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Multi-objective optimal input design for plant friendly identification

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Abstract— **In optimal input design problems, the designer seeks to solve for maximally informative inputs to be used as perturbation signals in system identification experiments. Plant-friendly identification experiments are those that satisfy plant or operator constraints on experiment time, input and output amplitudes or input move sizes. These have been reported to be in direct conflict with requirements for good identification. Hence plant-friendly input design is inherently multi-objective in nature. In this contribution, we present the use of two well known techniques of multi-objective optimization to solve for a plant friendly input design where the plant friendly objective is to keep input move sizes low. We relax the constraint on the input move sizes by constraining the variance of the move size instead. Both techniques result in convex optimization problems which can be solved efficiently using powerful algorithms.**

I. INTRODUCTION

System identification is the process of generating a dynamic model of a system using data and process knowledge. It is common practice to perturb the system of interest and use the resulting data to build the model. The problem of input design is to synthesize an input signal that is maximally informative for generating good quality models.

Identification is carried out on operating plants and hence it is important to ensure that the experiment is least hostile to operating conditions or equivalently, the input signal should be plant friendly. Plant friendliness requirements can be imposed in the form of input rate constraints, input and output magnitudes and experiment time [1]. We focus on input rate requirements in this contribution, i.e., the input move size has to be smaller than a pre-specified tolerance.

It has been reported that plant friendliness demands are often in conflict with requirements for good identification [1], [2]. Hence, plant friendly input design is inherently a multi-objective problem and this was demonstrated in [3]. In this contribution, we solve a relaxed version of the optimal plant friendly input design problem using two well-known techniques for multi-objective optimization.

II. PROBLEM DEFINITION

A. Definition of plant friendliness

- Given an input sequence $u_k, k = 1, \ldots, N$, plant friendliness was defined as follows [2]: Define a transition as an event when $u_k \neq u_{k-1}$ and n_t as the number of transitions. The plant friendliness f is defined as:

$$f = 100\left(1 - \frac{n_t}{N-1}\right) \tag{1}$$

  Note that for a constant sequence $f = 100$, while f of a sequence that changes at every instant is 0. While this is intuitively pleasing, this definition does not allow for a closed form expression nor is it clear how to incorporate this formally in a design formulation.
- In a previous contribution [3], the plant friendliness ϕ_f was defined as

$$\phi_f = 1 - \frac{\sum_{k=2}^{N}(u_k - u_{k-1})^2}{(N-1)\max(u_k - u_{k-1})^2}. \tag{2}$$

  In the same spirit, we define plant friendliness as:

$$P_f = \frac{\sum_{k=2}^{N}(u_k - u_{k-1})^2}{N-1}. \tag{3}$$

Remark 2.1: Note that compared to the previous definitions of plant friendliness, the above would actually correspond to "unfriendliness", i.e., a low value of P_f would imply that the input is plant friendly. However, since the term "Plant unfriendliness" is unwieldy, we continue to call P_f "Plant-friendliness".

Remark 2.2: Note that $|u(t) - u(t-1)| \leq b$ implies $P_f \leq b^2$. Hence, a constraint of the form: $P_f \leq c$ is a relaxation of the following constraint $|u(t)-u(t-1)| \leq b$, where $c = b^2$. Hence, P_f is used as a metric to judge the plant friendliness of the input signal.

B. Problem formulation

Following [4], we consider a SISO system with input u_k, output y_k parameterized as:

$$y_k = G_1(q^{-1}, \beta)u_k + G_2(q^{-1}, \gamma)e_k, \tag{4}$$

where q^{-1} is the backward shift operator, $G_1(q^{-1}, \beta)$ and $G_2(q^{-1}, \gamma)$ are rational transfer functions in q^{-1} parameterized by β and γ respectively, e_k is a discrete time Gaussian White Noise with mean zero and variance σ^2. $\theta = [\beta', \gamma', \sigma]'$ is the overall vector of parameters to be estimated. Supposing an identification experiment is performed with input $u_1, \ldots, u_N$ resulting in the output $y_1, \ldots, y_N$. $\hat{\theta}$ which is an estimate of the true parameters is a statistical quantify. The quality of the estimated parameters can be described by the bias and covariance of $\hat{\theta}$. Given an unbiased estimator, a lower bound on the covariance of $\hat{\theta}$ is given by the following Cramer-Rao inequality:

$$cov(\hat{\theta}) \geq (M_\theta)^{-1}, \tag{5}$$

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where M_θ is the Fisher information matrix. A typical problem in experiment design is to choose an input sequence $u_1, \ldots, u_N$ that minimizes a scalar function of M_θ subject to certain constraints. The following is the well-known D-optimal experiment design problem:

Problem 1:

$$\min_{u_1,\ldots,u_N} -\log\det(M_\theta), \text{ s.t. } u_i \in U.$$

The constraints $u_i \in U$ could be of the form: $|u_i| \leq \alpha$ (input amplitude) or $\sum u_i^2 \leq \alpha$ (input power). The plant friendly input design that constraints input move sizes is posed as:

Problem 2:

$$\min_{u_1,\ldots,u_N} -\log\det(M_\theta), \text{ s.t. } u_i \in U, |u(t)-u(t-1)| \leq b.$$

The plant friendliness of the input signal is measured in terms of P_f. Hence, we seek to solve the following multi-objective plant-friendly input design:

Problem 3:

$$\min_{u_1,\ldots,u_N} -\log\det(M_\theta), P_f, \text{ s.t } u_i \in U.$$

The above described optimization problems were formulated in the time domain and the size of the optimization problem is N, the number of time points. There are several well-known advantages in reformulating the problem in the frequency domain.

Following [4], we make the following assumptions:

- The experiment time N is large.
- The input is stationary with power spectrum (two-sided) $\overline{\Phi}_u(\omega)$ defined on $[-\pi, \pi]$. $r_u(\tau) = E(u_k u_{k+\tau})$ is the covariance at lag τ and forms a Fourier transform pair with $\overline{\Phi}_u(\omega)$. Corresponding to the two-sided power spectrum, we define an equivalent one-sided power spectrum $\underline{\Phi}_u(\omega)$ on $[0, \pi]$. The relationship between the two is described in [5]–[7]. In what follows, we shall refer almost exclusively to the one-sided power spectrum and simply denote it as $\Phi_u(\omega)$ and abusing notation, we refer to it as a design measure on $[0, \pi]$.
- The class of inputs is further constrained to those having unit input power:

$$\int_0^\pi \Phi_u(\omega) d\omega = 1. \tag{6}$$

Since N, the experiment time is large, the average information matrix is defined as:

$$\overline{M} = \lim_{N\to\infty} \frac{1}{N} M_\theta. \tag{7}$$

It is shown in [4] that $\overline{M}$ can be expressed as:

$$\overline{M} = \int_0^\pi \widetilde{M}(\omega)\Phi_u(\omega) d\omega + \overline{M}_c, \tag{8}$$

where

$$\begin{aligned}
\widetilde{M}(\omega) &= Re\left\{\frac{1}{\sigma}\frac{\partial G_1(e^{j\omega})}{\partial\theta}' \left|G_2(e^{j\omega})\right|^{-2} \frac{\partial G_1(e^{-j\omega})}{\partial\theta}\right\}, \\
\overline{M}_c &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\partial G_2(e^{j\omega})}{\partial\theta}' \left|G_2(e^{j\omega})\right|^{-2} \frac{\partial G_2(e^{-j\omega})}{\partial\theta} d\omega \\
&+ \frac{1}{2\sigma^2}\left(\frac{\partial\sigma}{\partial\theta}\right)'\left(\frac{\partial\sigma}{\partial\theta}\right).
\end{aligned} \tag{9}$$

It can be shown that P_f can be expressed in the frequency domain as follows [7]:

$$P_f = \int_0^\pi 2(1-\cos(\omega))\Phi_u(\omega) d\omega. \tag{10}$$

Thus, the frequency domain D-optimal design problem is as follows:

Problem 4:

$$\min -\log\det(\overline{M}), \text{ s.t} \int_0^\pi \Phi_u(\omega) d\omega = 1,$$

and the frequency domain version of the multi-objective problem is:

Problem 5:

$$\min -\log\det(\overline{M}), \; P_f, \text{ s.t} \int_0^\pi \Phi_u(\omega) d\omega = 1.$$

III. Multi-objective optimization

In single objective optimization problems, the aim is to determine the global optimal solution, if it exists. Unlike single objective optimization, in optimization with conflicting objectives, there is no single optimal solution. We consider two methods of solving the Problem 5. The first is the ϵ constraint method:

Problem 6:

$$\min -\log\det(\overline{M}), \text{ s.t} \left\{ \begin{array}{c} \int_0^\pi \Phi_u(\omega) d\omega = 1 \\ P_f \leq c, \end{array}\right.$$

where $\overline{M}$ and P_f are given in (8) and (10) respectively.

The second is the method of lexicographic optimization:

Problem 7:

$$\min_{\Phi_u} P_f, \text{ s.t } \left\{ \; \Phi_u(\omega) \text{ solves Problem 4}, \int \Phi_u(\omega) d\omega = 1. \right.$$

IV. ϵ constraint method

Theorem 1: [7] Problem 6 can be cast as a convex optimization problem.

It can be shown that it is possible to realize the optimal input spectrum $\Phi_u^*(\omega)$ (that solves the above convex optimization problem) with a finite number of frequencies [7]. Efficient parameterizations of the model and input spectrum and techniques for solving the optimization problem are available [5], [8]. Varying the value of c (which plays the role of ϵ in this formulation) gives rise to different input signals and the trade-off between the model quality as judged by $\det(M_\theta)$ and plant friendliness as judged by P_f.

V. Lexicographic optimization

The optimal input spectrum that solves Problem 1 can have several realizations in the time domain with possibly differing P_f. Lexicographic optimization is one technique of determining the most plant friendly input signals among these D-optimal input designs. Essentially, this implies that the multiple objectives (D-optimality and plant friendliness) are ranked in a decreasing order of importance. In this section, we characterize the lexicographic solutions for a certain class of systems using the theory of Tchebysheff systems which has been used to solve input design problems previously [5], [6].

A. Tchebysheff systems

Following [6], the transfer functions G_1 and G_2 in (4) are further parameterized according to:

$$y_k = \frac{B(q^{-1})}{A(q^{-1})}u_{k-d} + \frac{D(q^{-1})}{C(q^{-1})}e_k, \tag{11}$$

where:

$$A(q) = 1 + \sum_{j=1}^{n} a_j q^j, \qquad B(q) = \sum_{j=0}^{m} b_j q^j,$$
$$C(q) = 1 + \sum_{j=1}^{s} c_j q^j, \qquad D(q) = \sum_{j=0}^{r} d_j q^j, \tag{12}$$

where we assume that there are no pole-zero cancellations and $A(z)$ and $D(z)$ have no zeroes on the closed unit disk. In addition, we assume that $A(z)$ and $C(z)$ have no common roots. For the above dynamic system, it has been shown that M_θ can be partitioned as:

$$M_\theta = \begin{bmatrix} M_\beta(u) & 0 \\ 0 & M_\gamma \end{bmatrix}, \tag{13}$$

where $M_\beta \in \mathbb{R}^{p\times p}$ is related to the $p = m+n+1$ parameters in G_1 and dependent on the input. M_γ is related to the noise parameters and importantly, is independent of the input. Hence, it is pertinent to consider M_β. As before, we define an average information matrix related to β and it can be expressed as a function of the input spectrum in the following form: form [4], [6]:

$$\begin{aligned} \overline{M_\beta}(\Phi_u) &= \lim_{N\to\infty} \frac{1}{N} M_\beta \\ &= \sum_{i=1}^{p} L_i x_i, \end{aligned} \tag{14}$$

where L_1 is a constant $p \times p$ matrix and x_i is given by:

$$x_i = \int_0^\pi \left| \frac{C(e^{j\omega})}{D(e^{j\omega})A^2(e^{j\omega})} \right|^2 \cos^{i-1}(\omega)\Phi_u d\omega. \tag{15}$$

Thus, for this choice of parametrization of the system and information matrix, the entries in M_β are linear in x_i which can be thought of as the decision variables. Thus, the frequency domain D-optimal input design Problem 4 can be solved by optimizing over x_i subject to the input power constraint. It has been shown [6] that under certain conditions that only involve the model orders, the input power constraint $\int \Phi_u d\omega = 1$ is equivalent to the x_i lying on a hyperplane in $\mathbb{R}^p$ and is summarized in the following theorem:

Theorem 2: [6] The following are equivalent:

1. $s = 0,\ m \geq n + r$.
2. If $\int \Phi_u d\omega = 1$, x_i lie in a hyperplane in $\mathbb{R}^p$.
3. Given Φ'_u and Φ''_u such that $\overline{M}_\beta(\Phi'_u) = \overline{M}_\beta(\Phi''_u)$ and $\int \Phi'_u d\omega = 1, \implies \int \Phi''_u d\omega = 1$.

Hence, the original infinite dimensional D-optimal input design problem (Problem 4) can be converted to the following finite dimensional problem where the decision variables are x_i.

Problem 8:

$$\min_{x_i} -\log\det(\overline{M}_\beta), \text{ s.t.} \begin{cases} x_i \text{ as in (15)}, \\ \sum_{i=1}^{i=p} \alpha_i x_i = 1, \end{cases}$$

where the α_i depend on the transfer function polynomials D and A.

Definition 1: [6], [9] Let $u_1, \ldots, u_p$ denote continuous real-valued functions defined on a closed interval $[a, b]$. These functions constitute a Tchebysheff system (T-system) if the following determinant

$$\begin{vmatrix} u_1(t_1) & u_1(t_2) & \ldots & u_1(t_p) \\ \vdots & \vdots & \vdots & \\ u_p(t_1) & u_p(t_2) & \ldots & u_p(t_p) \end{vmatrix}$$

is strictly positive whenever $a \leq t_1 < t_2 < \ldots < t_p \leq b$.

Define

$$f(\omega)^{-1} = \left| \frac{C(e^{j\omega})}{D(e^{j\omega})A^2(e^{j\omega})} \right|^2 \tag{16}$$

$$v_i(\omega) = \frac{1}{f(\omega)} \cos^{i-1}(\omega) \tag{17}$$

Theorem 3: [6] $v_1, v_2, \ldots, v_p$ as defined above is a T-system on $[0, \pi]$ if $p(p-1)/2$ is even. Else, $\{v_1, v_2, \ldots, v_{p-1}, -v_p\}$ is a T-system on $[0, \pi]$.

Denote $\{\overline{v_i}\}_1^p$ to denote either $v_1, v_2, \ldots, v_p$ or $v_1, v_2, \ldots, -v_p$ that constitute the appropriate T-system. In view of the above theorem, we can clearly extend the T-system so generated with v_{p+1}, where $v_{p+1} = \pm f(\omega)^{-1}\cos^p(\omega)$ (as the case may be) and the resulting system $\{\overline{v_i}\}_1^{p+1}$ is also a T-system and will be referred to as the augmented T-system.

Definition 2: [9] Let $x = [x_1, \ldots, x_p]' \in \mathbb{R}^p$, with x_i defined as in (15). Then, $\mathcal{M}_p \in \mathbb{R}^p = \lambda x$, with $\lambda \geq 0$ is defined as a moment space with respect to $\{\overline{v_i}\}_i^p$.

Definition 3: [5], [6], [9] Let $x \in \mathcal{M}^p$ and $\Phi_u(\omega)$ induce x. The set of all measures that induce x is denoted by $V(x)$. The number of points in the support of $\Phi_u(\omega)$, with 0 and π counted as half is the index of $\Phi_u(\omega)$. Measures $\Phi_u(\omega)$ of index $p/2$ which induce x are called principal representations. If p is even, the lower principal representation $\Phi_u(\omega)$ that induces x contains $p/2$ frequencies in $(0, \pi)$ and the upper principal representation contains $p/2 + 1$ frequencies including 0 and π. If p is odd, the lower principal representation $\Phi_u(\omega)$ that induces x contains $(p + 1)/2$ frequencies including 0 and the upper principal representation contains $(p + 1)/2$ frequencies including π.

Theorem 4: [9] Let $\{\overline{v_i}\}_1^p$ and $\{\overline{v_i}\}_1^{p+1}$ be a T-system and augmented T-system respectively. Given $x \in int(\mathcal{M}_p)$, the maximum and minimum of $\int v_{p+1}\Phi_u d\omega$ such that $\Phi_u \in V(x)$ are attained at the upper and lower principal representations that induce x.

Theorem 5: Let x^* be the optimal solution to Problem 8. If $s = 0,\ m = n + r$, the maximum and minimum values of P_f occur at the principal representations of x^*. If $s = 0,\ m \geq n + r + 1$, all representations of x^* have the same value of P_f.

Proof: Recall

$$P_f = \int_0^\pi 2(1-\cos(\omega))\Phi_u d\omega = 2 - 2\int_0^\pi \cos(\omega)\Phi_u d\omega. \tag{18}$$

When $s = 0$, $f(\omega)$ as defined in (16) is a polynomial of degree $2n+r$ in $\cos(\omega)$ and denote it by $\sum_{i=0}^{2n+r} \alpha_i \cos^i(\omega)$.

$$\begin{aligned} \int_0^\pi \cos(\omega)\Phi_u d\omega &= \int_0^\pi \frac{f(\omega)}{f(\omega)} \cos(\omega)\Phi_u d\omega \\ &= \sum_{i=0}^{2n+r} \frac{\alpha_i \cos^{i+1}(\omega)}{f(\omega)} \Phi_u d\omega. \end{aligned} \tag{19}$$

Consider the case when $m = n+r$. Since, $p = m+n+1$, $2n+r = p-1$. P_f simplifies to:

$$\begin{aligned} P_f &= 2 - 2\sum_{i=2}^{p} \alpha_{i-2} x_i - 2\int \frac{\alpha_{p-1}}{f(\omega)} \cos^p(\omega)\Phi_u d\omega \\ &= 2 - 2\sum_{i=2}^{p} \alpha_{i-2} x_i - 2\alpha_{p-1} \int_0^\pi v_{p+1}\Phi_u d\omega. \end{aligned} \tag{20}$$

Since x^* solves the modified D-optimal problem 8, the maximum and minimum values of P_f amongst the D-optimal solutions depends on the last term in (20). Since $\{\overline{v_p}\}_1^{p+1}$ constitute an augmented T-system, from Theorem 4, the maximum and minimum values of P_f amongst D-optimal solutions are found at the principal representations of x^*.

Consider the case when $m \geq n+r+1$. We have $p = m+n+1 \geq 2n+r+2$. Hence, P_f simplifies to:

$$P_f = 2 - 2\sum_{i=2}^{2n+r+2} \alpha_{i-2} x_i. \tag{21}$$

Hence, if x^* is a D-optimal solution, then P_f is:

$$P_f = 2 - 2\sum_{i=2}^{2n+r+2} \alpha_{i-2} x_i^*, \tag{22}$$

and all D-optimal solutions have the same P_f.

∎

Thus, for a certain class of systems characterized by the model orders $s = 0, m = n+r$, the maximum and minimum values of P_f corresponding to the D-optimal solutions are explicitly characterized: they occur at the principal realizations of the D-optimal solution. Not only does this characterize the lexicographic solutions, it also gives an upper limit to P_f for all D-optimal solutions. When $s = 0, m \geq n+r+1$, all D-optimal solutions have the same P_f and hence, all D-optimal solutions are lexicographically equivalent. Given a point x^*, techniques to determine the corresponding principal representations are discussed in [5].

VI. Examples

Example 1: Consider the following example [4]:

$$y_k = b_1 u_{k-1} + b_2 u_{k-2} + b_3 u_{k-3} + e_k,$$

where e_k is white noise with variance σ^2. It is shown that [4]

$$\overline{M} = \frac{1}{\sigma}\begin{bmatrix} 1 & x_1 & x_2 & 0 \\ x_1 & 1 & x_1 & 0 \\ x_2 & x_1 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2\sigma} \end{bmatrix}, \tag{23}$$

where

$$x_1 = \int_0^\pi \cos(\omega)\Phi_u(\omega)d\omega \tag{24}$$

$$x_2 = \int_0^\pi \cos(2\omega)\Phi_u(\omega)d\omega, \tag{25}$$

and so $\overline{M}$ is the convex hull of

$$\overline{M} = \frac{1}{\sigma}\begin{bmatrix} 1 & \cos(\omega) & \cos(2\omega) & 0 \\ \cos(\omega) & 1 & \cos(\omega) & 0 \\ \cos(2\omega) & \cos(\omega) & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2\sigma} \end{bmatrix}. \tag{26}$$

We shall consider three related problems: the single objective classical D-optimal design problem (Problem 1), the ϵ constraint method (Problem 6) and the lexicographic method (Problem 7).

D-optimal design: The problem is to minimize $-\log(\det(\overline{M}))$ or equivalently, maximize $\det(\overline{M})$, where

$$\det(\overline{M}) = \frac{1}{2\sigma^4}(1 + 2x_1^2(x_2 - 1) - x_2^2),$$

subject to the constraint (24 and 25) on x_1 and x_2. The D-optimal solution is $x_1^* = x_2^* = 0$ [4], with the optimal cost function, $\det(\overline{M})$ being 1. There are an infinite number of solutions, one solution being to place equal weights at frequencies $\pi/4, 3\pi/4$ [4].

Lexicographic optimization: In this example, P_f can be conveniently expressed as $2(1-x_1)$. All D-optimal input designs, i.e., $x_1^* = x_2^* = 0$ have the same P_f since $P_f = 2(1-x_1) = 2$. This is also easily seen as a consequence of Theorem 5, as $s = n = r = 0, m = 2, m \geq n+r+1$.

ϵ constraint method: Now, the plant friendliness constraint is imposed:

$$\int_0^\pi 2(1-\cos(\omega))\Phi_u(\omega)d\omega \leq c$$

which in turn simplifies to:

$$x_1 \geq 1 - c/2, \tag{27}$$

which is simply a half-plane in $\mathbb{R}^2$. As long as $c \geq 2$, the solution is the same as the D-optimal input, i.e., $x_1^* = x_2^* = 0$.

However, when $c < 2$, the constraint $x_1 = 1 - c/2$ becomes active. It can be seen that the optimal solution is $(1-c/2, (1-c/2)^2)$. For example, let $c = 1$, i.e., $P_f = 1$. It is seen that the optimal solution is: $x_1^* = .5, x_2^* = .25$ and $\det(\overline{M}) = .5625$. Thus, the quality of the parameter estimates as judged by $\det(\overline{M})$ decreases, while the input is definitely more friendly. One solution for the optimal input is $.05 + 1.0505\cos(\pi/4t + \phi_1) + .6296\cos(3\pi/4t + \phi_2)$. When $c = 0$, we have a constant input and $\det(\overline{M}) = 0$, i.e., the parameters are not identifiable and the input is

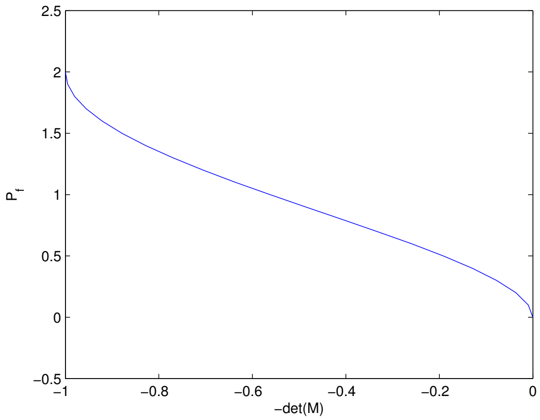

Fig. 1. Trade-off between input friendliness and parameter variance in Example 1 [7]

not persistently exciting. When $c < 0$, which is physically meaningless, clearly, there are no feasible solutions. Thus, the trade-off between plant friendliness and identifiability is directly quantified. Figure 1 shows the trade-off between P_f and $-\det(\overline{M})$.

Example 2: Consider the following example [10]:

$$y_k = (b_0 + b_1 q^{-1})u_k + (d_0 + d_1 q^{-1})e_k$$

The information matrix $\overline{M}$ is partitioned into:

$$\overline{M} = \begin{bmatrix} \overline{M}_\beta & 0 \\ 0 & \overline{M}_\gamma \end{bmatrix}, \quad (28)$$

where $\overline{M}_\beta$ is related to the parameters in G_1. $\overline{M}_\gamma$ is related to the parameters in G_2 and is independent of the input. Hence, while minimizing the parameter covariance, it is sufficient to consider the block $\overline{M}_\beta$. Further, it is shown that [10]

$$\overline{M}_\beta = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_1 \end{bmatrix}, \quad (29)$$

where

$$\begin{aligned} x_1 &= \int_0^\pi \frac{1}{d_0^2 + d_1^2 + 2d_0 d_1 \cos(\omega)} \Phi_u(\omega) d\omega, \\ x_2 &= \int_0^\pi \frac{\cos(\omega)}{d_0^2 + d_1^2 + 2d_1 d_2 \cos(\omega)} \Phi_u(\omega) d\omega. \quad (30) \end{aligned}$$

From the input energy constraint we have:

$$\begin{aligned} \int_0^\pi \Phi_u(\omega) d\omega &= 1 \\ \int_0^\pi \frac{d_0^2 + d_1^2 + 2d_1 d_2 \cos(\omega)}{d_0^2 + d_1^2 + 2d_0 d_1 \cos(\omega)} \Phi_u(\omega) d\omega &= 1 \\ (d_0^2 + d_1^2)x_1 + 2d_0 d_1 x_2 &= 1, \quad (31) \end{aligned}$$

which is a straight line in $\mathbb{R}^2$. Since, any feasible point (x_1, x_2) is a convex combination of single frequency designs, the feasible set is the segment connecting $(1,1)/(d_0 + d_1)^2$ and $(1,-1)/(d_0 + d_1)^2$ (part of the line $(d_0^2 + d_1^2)x_1 + 2d_0 d_1 x_2 = 1$). Thus, in this example, the feasible set of information matrices is well characterized and parameterized.

D-optimal input design: As before, consider the classical D-optimal problem. When $d_0 = 1, d_1 = 0.3$, the optimal solution is $x_1^* = 1.3163$, $x_2^* = -0.7246$, with $\det(\overline{M}_\beta) = 1.2076$ [10]. There are an infinite number of solutions to realize this in the time domain, the simplest being a single frequency design: $\delta(\omega - 2.1537)$ or in time domain: $u_k = 1.414\cos(2.1537k + \phi_1)$. Another solution concentrates power at $\omega = 0$ and $\omega = \pi$: $u_k = \sqrt{1/2} + \sqrt{1/2}\cos(\pi k)$.

Lexicographic optimization: There are an infinite number of D-optimal input designs that satisfy $x_1^* = 1.3163$, $x_2^* = -0.7246$, with different P_f. Also the model orders satisfy $m = n + r$ and so from Theorem 5 the maximum and minimum values of P_f occur at the principal realizations of $(1.3163, -.7246)$. In this case, the lower principal realization involves $\omega = 2.1537$ and the upper principal realization involves $0, \pi$. The upper principal realization corresponds to minimum P_f i.e., the input design that concentrates power at 0 and π, i.e., $\Phi_u(\omega) = .5\delta(\omega) + .5\delta(\omega - \pi)$, gives minimum P_f with $P_f = 2$. The lower principal realization corresponds to maximum P_f, i.e., For the single frequency design $\delta(\omega - 2.1537)$, $P_f = 3.1$.

ϵ constraint method: Now, consider the plant friendly input design problem: $P_f \leq c$. As before, when c is sufficiently high, the constraint is inactive and the solution is the same as the D-optimal design. From the above discussion, it is possible to find a solution where the constraint $P_f \leq c$ is not active when $c > 2$. When $c \leq 2$, the constraint becomes active. The optimal input is of the form: $\Phi_u(\omega) = \zeta\delta(\omega) + (1 - \zeta)\delta(\omega - \pi)$, $0 \leq \zeta \leq 1$, or in time domain:

$$u_k = \sqrt{\zeta} + \sqrt{(1 - \zeta)}\cos(\pi k),$$

where ζ is determined by solving the equation $P_f = c$, which in turn implies:

$$2(\zeta(1 - \cos(0)) + (1 - \zeta)(1 - \cos(\pi))) = c \quad (32)$$

For example, let $c = 1$. Solving the above results in $\zeta = .75$, $\det(\overline{M}) = .9057$. The plant friendly input is:

$$u_k = .866 + .5\cos(\pi k).$$

Figure 2 shows the trade-off between plant friendliness and quality of parameter estimates [7].

Example 3: Consider the following system [6]:

$$y_k = \frac{b_0 + b_1 q^{-1}}{1 + a_1 q^{-1}} u_k + e_k. \quad (33)$$

D-optimal solution: [6] has determined the D-optimal solutions to the above problem when $b_0 = 1, b_1 = .3, a_1 = .5$ and $\sigma^2 = .01$. The same can be determined by solving Problem 8 and using an LMI characterization of the feasible space [5]. The resulting convex program is solved numerically using YALMIP [11].

Lexicographic solution: The corresponding values of P_f are also tabulated. Since the model orders satisfy $m = n + r$, the maximum and minimum values of P_f for a given input design are attained at the principal realizations. In this case, they happen to occur at the upper and lower principal realizations respectively. The principal representations of the

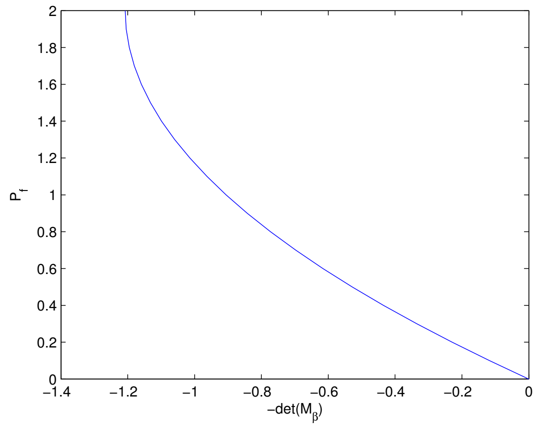

Fig. 2. Trade-off between input friendliness and parameter variance in Example 2 [7].

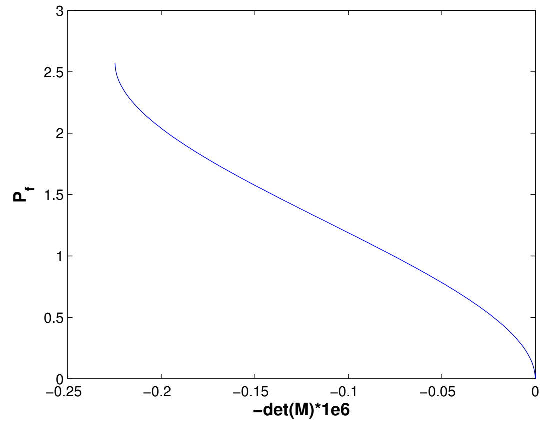

Fig. 3. Trade-off between input friendliness and parameter variance in Example 3.

D-optimal input and the corresponding values of P_f are shown in Table I [6]. It must be noted that the power distribution corresponds to the one-sided power spectrum.

ϵ constraint method: Figure 3 shows the trade-off between plant friendliness and quality of parameter estimates for the ϵ constraint method. The constraint $P_f \leq c$ can be expressed as a linear constraint and the resulting optimization problem is solved using YALMIP.

TABLE I

REPRESENTATIONS OF D-OPTIMAL INPUT DESIGNS FOR EXAMPLE 3

Property	Frequencies	Power	$\det(M)^{-1}$	P_f	Index
Lower Principal	0 2.76	1/3 2/3	4.45e-6	2.57	1.5
Upper Principal	2.09 π	2/3 1/3	4.45e-6	3.33	1.5
Canonical	0 2.30 π	0.15 0.55 0.30	4.45e-6	3.03	2

VII. CONCLUSIONS

We presented 2 multi-objective optimization formulations for plant friendly input design in the frequency domain: the ϵ constraint method and lexicographic optimization. The ϵ constraint formulation can be shown to be a convex optimization problem and can be solved efficiently. For a certain class of systems, the Tchebysheff system ideas were used to geometrically characterize the lexicographic solutions.

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