circ(\mathbf{id}+\omega_2)\circ\sigma_2(s) \leq \alpha_1(s), \forall s \in \mathbb{R}_+ \quad (34)$$

holds. Furthermore, an ISS Lyapunov function of Σ is given as (28) for some non-decreasing continuous functions $\lambda_1, \lambda_2 : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (29).

It can be verified that

$$\textit{(B1)} \vee \textit{(B2)} \Leftarrow \textit{(C1)} \vee \textit{(C2)} \vee \textit{(C3)} \Leftarrow \textit{(C1)}$$

holds under the assumption that there exist $\omega_i \in \mathscr{K}_\infty$, $i=1,2$ such that (32) holds. The statement about a Lyapunov function in Theorem 2 claims that along Σ we have

$$\exists \alpha_{cl} \in \mathscr{P}, \; \sigma_{cl} \in \mathscr{P}_0 \quad \text{s.t.} \quad \dot{V}_{cl}(t,x) \leq -\alpha_{cl}(|x|) + \sigma_{cl}(|r|), \; \forall x \in \mathbb{R}^n, r \in \mathbb{R}^k \quad (35)$$

Theorem 3 indicates $\alpha_{cl} \in \mathscr{K}_\infty$. Theorem 3 covers the worst case where (33) holds. If the exogenous signals affect systems through sufficiently small $\sigma_{ri} \notin \mathscr{K}_\infty$, the condition (34) is not always required, while (27) with $j=1$ is necessary.

The functions $\lambda_1(s)$ and $\lambda_2(s)$ for the Lyapunov function V_{cl} of the overall system are shown explicitly in Section V. There is a single pair $\{\lambda_1, \lambda_2\}$ applicable to all the theorems given above. Proof of the theorems consists of sufficiency and necessity parts. The sufficiency is the topic of Section V. Although detailed discussion on the necessity is omitted due to the space limitation, the computation of destabilizing systems is exactly the same as in [6].

It is stressed that $j=1$ and $j=2$ in (32) are not equivalent cases in general. They become equivalent if *(C1)* holds. The same remark applies to (27) and (34). In Theorem 1 and Theorem 3, the pair $\mathscr{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri})$, $i=1,2$ can replace $\mathscr{S}_{Vi}(n_i, \alpha_i, \sigma_i, \sigma_{ri})$, $i=1,2$ if $\underline{\alpha}_i = \bar{\alpha}_i$ holds for $i=1,2$.

Remark 1: In the case of *(C1)*, the inequalities (32) and (34) are identical with the nonlinear small-gain condition proposed by [9] which was originally a sufficient condition for stability of interconnected ISS systems. The slight difference between (32) and (34) arises from the difference of $\mathscr{S}_i$ and $\mathscr{S}_{Vi}$ as mentioned at the end of Section III. This paper deals with interconnected iISS systems and it includes the nonlinear small-gain theorem presented in [9] as a special case, i.e, the sufficiency part of Theorem 3. In contrast to an earlier result of the first author[5] on iISS systems, theorems in this paper do not require the small-gain conditions to be uniform contraction over $\mathbb{R}_+$. Linearity is not imposed on the amplification factors $\omega_i(s)$, $i=1,2$, in the small-gain conditions of (32) and (34). This comment also applies to (27) in view of the existence of ω_i. This paper demonstrates that the uniformity assumption on the small-gain conditions in [5] can be removed in the explicit construction of Lyapunov functions. Another novelty of this paper is unification of Lyapunov functions for iISS and ISS systems, which is presented in detail in Section V. The previous study[5] employs a Lyapunov function for iISS systems which is different from that for ISS systems. This paper merges those two Lyapunov functions into a single representation and takes non-linear $\omega_i(s)$'s into account. Furthermore, the small-gain condition (32) is simpler and less restrictive than the small-gain condition proposed in [5]. In fact, this paper proves that the single small-gain condition (32) applies to ISS systems and iISS systems equally. The fulfillment of the small-gain condition in [5] implies (32). The unification of Lyapunov functions and small-gain conditions allows the iISS result to include the ISS result totally as a special case. Moreover, both Theorem 1 and Theorem 2 cover larger classes of α_i and σ_i than [5].

Remark 2: The constraint *(B1)*$\vee$*(B2)*$\vee$*(B3)* is not a technical assumption. The assumption cannot be eliminated from Theorem 1. Consider the following class $\mathscr{K}$ functions α_i:

$$\alpha_1(s) = \frac{s}{1+s}, \;\; \sigma_1(s) = s, \;\; \sigma_2(s) = \alpha_2\left(\frac{s^2}{1+s+s^2}\right) \qquad (36)$$

$$\alpha_2(s) = \begin{cases} 2^{3-p}(p-2)s^3 + 2^{2-p}(3-p)s^2 & , s \in [0, 1/2) \\ s^p & , s \in [1/2, \infty) \end{cases} \qquad (37)$$

These functions are $\mathbf{C}^1$ on $[0,\infty)$, and satisfy $\alpha_i \in \mathscr{O}(>1)$ and $\sigma_i \in \mathscr{O}(>0)$ for $0<p\le 3$. In the absence of r_1 and r_2, a pair $\{\Sigma_1,\Sigma_2\}$ satisfying (20) and (21) is

$$\dot{x}_1 = -\alpha_1(x_1)+\sigma_1(x_2), \quad (x_1(0),x_2(0))\in\mathbb{R}^2_+ \quad (38)$$
$$\dot{x}_2 = -\alpha_2(x_2)+\sigma_2(x_1), \quad V_1=x_1, \quad V_2=x_2 \quad (39)$$

defined on $x\in\mathbb{R}^2_+$. This system has unbounded trajectories in the case of $0<p<1$ [1]. However, the functions α_i, σ_i, $i=1,2$, in (36) and (37) satisfy (27) with $j=1$. Indeed, the functions fulfill ¬*(B1)* ∧ ¬*(B2)* ∧ ¬*(B3)*.

V. Construction of Lyapunov Functions

To construct Lyapunov functions, we seek a pair $\{\lambda_1,\lambda_2\}$ so that the composite function V_{cl} in (28) fulfills (30)and (35).

A. Existence and Construction of λ_i

Theorem 4: Suppose that continuous functions $V_i:(t,x_i)\in\mathbb{R}_+\times\mathbb{R}^{n_i}\to\mathbb{R}_+$, $i=1,2$, satisfy (22) for some $\underline{\alpha}_i$, $\bar{\alpha}_i\in\mathscr{K}_\infty$. Let ρ_i, $i=1,2$, be

$$\rho_i(x_i,u_i,r_i)=-\alpha_i(|x_i|)+\sigma_i(|u_i|)+\sigma_{ri}(|r_i|) \quad (40)$$

which consist of $\alpha_1,\alpha_2,\sigma_1,\sigma_2\in\mathscr{K}$ and $\sigma_{r1},\sigma_{r2}\in\mathscr{P}_0$ satisfying one of the following conditions:

(D1) $\lim_{s\to\infty}\alpha_2(s)=\infty \quad\wedge\quad \lim_{s\to\infty}\alpha_1(s)=\infty$
(D2) $\lim_{s\to\infty}\alpha_2(s)=\infty \quad\wedge\quad \lim_{s\to\infty}\sigma_2(s)<\infty$
(D3) $\lim_{s\to\infty}\sigma_1(s)<\infty \quad\wedge\quad \lim_{s\to\infty}\sigma_2(s)<\infty$

If there exist $\omega_i\in\mathscr{K}_\infty$, $i=1,2$ such that

$$(\mathbf{id}+\omega_1)\circ\sigma_1\circ\underline{\alpha}_2^{-1}\circ\bar{\alpha}_2\circ\alpha_2^{-1}\circ(\mathbf{id}+\omega_2)\circ\sigma_2(s) \le \alpha_1\circ\bar{\alpha}_1^{-1}\circ\underline{\alpha}_1(s), \quad \forall s\in\mathbb{R}_+ \quad (41)$$

is satisfied, then the following hold.

(4a) There exist $\hat{\alpha}_1,\hat{\sigma}_1\in\mathscr{K}$ and $\hat{\omega}_1$, $\hat{\omega}_2\in\mathscr{K}_\infty$ such that

$$(\mathbf{id}+\hat{\omega}_1)\circ\hat{\sigma}_1\circ\underline{\alpha}_2^{-1}\circ\bar{\alpha}_2\circ\alpha_2^{-1}\circ(\mathbf{id}+\hat{\omega}_2)\circ\sigma_2(s) \le \hat{\alpha}_1\circ\bar{\alpha}_1^{-1}\circ\underline{\alpha}_1(s), \ \forall s\in\mathbb{R}_+ \quad (42)$$
$$\sigma_1(s)\le\hat{\sigma}_1(s), \ \forall s\in\mathbb{R}_+ \quad (43)$$
$$\hat{\alpha}_1(s)\le\alpha_1(s), \ \forall s\in\mathbb{R}_+ \quad (44)$$
$$\lim_{s\to\infty}\hat{\sigma}_1(s)\ge\lim_{s\to\infty}\hat{\alpha}_1(s) \quad (45)$$
$$\lim_{s\to\infty}\alpha_2(s)=\infty \ \Rightarrow\ \hat{\alpha}_1=\alpha_1 \quad (46)$$
$$\lim_{s\to\infty}\alpha_2(s)<\infty \ \Rightarrow\ \hat{\sigma}_1=\sigma_1 \quad (47)$$

(4b) There exists a continuous function $\rho_e(x,r)$ of the form

$$\rho_e(x,r)=\sum_{i=1}^{2}-\alpha_{cl,i}(|x_i|)+\sigma_{cl,i}(|r_i|) \quad (48)$$
$$\alpha_{cl,1},\alpha_{cl,2}\in\mathscr{K}, \quad \sigma_{cl,1},\sigma_{cl,2}\in\mathscr{P}_0 \quad (49)$$

such that

$$\lambda_1(V_1(t,x_1))\rho_1(x_1,x_2,r_1) + \lambda_2(V_2(t,x_2))\rho_2(x_2,x_1,r_2)\le\rho_e(x,r), \quad \forall x_1\in\mathbb{R}^{n_1},x_2\in\mathbb{R}^{n_2},r_1\in\mathbb{R}^{m_1},r_2\in\mathbb{R}^{m_2},t\in\mathbb{R}_+ \quad (50)$$

holds with

$$\lambda_1(s)=\left[\delta_2\circ\underline{\omega}_2\circ\tau_2^{-1}\circ\alpha_2\circ\bar{\alpha}_2^{-1}\circ\underline{\alpha}_2\circ\hat{\sigma}_1^{-1}\circ\tau_1^{-1}\circ\hat{\alpha}_1\circ\bar{\alpha}_1^{-1}(s)\right]\left[\nu\circ\tau_1^{-1}\circ\hat{\alpha}_1\circ\bar{\alpha}_1^{-1}(s)\right]\left[\psi\circ\tau_1^{-1}\circ\hat{\alpha}_1\circ\bar{\alpha}_1^{-1}(s)\right] \quad (51)$$
$$\lambda_2(s)=\hat{\sigma}_1\circ\underline{\alpha}_2^{-1}(s)\left[\nu\circ\hat{\sigma}_1\circ\underline{\alpha}_2^{-1}(s)\right]\left[\psi\circ\hat{\sigma}_1\circ\underline{\alpha}_2^{-1}(s)\right] \quad (52)$$
$$\tau_i=\mathbf{id}+\underline{\omega}_i, \quad i=1,2 \quad (53)$$

where $\underline{\omega}_i\in\mathscr{K}_\infty$ and $\delta_i\in\mathscr{K}$ are any continuous functions satisfying

$$\underline{\omega}_i(s)\le\hat{\omega}_i(s), \ \forall s\in\mathbb{R}_+, \quad i=1,2 \quad (54)$$
$$\mathbf{id}-\delta_i\in\mathscr{K}_\infty, \quad i=1,2\ , \quad (55)$$

and $\nu,\psi:[0,L)\to\mathbb{R}_+$ are any continuous functions which satisfy

$$0<\nu(s)<\infty, \ 0<\psi(s)<\infty \quad (56)$$
$$L:=\lim_{s\to\infty}\hat{\sigma}_1(s)$$

for all $s\in(0,L)$, and the properties

$$\left[\delta_2\circ\underline{\omega}_2\circ\tau_2^{-1}\circ\alpha_2\circ\bar{\alpha}_2^{-1}\circ\underline{\alpha}_2(s)\right]\left[\nu\circ\hat{\sigma}_1(s)\right]\left[\psi\circ\hat{\sigma}_1(s)\right] \ : \text{non-decreasing} \quad (57)$$
$$\hat{\sigma}_1(s)\left[\nu\circ\hat{\sigma}_1(s)\right]\left[\psi\circ\hat{\sigma}_1(s)\right] \ : \text{non-decreasing} \quad (58)$$
$$\left[\delta_2\circ\underline{\omega}_2\circ\tau_2^{-1}\circ\alpha_2\circ\bar{\alpha}_2^{-1}\circ\underline{\alpha}_2\circ\hat{\sigma}_1^{-1}\circ\tau_1^{-1}\circ\hat{\alpha}_1\circ\bar{\alpha}_1^{-1}(s)\right]\left[\nu\circ\tau_1^{-1}\circ\hat{\alpha}_1\circ\bar{\alpha}_1^{-1}(s)\right] \ : \text{non-decreasing} \quad (59)$$
$$\left[\psi\circ\hat{\sigma}_1\circ\underline{\alpha}_2^{-1}\circ\bar{\alpha}_2\circ\alpha_2^{-1}\circ\tau_2\circ\sigma_2(s)\right]\left[\hat{\sigma}_1\circ\underline{\alpha}_2^{-1}\circ\bar{\alpha}_2\circ\alpha_2^{-1}\circ\tau_2\circ\sigma_2(s)\right]\sigma_2(s) \le\left[\psi\circ\tau_1^{-1}\circ\hat{\alpha}_1\circ\bar{\alpha}_1^{-1}\circ\underline{\alpha}_1(s)\right]\left[\delta_2\circ\underline{\omega}_2\circ\sigma_2(s)\right]\left[\delta_1\circ\underline{\omega}_1\circ\tau_1^{-1}\circ\hat{\alpha}_1\circ\bar{\alpha}_1^{-1}\circ\underline{\alpha}_1(s)\right] \quad (60)$$

for all $s\in\mathbb{R}_+$, and

$$\hat{\alpha}_1,\alpha_2\notin\mathscr{K}_\infty \ \Rightarrow\ \lim_{s\to L}\nu(s)\psi(s)<\infty \quad (61)$$

Furthermore, the inequality (50) is achieved with $\alpha_{cl,i}$ and $\sigma_{cl,i}$ satisfying

$$\textit{(D1)} \ \Rightarrow\ \alpha_{cl,1},\alpha_{cl,2}\in\mathscr{K}_\infty \quad (62)$$
$$\sigma_{r,i}(s)\equiv 0 \ \Rightarrow\ \sigma_{cl,i}(s)\equiv 0 \quad (63)$$
$$\sigma_{r,i}(s)\not\equiv 0 \ \Rightarrow\ \sigma_{cl,i}(s)\in\mathscr{K} \quad (64)$$

for each $i=1,2$.

It is important that there always exist functions ν and ψ fulfilling (56), (57), (58), (59), (60) and (61). The existence and the construction are addressed in Subsection V-B. The task of finding $\{\lambda_1,\lambda_2\}$ which solves (50) is referred to as a state-dependent scaling problem in [5]. When (50) holds, the property (35) leads us to the iISS property of Σ. The above theorem and remaining parts of this section demonstrate that the single pair $\{\lambda_1,\lambda_2\}$ given in (51) and (52) can cover larger classes of $\{\alpha_1,\ \alpha_2,\ \sigma_1,\ \sigma_2,\ \sigma_{r1},\ \sigma_{r2}\}$ than [5].

It is stressed that (41) with ω_1, $\omega_2\in\mathscr{P}$ requires

$$\lim_{s\to\infty}\alpha_2(s)=\infty \quad\vee\quad \lim_{s\to\infty}\alpha_2(s)>\lim_{s\to\infty}\sigma_2(s) \quad (65)$$

The property (45) guarantees $L=\lim_{s\to\infty}\hat{\sigma}_1(s)\ge\lim_{s\to\infty}\tau_1^{-1}\circ\hat{\alpha}_1(s)$ so that functions in (51) make sense.

Remark 3: When ω_1 and ω_2 are restricted to linear functions, the functions in (51) and (52) reduce to ones given in [5] for the ISS case *(D1)*(See also the end of Subsection V-B). In the case of linear ω_i's, the paper [5] also deals with some iISS cases where *(D1)* does not hold. The previous iISS result employs a pair of λ_1 and λ_2 which are different from (51) and (52). The iISS Lyapunov function given in [5] for linear ω_i's can be obtained from (51)-(52) with $\nu(s) = \tilde{\lambda}_2 \circ \underline{\alpha}_2 \circ \hat{\sigma}_1^{-1}(s)/s\psi(s)$, where $\tilde{\lambda}_2$ denotes the function λ_2 derived in [5]. Hence, the pair (51)-(52) includes all solutions given in the previous study [5] and covers non-linear ω_i's. It is also worth mentioning that the small-gain condition (41) is less conservative and simpler than the corresponding condition (27) of [5].

Remark 4: The sum of *(D1)*, *(D2)*, and *(D3)* does not cover

(E1) $\infty = \lim_{s\to\infty} \alpha_2(s) = \lim_{s\to\infty} \sigma_2(s) \ \wedge \ \lim_{s\to\infty} \alpha_1(s) > \lim_{s\to\infty} \sigma_1(s)$

(E2) $\infty > \lim_{s\to\infty} \alpha_2(s) > \lim_{s\to\infty} \sigma_2(s) \ \wedge \ \lim_{s\to\infty} \sigma_1(s) = \infty$

In other words, we have

$$\{\alpha_1, \alpha_2, \sigma_1, \sigma_2 \in \mathscr{K}\} \wedge \{\exists \omega_1, \omega_2 \in \mathscr{P} \text{ s.t. } (41)\} = \textit{(D1)} \vee \textit{(D2)} \vee \textit{(D3)} \vee \textit{(E1)} \vee \textit{(E2)}$$

We can deal with *(E1)* and *(E2)* if influence of exogenous inputs on subsystems is small enough.

The amplification factors ω_1, ω_2 in the small-gain condition (41) can be replaced by a strict inequality sign as far as UGAS is concerned. Defining

(F1) $\lim_{s\to\infty} \frac{\sigma_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ \sigma_2(s)}{\alpha_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)} \le 1 \wedge \lim_{s\to\infty} \alpha_2(s) = \lim_{s\to\infty} \sigma_2(s)$

(F2) $\lim_{s\to\infty} \frac{\sigma_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ \sigma_2(s)}{\alpha_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)} < 1 \wedge \lim_{s\to\infty} \alpha_2(s) > \lim_{s\to\infty} \sigma_2(s)$

for the sake of concise explanation, we obtain the following.

Theorem 5: Suppose that continuous functions $V_i : (t, x_i) \in \mathbb{R}_+ \times \mathbb{R}^{n_i} \to \mathbb{R}_+$, $i = 1, 2$, satisfy (22) for some $\underline{\alpha}_i$, $\bar{\alpha}_i \in \mathscr{K}_\infty$. Let ρ_i, $i = 1, 2$, be

$$\rho_i(x_i, u_i) = -\alpha_i(|x_i|) + \sigma_i(|u_i|) \tag{66}$$

which consist of $\alpha_1, \alpha_2, \sigma_1, \sigma_2 \in \mathscr{K}$ satisfying one of the following conditions:

(G1) $\lim_{s\to\infty} \alpha_2(s) = \lim_{s\to\infty} \sigma_2(s)$

(G2) $\lim_{s\to\infty} \frac{\sigma_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ \sigma_2(s)}{\alpha_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)} < 1$

If

$$\sigma_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ \sigma_2(s) < \alpha_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s), \ \forall s \in (0, \infty) \tag{67}$$

is satisfied, then the following hold.

(5a) In the case of *(F1)*, there exist $\hat{\alpha}_1 \in \mathscr{K}$ and $\hat{\omega}_1$, $\hat{\omega}_2 \in \mathscr{P}_0$ such that (42), (44) and

$$\hat{\omega}_1 \circ \hat{\sigma}_1(s) > 0, \ \hat{\omega}_2 \circ \sigma_2(s) > 0, \quad \forall s \in (0, \infty) \tag{68}$$

$$\lim_{s\to\infty} \frac{\hat{\sigma}_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ \sigma_2(s)}{\hat{\alpha}_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)} = 1 \tag{69}$$

hold with $\hat{\sigma}_1 = \sigma_1$. In the *(F2)* case, *(4a)* holds.

(5b) There exists a continuous function $\rho_e(x)$ of the form

$$\rho_e(x) = -\sum_{i=1}^{2} \alpha_{cl,i}(|x_i|), \quad \alpha_{cl,1}, \alpha_{cl,2} \in \mathscr{P} \tag{70}$$

such that

$$\lambda_1(V_1(t, x_1))\rho_1(x_1, x_2) + \lambda_2(V_2(t, x_2))\rho_2(x_2, x_1) \le \rho_e(x), \quad \forall x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}, t \in \mathbb{R}_+ \tag{71}$$

holds with (51) and (52), where $\underline{\omega}_i \in \mathscr{P}_0$ and $\delta_i \in \mathscr{K}$ are any continuous functions satisfying (54) and

$$\tau_i := \mathbf{id} + \underline{\omega}_i \in \mathscr{K}_\infty, \quad i = 1, 2 \tag{72}$$

$$\underline{\omega}_i \circ \tau_i^{-1} \circ \hat{\alpha}_i - \delta_i \circ \underline{\omega}_i \circ \tau_i^{-1} \circ \hat{\alpha}_i \circ \bar{\alpha}_i^{-1} \circ \underline{\alpha}_i \in \mathscr{P}, \ i = 1, 2 \tag{73}$$

$$\underline{\omega}_1 \circ \hat{\sigma}_1(s) > 0, \ \underline{\omega}_2 \circ \sigma_2(s) > 0, \quad \forall s \in (0, \infty) \tag{74}$$

and $\nu, \psi : [0, L) \to \mathbb{R}_+$ are any continuous functions which satisfy (56) for all $s \in (0, L)$, and the properties (57), (58), (59) and (60) for all $s \in \mathbb{R}_+$. In the case of *(F2)*, $\underline{\omega}_i \in \mathscr{P}_0$ is replaced by $\underline{\omega}_i \in \mathscr{K}_\infty$.

It can be verified that

$$\textit{(F1)} \vee \textit{(F2)} = \textit{(G1)} \vee \textit{(G2)}$$

holds under (67). Note that the achievement of (67) requires

$$\lim_{s\to\infty} \alpha_2(s) \ge \lim_{s\to\infty} \sigma_2(s) \tag{75}$$

Its necessity is proved in [6]. In the case of *(F1)*, the property (69) implies $\hat{\alpha}_1(\infty) = \hat{\sigma}_1(\infty) = L$ and we have $\alpha_2^{-1} \circ \tau_2 \circ \sigma_2 \in \mathscr{K}_\infty$. Note that (74) guarantees

$$\left[\delta_2 \circ \underline{\omega}_2 \circ \tau_2^{-1} \circ \alpha_2 \circ \bar{\alpha}_2^{-1} \circ \underline{\alpha}_2 \circ \hat{\sigma}_1^{-1} \circ \tau_1^{-1} \circ \hat{\alpha}_1 \circ \bar{\alpha}_1^{-1}(s)\right] \in \mathscr{P}$$

since $\tau_2^{-1} \circ \alpha_2 \circ \bar{\alpha}_2^{-1} \circ \underline{\alpha}_2(\infty) = \sigma_2 \circ \underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \hat{\alpha}_1^{-1} \circ \tau_1 \circ \hat{\sigma}_1(\infty)$ follows from (42), (72), (69) and (74).

Remark 5: It is verified that all situations considered in Theorem 1 of [1] with their small-gain condition are covered by Theorem 5. Thus, this paper gives an interpretation of the result of [1] in terms of construction of Lyapunov functions.

Remark 6: Due to (57) and (52), the function V_{cl} constructed with λ_1 and λ_2 for (69) increases toward infinity as x_2 approaches the boundary where $V_2(x_2) = X_2$ holds for $X_2 := \lim_{s\to\infty} \underline{\alpha}_2 \circ \hat{\sigma}_1^{-1} \circ \hat{\alpha}_1(s)$. Thus, the function V_{cl} is not qualified as a Lyapunov function of global asymptotic stability of the interconnected system if X_2 is finite. The assumption $\alpha_2(\infty) = \sigma_2(\infty)$ in *(F1)* guarantees $X_2 = \infty$. The situation (69) with $\alpha_2(\infty) > \sigma_2(\infty)$ is referred to as the no gap case in [1]. The information about gain is not enough to discriminate between stable and unstable behavior in the no gap case as discussed in [1]. Indeed, the example given by (36)-(39) satisfies (69) and $\alpha_2(\infty) > \sigma_2(\infty)$. Although the small-gain condition (67) is fulfilled, there exist unbounded trajectories for $0 < p < 1$. This fact justifies the assumption of *(G1)* $\vee$ *(G2)* in Theorem 5, and this paper gives a Lyapunov interpretation to the no gap case.

B. Existence and Construction of ψ

Once a function ψ is given, we can always select a function ν required in Theorem 4 and Theorem 5 straightforwardly. This section demonstrates that we can always explicitly construct ψ satisfying (56), (60) and (61).

Lemma 1: Suppose that

$$b \in \mathscr{P}, \ d \in \mathscr{K}, \ \eta, \alpha \in \mathscr{K}, \ \omega \in \mathscr{P}_0, \ \tau \in \mathscr{K}_\infty \tag{76}$$

$$(\mathbf{id} + \omega) \circ \eta(s) \le \alpha(s), \ \forall s \in \mathbb{R}_+ \tag{77}$$

hold. Define $S = \lim_{s\to\infty}\eta(s)$ and $R = \lim_{s\to\infty}\tau^{-1}\circ\alpha(s)$. Then, the following properties hold.

(La) Assume that

$$\lim_{s\to\infty}(\mathbf{id}+\omega)\circ\eta(s) = \lim_{s\to\infty}\alpha(s) \tag{78}$$

holds. If τ satisfies

$$s \leq \tau(s), \quad \forall s\in\mathbb{R}_+ \ , \tag{79}$$

$$\tau(s) < s+\omega(s), \quad \forall s\in(0,S) \tag{80}$$

$$\lim_{s\to S}\tau(s) = \lim_{s\to S}\{s+\omega(s)\} \ , \tag{81}$$

then $S = R$ holds and there exists a continuous function ψ satisfying

$$0 < \psi(s) < \infty, \quad \forall s\in(0,R) \tag{82}$$

$$\left[\psi\circ\eta(s)\right]d(s) \leq \left[\psi\circ\tau^{-1}\circ\alpha(s)\right]b(s), \forall s\in(0,\infty) \tag{83}$$

(Lb) Assume that

$$S<\infty, \quad \lim_{s\to\infty}b(s)\neq 0, \quad d\notin\mathcal{K}_\infty \tag{84}$$

hold. If τ satisfies (79) and

$$\tau(s) < s+\omega(s), \quad \forall s\in(0,S] \ , \tag{85}$$

then there exists a continuous function ψ such that

$$0<\psi(s)<\infty, \ \forall s\in(0,\infty), \quad \lim_{s\to\infty}\psi(s)<\infty \tag{86}$$

and (83) hold.

This lemma generalizes Lemma 1 of [7] by replacing $b\in\mathcal{K}$ and $\omega\in\mathcal{K}_\infty$ with $b\in\mathcal{P}$ and $\omega\in\mathcal{P}_0$. The formula for constructing ψ explicitly is omitted since it remains the same. Lemma 1 is applied to Theorem 4 and Theorem 5 with

$$\alpha(s) = \hat{\alpha}_1\circ\bar{\alpha}_1^{-1}\circ\underline{\alpha}_1(s), \quad \omega(s) = \hat{\omega}_1(s)$$
$$\tau(s) = \tau_1(s) = s+\underline{\omega}_1(s)$$
$$b(s) = \left[\delta_1\circ\underline{\omega}_1\circ\tau_1^{-1}\circ\hat{\alpha}_1\circ\bar{\alpha}_1^{-1}\circ\underline{\alpha}_1(s)\right]\left[\delta_2\circ\underline{\omega}_2\circ\sigma_2(s)\right]$$
$$d(s) = \left[\hat{\sigma}_1\circ\underline{\alpha}_2^{-1}\circ\bar{\alpha}_2\circ\alpha_2^{-1}\circ\tau_2\circ\sigma_2(s)\right]\sigma_2(s)$$
$$\eta(s) = \hat{\sigma}_1\circ\underline{\alpha}_2^{-1}\circ\bar{\alpha}_2\circ\alpha_2^{-1}\circ\tau_2\circ\sigma_2(s)$$

which is slightly different from [7]. Since we have

$$\textit{(F2)} \Rightarrow \textit{(D2)}\vee\textit{(D3)}\vee\textit{(E2)}$$
$$\textit{(D1)}\wedge\neg\textit{(D2)} \Rightarrow \textit{(F1)}$$

under the assumption of (67), all the cases in Theorem 4 and Theorem 5 are covered by the following four situations:

(D2)' $\{\alpha_2(\infty)=\infty \ \wedge \ \sigma_2(\infty)<\infty\}$ with $\hat{\omega}_1,\hat{\omega}_2\in\mathcal{K}_\infty$
(D3)' $\{\hat{\sigma}_1(\infty)<\infty \ \wedge \ \sigma_2(\infty)<\infty\}$ with $\hat{\omega}_1,\hat{\omega}_2\in\mathcal{K}_\infty$
(E2)' $\{\infty>\alpha_2(\infty)>\sigma_2(\infty) \ \wedge \ \hat{\sigma}_1(\infty)=\infty\}$ with $\hat{\omega}_1,\hat{\omega}_2\in\mathcal{K}_\infty$
(F1)' $\{$ (69) $\wedge \ \alpha_2(\infty)=\sigma_2(\infty)\}$ with $\hat{\omega}_1,\hat{\omega}_2\in\mathcal{P}_0$

The inequality (42) corresponds to (77). The existence of ψ in the *(F1)*' case is guaranteed by *(La)* of Lemma 1. The claim *(Lb)* provides ψ for *(D2)*', *(D3)*' and *(E2)*'.

When supply rates (40) for Σ_i, $i=1,2$ are replaced by

$$\rho_i = -\alpha_i(V_i(t,x_i))+\sigma_i(V_{3-i}(t,x_{3-i}))+\sigma_{ri}(|r_i|),$$

everything in Section V remains valid by replacing $\underline{\alpha}_i,\bar{\alpha}_i$ with identity maps, and replacing $|x_i|$ with V_i.

When ω_i's are linear, there exists $K>0$ such that $\psi(s)=s^K$ is a solution. Then, λ_1 and λ_2 reduce to the ones used in [5] dealing with uniformly contractive loop gains for ISS.

VI. Concluding Remarks

This paper has investigated small-gain technique for the stability of interconnected systems consisting of iISS subsystems in view of necessity as well as sufficiency. This paper has derived an explicit Lyapunov function characterizing iISS, ISS and global asymptotic stability of an interconnected system whenever a small-gain condition is satisfied. The condition is not necessarily uniform contraction. The Lyapunov approach is not conservative since the small-gain condition is shown to be necessary for stability of sets of interconnected iISS systems defined with individual supply rates. This paper has eliminated technical assumptions used in the previous studies and enlarged the applicable class of systems.

The difference between iISS property and uniform global asymptotic stability is known to be delicate. This paper does not address the necessity of the assumptions in Theorem 2 for the iISS property of the interconnection. Investigation of some necessary condition which is more restrictive than the uniform global asymptotic stability case and milder than the ISS case is a subject of future study. Our future work will also be directed at the stability analysis of an interconnection of IOS (input-to-output stable) systems, instead of ISS systems.

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