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Adaptive H_∞ Control of Nonholonomic Mobile Robot Based on Inverse Optimality

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Abstract— **This paper deals with the problem of constructing adaptive trajectory control scheme of nonholonomic mobile robots based on H_∞ control strategy. Both kinematics control laws and dynamics control ones are developed based on H_∞ criterion and for processes with unknown parameters. It is shown that the resulting control signals are derived as solutions of certain H_∞ control problems where tracking errors of controlled velocities and estimation errors of tuning parameters are regarded as external disturbances to control systems.**

I. INTRODUCTION

Control problems of wheeled mobile robots have been investigated extensively by several researchers. Those mobile robots are considered as nonholonomic systems with nonintegrable constraints, and cannot be stabilized via continuous feedbacks. The control problems of nonholonomic mobile robots are divided into two categories; one is a stabilizing control problem, and the other is a tracking control one. In the stabilization of nonholonomic mobile robots, chained forms and/or state scaling transformations are introduced, and several discontinuous stabilizing control strategies have been proposed in [1], [2], [3], [4], [5], [6], [7]. On the contrary, the tracking control problems of nonholonomic mobile robots are known to be practical issues, and kinematics control (path tracking control of kinematics models via velocity inputs) and dynamics control (path tracking control for dynamic models via torque inputs) methodologies have been developed and uncertainties of dynamical models (actuator dynamics) have been dealt with by utilizing adaptive control schemes. Those were reported in [8], [9], [10], [11], [12], [13].

In the present paper, we consider adaptive trajectory control schemes of uncertain nonholonomic mobile robots based on H_∞ control strategy. Both kinematics control laws and dynamics control ones are developed based on H_∞ criterion and for processes with unknown parameters. It is shown that the resulting control signals are derived as solutions of certain H_∞ control problems where tracking errors of controlled velocities and estimation errors of tuning parameters are regarded as external disturbances to control systems.

II. PROBLEM STATEMENT

We consider trajectory control (path tracking control) problems of double wheeled mobile robots with pure rolling. A kinematics model of mobile robots is given by the following equations.

$$\begin{cases} \dot{x} = v\cos\theta \\ \dot{y} = v\sin\theta \\ \dot{\theta} = \omega, \end{cases} \tag{1}$$

where (x, y, θ) denotes the posture of the mobile, and (v, ω) is a control input; v is a linear velocity, and ω is an angular velocity, respectively. The control objective is to make the posture of the mobile (x, y, θ) follow the desired trajectory (x_r, y_r, θ_r) described as follows:

$$\begin{cases} \dot{x}_r = v_r\cos\theta_r \\ \dot{y}_r = v_r\sin\theta_r \\ \dot{\theta}_r = \omega_r. \end{cases} \tag{2}$$

Here we define path trecking errors $\tilde{x}$, $\tilde{y}$, $\tilde{\theta}$ by

$$\begin{cases} \tilde{x} = x_r - x \\ \tilde{y} = y_r - y \\ \tilde{\theta} = \theta_r - \theta, \end{cases} \tag{3}$$

and introduce new variables (e_1, e_2, e_3),

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{\theta} \end{bmatrix}, \tag{4}$$

then, the next error dynamics of the kinematics model are obtained.

$$\begin{cases} \dot{e}_1 = \omega e_2 - v + v_r\cos e_3 \\ \dot{e}_2 = -\omega e_1 + v_r\sin e_3 \\ \dot{e}_3 = \omega_r - \omega. \end{cases} \tag{5}$$

III. CONTROL OF KINEMATICS MODEL VIA VELOCITY INPUTS

First, we consider path tracking control problems of kinematics models (5) via velocity inputs (v, ω). Various methodologies have been developed to solve those problems by several researchers including Kanayama *et al.* [8], Jiang & Nijmeijer [10], Lefeber [11], Fukao *et al.* [12], and Wang & Tsai [13]. Among those, Wang & Tsai [13] recently proposed the following control strategy.

[Wang & Tsai Method (Kinematics Control)] [13] Input velocities v and ω are synthesized in the following way:

$$v = k_{01}e_1 + v_r\cos e_3 + \alpha\omega\sin e_3, \tag{6}$$

$$\omega = \frac{1}{1+\alpha e_1}\left\{k_{02}\, e_3\,\mathrm{sgn}(e_3\sin e_3) + v_r e_2 + \alpha v_r\sin e_3 + \omega_r\right\}, \tag{7}$$

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where it is assumed that $1+\alpha e_1 \neq 0$, and k_{01}, k_{02}, α are chosen such that $k_{01}>0$, $k_{02}>0$ and $\alpha v_r \geq 0$. A locally defined Lyapunov function candidate W_0 is introduced.

$$W_0 = \frac{1}{2}\left(e_1^2+e_2^2\right)+\left(1-\cos e_3\right). \tag{8}$$

The time derivative of W_0 along the trajectory of (e_1, e_2, e_3) is given by

$$\dot{W}_0 = -k_{01}e_1^2 - k_{02}|e_3 \sin e_3| - \alpha v_r \sin^2 e_3 \leq 0. \tag{9}$$

Then, it follows that e_1, e_2, $e_3 \in \mathcal{L}^\infty$, $e_1 \in \mathcal{L}^2$, $e_3 \sin e_3 \in \mathcal{L}^1$, $\sin e_3 \in \mathcal{L}^2$ (when $\alpha v_r \neq 0$), and it is also shown that $\dot{e}_1$, $\dot{e}_2$, $\dot{e}_3 \in \mathcal{L}^\infty$, and therefore $\lim_{t\to\infty} e_1 = \lim_{t\to\infty} e_3 = 0$ are deduced. Furthermore, since $\lim_{t\to\infty}\dot{e}_3 = -\lim_{t\to\infty} v_r e_2 = 0$, it is shown that $\lim_{t\to\infty} e_2 = 0$ when $\lim_{t\to\infty} v_r(t) \neq 0$. Or since $\lim_{t\to\infty}\dot{e}_1 = \lim_{t\to\infty}\omega_r e_2 = 0$, it follows that $\lim_{t\to\infty} e_2 = 0$ when $\lim_{t\to\infty}\omega_r(t) \neq 0$. Then, we obtain the next theorem [13].

Theorem 1 [13] *The control laws (6), (7) are applied to the kinematics model (5) under the conditions that* $1+\alpha e_1 \neq 0$ *and* $\alpha v_r \geq 0$. *Then the state* (e_1, e_2, e_3) *of (5) is locally bounded, and* (e_1, e_2, e_3) *converges to zero asymptotically (10), if either* $\lim_{t\to\infty} v_r(t) \neq 0$ *or* $\lim_{t\to\infty}\omega_r(t) \neq 0$.

$$\lim_{t\to\infty} e_1(t) = \lim_{t\to\infty} e_2(t) = \lim_{t\to\infty} e_3(t) = 0. \tag{10}$$

For Wang & Tsai method, the conditions $1+\alpha e_1 \neq 0$ and $\alpha v_r \geq 0$ are needed, which make the application of that method, too restrictive. Next, we propose a new kinematics control methodology based on H_∞ control criterion.

[Proposed Method (Kinematics Control)] When v and ω are not specified, the time derivative of W_0 along the trajectory of (5) is given by

$$\begin{aligned}\dot{W}_0 = &-e_1 v + e_1 v_r \cos e_3 \\ &+ e_2 v_r \sin e_3 + \omega_r \sin e_3 - \omega \sin e_3. \end{aligned}\tag{11}$$

From that, v and ω are chosen in the following way.

$$v = v_r \cos e_3 + v_0, \tag{12}$$

$$\omega = \omega_r + v_r e_2 + \omega_0, \tag{13}$$

where v_0 and ω_0 are stabilizing signal to be determined later, based on H_∞ control criterion. Then, $\dot{W}_0$ becomes

$$\dot{W}_0 = -e_1 v_0 - \omega_0 \sin e_3. \tag{14}$$

Here we add virtual external disturbances ξ_1, $\xi_2 \in \mathcal{L}^2$ to the input ports such that

$$\dot{W}_0 = -e_1(v_0+\xi_1) - (\omega_0+\xi_2)\sin e_3, \tag{15}$$

and introduce the following virtual system.

$$\frac{d}{dt}\begin{bmatrix} e_1 \\ e_3 \end{bmatrix} = \begin{bmatrix} -v_0-\xi_1 \\ -\omega_0-\xi_2 \end{bmatrix} \equiv F_0 + g_{01}d_0 + g_{02}V_0, \tag{16}$$

$$F_0 \equiv 0, \quad g_{01} = g_{02} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{17}$$

$$d_0 \equiv \begin{bmatrix} \xi_1 & \xi_2 \end{bmatrix}^T, \tag{18}$$

$$V_0 \equiv \begin{bmatrix} v_0 & \omega_0 \end{bmatrix}^T. \tag{19}$$

It should be noted that the time derivative of $\bar{W}_0$

$$\bar{W}_0 = \frac{1}{2}e_1^2 + (1-\cos e_3), \tag{20}$$

along the trajectory of the virtual system (16), (17), (18), (19) is equal to $\dot{W}_0$ (15). We are to stabilize that virtual system via a control input V_0 by H_∞ control criterion, where d_0 is regarded as an external disturbance to the process. For such purpose, we introduce the following Hamilton-Jacobi-Isaacs (HJI) equation.

$$\begin{aligned}\mathcal{L}_{F_0}W + \frac{1}{4}\left\{\frac{\|\mathcal{L}_{g_{01}}W\|^2}{\gamma_0^2} - (\mathcal{L}_{g_{02}}W)R_0^{-1}(\mathcal{L}_{g_{02}}W)^T\right\} \\ +q_0 = 0, \end{aligned}\tag{21}$$

where the solution W is given by $W = \bar{W}_0$ (8). Here we adopt inverse optimal control policy [14], [15], [16], [7], and obtain a symmetric positive definite matrix R_0 and a positive function q_0 from the inequality (21) for the given solution $W = \bar{W}_0$ and the positive constant γ_0. The substitution of $W = \bar{W}_0$ (20) into HJI equation (21) yields

$$\frac{1}{4}\left(\frac{\|s\|^2}{\gamma_0^2} - s^T R_0^{-1} s\right) + q_0 = 0, \tag{22}$$

$$s = \begin{bmatrix} e_1 & \sin e_3 \end{bmatrix}^T. \tag{23}$$

Then, R_0 and q_0 are given by

$$R_0 = \left(\frac{I}{\gamma_0^2} + K_0\right)^{-1}, \tag{24}$$

$$q_0 = \frac{1}{4}s^T K_0 s, \tag{25}$$
$$(K_0 = K_0^T > 0),$$

and the control input V_0 is derived as an optimal solution for the corresponding H_∞ control problem such that

$$\begin{aligned} V_0 &= -\frac{1}{2}R_0^{-1}\left(\mathcal{L}_{g_{02}}W_0\right)^T \\ &= \frac{1}{2}R_0^{-1}s = \frac{1}{2}\left(\frac{I}{\gamma_0^2}+K_0\right)s. \end{aligned}\tag{26}$$

By considering the relations of (21), (22), (23), (24), (25), we come back to stability analysis of the original process (5). From HJI equation (21), $\dot{W}_0$ (15) along the trajectory of (5) is evaluated by

$$\begin{aligned}\dot{W}_0 &= -s^T(V_0+d_0) \\ &= \left(V_0 - \frac{1}{2}R_0^{-1}s\right)^T R_0 \left(V_0 - \frac{1}{2}R_0^{-1}s\right) \\ &\quad -q_0 - V_0^T R_0 V_0 \\ &\quad -\gamma_0^2\left\|d_0 + \frac{s}{2\gamma_0^2}\right\|^2 + \gamma_0^2\|d_0\|^2, \end{aligned}\tag{27}$$

and it is shown that V_0 is an optimal solution which minimizes the following cost functional.

$$\begin{aligned}J_0 = \sup_{d_0\in\mathcal{L}^2}\left\{\int_0^t (q_0 + V_0^T R_0 V_0)d\tau + W_0(t)\right. \\ \left. -\gamma_0^2\int_0^t \|d_0\|^2 d\tau\right\}. \end{aligned}\tag{28}$$

Also, we have the next inequality.

$$\int_0^t (q_0 + V_0^T R_0 V_0)d\tau + W_0(t) \leq \gamma_0^2 \int_0^t \|d_0\|^2 d\tau + W_0(0). \qquad (29)$$

Hence, since $d_0 \in \mathcal{L}^2$ (in this section, $d_0 \equiv 0$ actually), the inequality (29) holds, and the right side of (29) is bounded from above, even for arbitrary t including $t = \infty$. Therefore, it follows that e_1, e_2, $e_3 \in \mathcal{L}^\infty$, e_1, $\sin e_3 \in \mathcal{L}^2$, and the same result as Wang & Tsai Method is derived. However, the conditions $1 + \alpha e_1 \neq 0$, $\alpha v_r \geq 0$ are not necessary in the proposed methodology.

Theorem 2 *The control laws (12), (13), (26) are applied to the kinematics model (5). Then the state (e_1, e_2, e_3) of (5) is locally bounded, and (e_1, e_2, e_3) converges to zero asymptotically, if either $\lim_{t\to\infty} v_r(t) \neq 0$ or $\lim_{t\to\infty} \omega_r(t) \neq 0$. Furthermore, V_0 is an optimal solution which minimizes the cost functional J_0 (28), and the inequality (29) holds.*

Remark 1 In our proposed methodology, $K_0 (= K_0^T > 0)$ is a free parameter. An example of it is given below:

$$K_0 = \begin{bmatrix} k_{011} + k_{012}e_1^2 & 0 \\ 0 & k_{021} + k_{022}e_3^2 \end{bmatrix}, \qquad (30)$$
$$(k_{011}, k_{021} > 0, \quad k_{012}, k_{022} \geq 0).$$

Especially, when $k_{012} = k_{022} = 0$, the proposed methodology is essentially equivalent to the method by Kanayama *et al.* [8]. However, the proposed scheme is derived as a solution for certain H_∞ control problem, and it possesses disturbance attenuation property represented by (29). That is an important point, when dynamical actuator models (dynamic models) are considered in the following sections.

Remark 2 In most of the previous related works, the condition that either $\lim_{t\to\infty} v_r(t) \neq 0$ or $\lim_{t\to\infty} \omega_r(t) \neq 0$, is necessary for asymptotic zero-tracking of (e_1, e_2, e_3) [8], [10], [12]. This is owing to stability analysis based on Lyapunov functions. On the contrary, Lefeber [11] proposed a different control scheme of the same problem by introducing the cascaded design, where the previous condition on v_r and ω_r is replaced by the new one that ω_r is persistently exciting. Hence, his approach cannot be *directly* applied to constant ω_r.

Remark 3 Similar to [8], [12], [13], the proposed control scheme is stable in the local sense. On the contrary, the globally stable control strategies were proposed in [10], [11]. The difference between those two research results is owing to the difference of Lyapunov functions used in stability analysis. Although the proposed methodology is locally stable, that can be extended to a globally stable control scheme via the new Lyapunov function, which will be shown in the future study.

IV. CONTROL OF DYNAMIC MODEL VIA TORQUE INPUTS

In Section III, the path tracking of the kinematics model is attained via velocity inputs v and ω. In the present section, we consider actuator dynamics, and control the velocity of the dynamic model via torque inputs, which is described as follows [6], [13]:

$$M\dot{V} + FV = B\tau, \qquad (31)$$
$$M = \begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix}, \quad F = \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix},$$
$$B = \frac{1}{r_0}\begin{bmatrix} 1 & 1 \\ l/2 & -l/2 \end{bmatrix}, \qquad (32)$$
$$V = \begin{bmatrix} v & \omega \end{bmatrix}^T, \qquad (33)$$
$$\tau = \begin{bmatrix} \tau_1 & \tau_2 \end{bmatrix}^T, \qquad (34)$$

where m is a mass of the mobile, I is an inertia moment of the rotation of the mobile, f_1 and f_2 are friction constants of two wheels, r_0 is a diameter of the wheels, l is a width of the mobile, and τ_1 and τ_2 are input torques of the two wheels. In this manuscript, r_0 and l (physical configuration of the mobile) are known a priori, and hence B is a known matrix. However, m, I, f_1 and f_2 are assumed to be unknown. Then, the dynamic model of the mobile is rewritten into the following form.

$$B\tau = M\dot{V} + FV = Y(V, \dot{V})\phi, \qquad (35)$$
$$Y(V, \dot{V}) = \begin{bmatrix} \dot{v} & 0 & v & 0 \\ 0 & \dot{\omega} & 0 & \omega \end{bmatrix}, \qquad (36)$$
$$\phi = \begin{bmatrix} m & I & f_1 & f_2 \end{bmatrix}^T, \qquad (37)$$

where ϕ is an unknown parameter vector.

In order to make the velocity of the mobile $V = \begin{bmatrix} v & \omega \end{bmatrix}^T$ follow the desired reference velocity $V_d = \begin{bmatrix} v_d & \omega_d \end{bmatrix}^T$, the input torque τ is synthesized as follows [6], [13]:

$$B\tau = Y(V, \dot{V}_d)\hat{\phi} - K_1\xi, \quad (K_1 = K_1^T > 0), \qquad (38)$$
$$\xi = V - V_d = \begin{bmatrix} \xi_1 & \xi_2 \end{bmatrix}^T, \qquad (39)$$
$$Y(V, \dot{V}_d) = \begin{bmatrix} \dot{v}_d & 0 & v & 0 \\ 0 & \dot{\omega}_d & 0 & \omega \end{bmatrix}, \qquad (40)$$

where $\hat{\phi}$ is a current estimate of ϕ. Then, we obtain

$$M\dot{\xi} + K_1\xi = Y(V, \dot{V}_d)\tilde{\phi}, \qquad (41)$$
$$\tilde{\phi} = \hat{\phi} - \phi. \qquad (42)$$

The estimate $\hat{\phi}$ is tuned by the next adaptation law [6], [13].

$$\dot{\hat{\phi}} = -\Gamma Y(V, \dot{V}_d)^T \xi, \quad (\Gamma = \Gamma^T > 0). \qquad (43)$$

For stability analysis of the adaptive control systems, we introduce a positive function W_1

$$W_1 = \frac{1}{2}\xi^T M \xi + \frac{1}{2}\tilde{\phi}^T \Gamma^{-1} \tilde{\phi}, \qquad (44)$$

and take the time derivative of W_1 along the trajectory of $(V, \tilde{\phi})$.

$$\dot{W}_1 = -\xi^T K_1 \xi \leq 0. \qquad (45)$$

Then, it is shown that ξ, $\hat{\phi} \in \mathcal{L}^\infty$, $\xi \in \mathcal{L}^2$, and $\dot{\xi} \in \mathcal{L}^\infty$ is deduced, and it follows that $\lim_{t\to\infty} \xi(t) = 0$.

Next, we combine two control schemes, the path tracking control via velocity inputs (kinematics control) and the velocity control via torque inputs (dynamics control), and construct the path tracking control scheme via torque inputs (dynamics control).

[Path Tracking Control Strategy of Dynamic Model via Wang & Tsai Method (Kinematics Control)] The substitution of the velocity tracking errors (ξ_1, ξ_2)

$$\xi_1 = v - v_d, \quad \xi_2 = \omega - \omega_d, \tag{46}$$

into the kinematics model (e_1, e_2, e_3) (5) yields

$$\begin{cases} \dot{e}_1 = \omega e_2 - v_d - \xi_1 + v_r \cos e_3 \\ \dot{e}_2 = -\omega e_1 + v_r \sin e_3 \\ \dot{e}_3 = \omega_r - \omega_d - \xi_2. \end{cases} \tag{47}$$

Here we apply Wang & Tsai Method (kinematics control) [13] to generate v_d and ω_d such that

$$v_d = k_{01} e_1 + v_r \cos e_3 + \alpha \omega \sin e_3, \tag{48}$$

$$\omega_d = \frac{1}{1 + \alpha e_1} \{ k_{02} e_3 \text{sgn}(e_3 \sin e_3) + v_r e_2 + \alpha v_r \sin e_3 + \omega_r \}. \tag{49}$$

The time derivative of W_0 (8) along the trajectory of (e_1, e_2, e_3) is given by

$$\dot{W}_0 = -k_{01} e_1^2 - k_{02} |e_3 \sin e_3| - \alpha v_r \sin^2 e_3 - \xi_1 e_1 - \xi_2 \sin e_3. \tag{50}$$

For stability analysis of the path tracking control scheme of dynamic models via torque inputs, a positive function W_2 is newly defined by

$$W_2 = W_0 + \beta W_1, \quad (\beta > 0). \tag{51}$$

The time derivative of W_2 along the trajectory of $(e_1, e_2, e_3, \xi_1, \xi_2, \hat{\phi}^T)$ is given as follows:

$$\begin{aligned} \dot{W}_2 \ =& \ -k_{01} e_1^2 - k_{02} |e_3 \sin e_3| - \alpha v_r \sin^2 e_3 \\ & -\xi_1 e_1 - \xi_2 \sin e_3 - \beta \xi^T K_1 \xi \\ \leq& \ -k_{01} e_1^2 - k_{02} |e_3 \sin e_3| - \alpha v_r \sin^2 e_3 \\ & -\xi_1 e_1 - \xi_2 \sin e_3 - \beta \lambda_{\min}(K_1)(\xi_1^2 + \xi_2^2) \\ \leq& \ -\frac{1}{2} k_{01} e_1^2 - k_{02} |e_3 \sin e_3| - \frac{1}{2} \alpha v_r \sin^2 e_3 \\ & - \left(\beta \lambda_{\min}(K_1) - \frac{1}{2k_{01}} \right) \xi_1^2 \\ & - \left(\beta \lambda_{\min}(K_1) - \frac{1}{2\alpha v_r} \right) \xi_2^2, \end{aligned} \tag{52}$$

where $\lambda_{\min}(K_1)$ is the minimal eigenvalue of K_1. Then, for β satisfying

$$\beta > \max \left(\frac{1}{2k_{01} \lambda_{\min}(K_1)}, \frac{1}{2\alpha v_r \lambda_{\min}(K_1)} \right), \tag{53}$$

it follows that $\dot{W}_2 \leq 0$, and the next theorem is deduced.

Theorem 3 *The velocity control scheme is constructed via (38), (43), and V_d is synthesized via Wang & Tsai Method (path tracking control of kinematics model) (48), (49). Then, the velocity tracking error and the estimate of ϕ, $(\xi^T, \hat{\phi}^T)$ are globally bounded, and $\xi(t)$ converges to zero asymptotically.*

$$\lim_{t \to \infty} \xi(t) = 0, \quad (\xi \in \mathcal{L}^2). \tag{54}$$

Furthermore, (e_1, e_2, e_3) are locally bounded under the conditions $1 + \alpha e_1 \neq 0$, $\alpha v_r > 0$, and converge to zero asymptotically (10) if either $\lim_{t \to \infty} v_r(t) \neq 0$ or $\lim_{t \to \infty} w_r(t) \neq 0$.

The condition $\alpha v_r \neq 0$ is newly added from (53). Therefore, the path tracking control strategy of dynamic models via Wang & Tsai method has many restrictions such as $1 + \alpha e_1 \neq 0$ and $\alpha v_r > 0$. Additionally, it should be also noted that β is not a design parameter, or β need not be determined explicitly. Instead, it is sufficient for β to exist under the condition (53) and for given design parameters k_{01}, α, K_1, v_r.

Next, we provide the path tracking control strategy of dynamic models via the proposed methodology (kinematics control).

[Path Tracking Control Strategy of Dynamic Models via Proposed Method (Kinematics Control)] The proposed methodology (12), (13), (26) (kinematics control) is applied to generate v_d and ω_d such that

$$v_d = v_r \cos e_3 + v_0, \tag{55}$$

$$\omega_d = \omega_r + v_r e_2 + \omega_0, \tag{56}$$

$$V_0 = \begin{bmatrix} v_0 \\ \omega_0 \end{bmatrix} = \frac{1}{2} R_0^{-1} \begin{bmatrix} e_1 \\ \sin e_3 \end{bmatrix}. \tag{57}$$

Then the time derivative of W_0 along the trajectory of (e_1, e_2, e_3) is given by

$$\dot{W}_0 = -e_1(v_0 + \xi_1) - (\omega_0 + \xi_2) \sin e_3. \tag{58}$$

This is equivalent to the previous equation (15) by setting $d_0 = \xi$. Hence W_0 is evaluated as follows:

$$\begin{aligned} \dot{W}_0 \ =& \ -s^T (V_0 + \xi) \\ =& \ \left(V_0 - \frac{1}{2} R_0^{-1} s \right)^T R_0 \left(V_0 - \frac{1}{2} R_0^{-1} s \right) \\ & -q_0 - V_0^T R_0 V_0 \\ & -\gamma_0^2 \left\| \xi + \frac{s}{2\gamma_0^2} \right\|^2 + \gamma_0^2 \|\xi\|^2. \end{aligned} \tag{59}$$

Since $\xi \in \mathcal{L}^2$, it follows that

$$\begin{aligned} & \int_0^\infty (q_0 + V_0^T R_0 V_0) d\tau + W_0(\infty) \\ & \qquad \leq \gamma_0^2 \int_0^\infty \|\xi\|^2 d\tau + W_0(0) < \infty, \end{aligned} \tag{60}$$

and the similar result to the previous case is obtained.

Theorem 4 *The velocity control scheme is constructed via (38), (43), and V_d is synthesized via Proposed Method (path tracking control of kinematics model) (55), (56), (57). Then, the velocity tracking error and the estimate of ϕ, $(\xi^T, \hat{\phi}^T)$ are globally bounded, and $\xi(t)$ converges to zero asymptotically (54). Also, (e_1, e_2, e_3) are locally bounded, and converge to zero asymptotically (10) if either $\lim_{t \to \infty} v_r(t) \neq 0$ or $\lim_{t \to \infty} w_r(t) \neq 0$. Furthermore, V_d is an optimal solution which minimizes the cost functional J_0 (28) for the velocity tracking error $d_0 = \xi$, and the inequality (60) also holds.*

In the path tracking control strategy of dynamic models via proposed method (kinematics control), stability analysis of the path tracking control (kinematics model) and velocity control (dynamic model) can be discussed separately because of H_∞ control property of the path tracking control scheme (kinematics control). Or total stability analysis can be also carried out by utilizing W_2 (51). Since it holds that

$$\dot{W}_2 \le -q_0 - V_0^T R_0 V_0 - \left(\beta\lambda_{\min}(K_1) - \gamma_0^2\right)\|\xi\|^2, \quad (61)$$

then, for β satisfying

$$\beta > \frac{\gamma_0^2}{\lambda_{\min}(K_1)}, \quad (62)$$

it follows that $\dot{W}_2 \le 0$, and the same result as Theorem 4 is obtained. It should be noted that the conditions of Wang & Tsai Method ($\alpha v_r > 0$ and $1 + \alpha e_1 \ne 0$) are not necessary in the proposed methodology (Theorem 4).

V. CONTROL OF DYNAMIC MODEL VIA TORQUE INPUTS BASED ON H_∞ CONTROL CRITERION

Next, we construct velocity control scheme (dynamics control) based on H_∞ control criterion, and provide the path tracking control scheme via torque inputs based on H_∞ control strategy. First, input torques are determined in the following way.

$$B\tau = Y(V, \dot{V}_d)\hat{\phi} + B\tau_0, \quad (63)$$

where τ_0 is a stabilizing signal to be determined later based on H_∞ criterion. Then, the dynamic model of the mobile is written by

$$\begin{aligned} \dot{\xi} &= M^{-1}Y(V, \dot{V}_d)\tilde{\phi} + M^{-1}B\tau_0 \\ &\equiv F_1 + g_{11}d_1 + g_{12}u_0, \quad (64) \\ F_1 &= 0, \quad g_{11} = M^{-1}Y(V, \dot{V}_d), \quad g_{12} = M^{-1}, \\ d_1 &= \tilde{\phi}, \quad u_0 = B\tau_0. \quad (65) \end{aligned}$$

Here we are to stabilize the process via an input u_0 based on H_∞ control strategy where $d_1 = \tilde{\phi}$ is regarded as an external disturbance to the process. For such purpose, the next HJI equation is introduced.

$$\mathcal{L}_{F_1}W + \left\{ \frac{\|\mathcal{L}_{g_{11}}W\|^2}{\gamma_1^2} - (\mathcal{L}_{g_{12}}V)R_1^{-1}(\mathcal{L}_{g_{12}}V)^T \right\} + q_1 = 0. \quad (66)$$

R_1 is a symmetric positive definite matrix and q_1 is a positive function, and those are derived based on the notion of inverse optimality [14], [15], [16], [7] for the given positive constant $\gamma_1 > 0$ and for the solution $W = W_3$ given by

$$W_3 = \frac{1}{2}\xi^T M \xi. \quad (67)$$

The substitution of the solution $W = W_3$ (67) into HJI equation (66) yields

$$\frac{1}{4}\left(\frac{\|\xi^T Y(V, \dot{V}_d)\|^2}{\gamma_1^2} - \xi^T R_1^{-1}\xi \right) + q_1 = 0. \quad (68)$$

Therefore, R_1 and q_1 are obtained such as

$$R_1 = \left(\frac{1}{\gamma_1^2} Y(V, \dot{V}_d) Y(V, \dot{V}_d)^T + K_1 \right)^{-1}, \quad (69)$$

$$q_1 = \frac{1}{4}\xi^T K_1 \xi, \quad (70)$$
$$(K_1 = K_1^T > 0),$$

and u_0 is derived as an optimal solution for the corresponding H_∞ control problem.

$$\begin{aligned} u_0 &= B\tau_0 = -\frac{1}{2}R_1^{-1}\left(\mathcal{L}_{g_{12}}W_3\right)^T = -\frac{1}{2}R_1^{-1}\xi \\ &= -\frac{1}{2}\left(\frac{1}{\gamma_1^2} Y(V, \dot{V}_d) Y(V, \dot{V}_d)^T + K_1 \right)\xi. \quad (71) \end{aligned}$$

Then, $\dot{W}_3$ is evaluated by

$$\begin{aligned} \dot{W}_3 &= \xi^T \left\{ u_0 + Y(V, \dot{V}_d)\tilde{\phi} \right\} \\ &= \left(u_0 + \frac{1}{2}R_1^{-1}\xi \right)^T R_1 \left(u_0 + \frac{1}{2}R_1^{-1}\xi \right) \\ &\quad -q_1 - u_0^T R_1 u_0 \\ &\quad -\gamma_1^2 \left\| \tilde{\phi} - \frac{Y(V, \dot{V}_d)^T\xi}{2\gamma_1^2} \right\|^2 + \gamma_1^2\|\tilde{\phi}\|^2, \quad (72) \end{aligned}$$

and it is shown that u_0 is an optimal solution which minimizes the following cost functional J_1.

$$J_1 = \sup_{\tilde{\phi}\in\mathcal{L}^2} \left\{ \int_0^t (q_1 + u_0^T R_1 u_0) d\tau + W_3(t) - \gamma_1^2 \int_0^t \|\tilde{\phi}\|^2 d\tau \right\}. \quad (73)$$

Also, the next inequality holds.

$$\int_0^t (q_1 + u_0^T R_1 u_0) d\tau + W_3(t) \le \gamma_1^2 \int_0^t \|\tilde{\phi}\|^2 d\tau + W_3(0). \quad (74)$$

The estimation parameter $\hat{\phi}$ is tuned by the same adaptation law (43). Then, $\dot{W}_1$ is evaluated by

$$\dot{W}_1 = -\frac{1}{2}\xi^T R_1^{-1}\xi \le 0, \quad (75)$$

and it is shown that $\xi, \hat{\phi} \in \mathcal{L}^\infty$, $\xi \in \mathcal{L}^2$, and $\dot{\xi}$ is derived, and it follows that $\lim_{t\to\infty}\xi(t) = 0$.

Here, we combine two control schemes based on H_∞ criterion, the path tracking control via velocity inputs (kinematics control) (55), (56), (57), and the velocity control via torque inputs (dynamics control) (63), (71), (43), and provide the path tracking control scheme via torque inputs. Then, V_d is synthesized by considering velocity tracking error ξ, and u_0 is generated by considering estimation error $\tilde{\phi}$. Both control strategies are determined based on H_∞ optimal control criterions. That total concept is shown in the following equations. For W_4 defined by

$$W_4 = W_0 + W_3, \quad (76)$$

$\dot{W}_4$ is evaluated as follows:

$$\begin{aligned}
\dot{W}_4 &= -s^T(V_0+\xi)+\xi^T\left\{u_0+Y(V,\dot{V}_d)\tilde{\phi}\right\} \\
&= \left(V_0-\frac{1}{2}R_0^{-1}s\right)^T R_0\left(V_0-\frac{1}{2}R_0^{-1}s\right) \\
&\quad -q_0-V_0^TR_0V_0 \\
&\quad -\gamma_0^2\left\|\xi+\frac{s}{2\gamma_0^2}\right\|^2+\gamma_0^2\|\xi\|^2 \\
&\quad +\left(u_0+\frac{1}{2}R_1^{-1}\xi\right)^T R_1\left(u_0+\frac{1}{2}R_1^{-1}\xi\right) \\
&\quad -q_1-u_0^TR_1u_0 \\
&\quad -\gamma_1^2\left\|\tilde{\phi}-\frac{Y(V,\dot{V}_d)^T\xi}{2\gamma_0^2}\right\|^2+\gamma_1^2\|\tilde{\phi}\|^2. \qquad (77)
\end{aligned}$$

Then, it is shown that V_0 and u_0 are optimal solutions which minimize the following cost functional J_2.

$$\begin{aligned}
J_2 = \sup_{\xi,\,\tilde{\phi}\in\mathcal{L}^2}&\left\{\int_0^t(q_0+V_0^TR_0V_0)d\tau\right. \\
&+\int_0^t(q_1+u_0^TR_1u_0)d\tau+W_4(t) \\
&\left.-\gamma_0^2\int_0^t\|\xi\|^2d\tau-\gamma_1^2\int_0^t\|\tilde{\phi}\|^2\right\}. \qquad (78)
\end{aligned}$$

Also, the next inequality holds.

$$\begin{aligned}
&\int_0^t(q_0+V_0^TR_0V_0)d\tau+\int_0^t(q_1+u_0^TR_1u_0)d\tau+W_4(t) \\
&\quad\le \gamma_0^2\int_0^t\|\xi\|^2d\tau+\gamma_1^2\int_0^t\|\tilde{\phi}\|^2d\tau+W_4(0). \qquad (79)
\end{aligned}$$

Stability of the proposed adaptive control systems is assured similarly to the previous cases, by considering $\xi\in\mathcal{L}^2$. Or $\dot{W}_2$ is evaluated by

$$\dot{W}_2\le -q_0-V_0^TR_0V_0-\xi^T\left(\frac{\beta}{2}R_1^{-1}-\gamma_0^2I\right)\xi, \qquad (80)$$

and for $\beta>0$ satisfying

$$\beta R^{-1}>2\gamma_0^2I, \qquad (81)$$

it follows that $\dot{W}_2\le 0$, and then stability of the proposed adaptive system is assured.

Theorem 5 *The velocity control scheme is constructed via (63), (71), (43), and V_d is synthesized via Proposed Method (path tracking control of kinematics model) (55), (56), (57). Then, the velocity tracking error and the estimate of ϕ, $(\xi^T,\hat{\phi}^T)$ are globally bounded, and $\xi(t)$ converges to zero asymptotically (54). Also, (e_1,e_2,e_3) are locally bounded, and converge to zero asymptotically (10) when $\lim_{t\to\infty}v_r(t)\neq 0$ or $\lim_{t\to\infty}w_r(t)\neq 0$. Furthermore, V_d and u_0 are optimal solutions which minimize the cost functional J_2 (78) for the velocity tracking error ξ and the estimation error $\tilde{\phi}$, and the inequality (79) also holds.*

Remark 4 Of course, J_2 (78) is a fictitious cost functional, since $\tilde{\phi}$ is not actually an external disturbance but an error of the tuning parameter, and since that is not generally included in $L^2[0,\infty)$. Nevertheless, V_0, which is derived as a solution for that fictitious H_∞ control problem, attain the inequality (79), and it means that the L^2 gains from the disturbances ξ, $\tilde{\phi}$ to the generalized output $\sqrt{q_0+V_0^TRV_0}$ are prescribed by positive constants γ_0,γ_1. Additionally, it should be noted that boundedness of $\tilde{\phi}$ is assured in the stability analysis as the adaptive control schemes.

VI. Concluding Remarks

The problem of constructing adaptive H_∞ trajectory control scheme of uncertain nonholonomic mobile robots, is discussed in this paper. Both kinematics and dynamics control laws are developed based on H_∞ criterion and for processes with unknown parameters. In the present work, the matrix B of the dynamical models of the wheeled mobiles (physical configuration of the mobiles) are assumed to be known a priori for simplicity of notation. However, the present control schemes can be easily extended to the case where even B is unknown. That result together with the globally stable control scheme will be shown in our future research.

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