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Reliable Robust H_∞ Tracking Control for $Lur'e$ Singular Systems with Parameter Uncertainties

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***Abstract*— The problem of reliable robust H_∞ tracking control for a class of *Lur'e* singular systems with parameter uncertainties is studied. The uncertainty is assumed to be convex polytopic. A practical and general failure model of actuator and sensor is considered. Some sufficient conditions about reliable robust H_∞ tracking control are presented for the case of actuator and sensor failures in terms of Linear Matrix Inequalities(*LMIs*). The resultant control systems are reliable in that they guarantee closed-loop system regular, impulse-free, stable with H_∞ performance and the output tracking the reference signal without steady-state error when all control components are operational as well as when some control components experience failures. Finally, a numerical example is given to show the effectiveness of the proposed methods.**

I. INTRODUCTION

Control of singular systems has been extensively studied in the past years due to the fact that singular systems better describe physical systems than regular ones. A great number of results based on the theory of regular systems (or state-space systems) have been extended to the area of singular systems, such as, robust stability and robust stabilization problem [1], [2], robust H_∞ control problem [3]–[5], robust H_∞ filtering problem [6] etc.

However, all the aforementioned results are under a full reliability assumption that all control components of the systems are in good working condition. As is well known, failures of actuators or sensors, in practical engineering systems, often occur, which may lead to intolerable performance of the closed-loop system. Therefore, from a safety as well as performance point of view, it is required to have a reliable controller design which can tolerate actuator or sensor failures and guarantee the stability and performance of resultant closed-loop system. Furthermore, there are lots of results about reliable control for state-space systems [7]–[10]. For singular systems, [11] studied the reliable H_∞ control problem with actuator failures and multiple time delays, but the reliable controller design method is based on a basic assumption that control component failures are modeled as outages, that is, when a failure occurs, the signal(in the case of sensors) or the control action (in the case of actuators) simply becomes zero. In this paper, a more practical and general failure model is adopted for sensor and actuator failures, which consists of a scaling factor with upper and lower bounds to the signal to be measured or to the control action, plus a disturbance. This model was introduced by [12]. For actuators, this general failure model could represent faults in the driving circuitry before the final control actuator. For sensors, it could represent faults in the signal conditioning circuitry at the sensors. To the author's knowledge, the reliable control problem of singular systems based on such failure model has not yet been fully investigated.

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On the other hand, designing a controller which can guarantee the output of controlled system tracking the reference signal is of theoretical and practical meaning. A few results were presented on the problem of reliable tracking control [13], [14]. To the author's knowledge, the study of reliable robust H_∞ tracking control for singular systems with actuator or sensor failures has not yet been tackled. Therefore, this topic remains challenging.

In this paper, we investigate the problem of reliable robust H_∞ tracking controller design for uncertain singular systems with actuator or sensor failures. The parameter uncertainties under consideration are possibly time-varying convex polytopic. We aim to design a linear memoryless controller such that, for all admissible uncertainties and actuator or sensor failures, the resulting closed-loop system is regular, impulse-free, stable with an H_∞ norm bound constraint, and the output of the resultant closed-loop system tracking the reference signal without steady-state error. Finally, a numerical example is also given to illustrate the effectiveness of our method.

II. PROBLEM FORMULATION AND PRELIMINARIES

Consider the following uncertain Lur'e singular system

$$\begin{cases} E\dot{x}(t) = A(\theta)x(t) + B_1(\theta)u(t) + D_1(\theta)\omega(t) \\ \qquad\qquad + E_1 f(\sigma(t)), \quad \sigma(t) = Cx(t), \\ y(t) = C_1(\theta)x(t) + D_2(\theta)\eta(t), \\ z(t) = C_2(\theta)x(t) + B_2(\theta)u(t) + D_3(\theta)\omega(t), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector, $\omega(t) \in \mathbb{R}^p$ is the disturbance input vectors from $L_2[0,\infty)$, $\eta(t) \in \mathbb{R}^q$ is the square-integrable measurement noise, $z(t) \in \mathbb{R}^l$ is the control output vector, $y(t) \in \mathbb{R}^q$ is the measured output vector. The matrix $E \in \mathbb{R}^{n\times n}$ may be singular, we shall also assume the $\operatorname{rank} E = r \leq n$. $A(\theta)$, $B_1(\theta)$, $B_2(\theta)$, $C_1(\theta)$, $C_2(\theta)$, $D_1(\theta)$, $D_2(\theta)$ and

$D_3(\theta)$ denote the unknown possibly time-varying polytopic uncertainties which satisfy

$$\begin{aligned} A(\theta) &= \sum_{i=1}^{k} A_i\theta_i,\ B_1(\theta) = \sum_{i=1}^{k} B_{1i}\theta_i, \\ B_2(\theta) &= \sum_{i=1}^{k} B_{2i}\theta_i,\ C_1(\theta) = \sum_{i=1}^{k} C_{1i}\theta_i, \\ C_2(\theta) &= \sum_{i=1}^{k} C_{2i}\theta_i,\ D_1(\theta) = \sum_{i=1}^{k} D_{1i}\theta_i, \\ D_2(\theta) &= \sum_{i=1}^{k} D_{2i}\theta_i,\ D_3(\theta) = \sum_{i=1}^{k} D_{3i}\theta_i, \end{aligned} \tag{2}$$

where $A_i,\ B_{1i},\ B_{2i},\ C_{1i},\ C_{2i},\ D_{1i},\ D_{2i}$ and $D_{3i}(i = 1, \cdots, k)$ are known constant matrices of appropriate dimensions. $\theta = [\theta_1, \theta_2, \cdots, \theta_k]^T \in \mathbb{R}^k$ is the uncertain possibly time-varying constant parameter vector satisfying

$$\theta = \Omega = \{\theta \in \mathbb{R}^k : \theta_i \geq 0, \sum_{i=1}^{k}\theta_i = 1\} \tag{3a}$$

$$\theta(t) = \Omega = \{\theta(t) \in \mathbb{R}^k : \theta_i(t) \geq 0, \sum_{i=1}^{k}\theta_i(t) = 1\} \tag{3b}$$

$\sigma = [\sigma_1(t)\ \ \sigma_2(t)\ \ \cdots\ \ \sigma_n(t)] \in \mathbb{R}^n$, $f(\sigma) = [f_1(\sigma_1)\ \ f_2(\sigma_2)\ \ \cdots\ \ f_n(\sigma_n)] \in \mathbb{R}^n$ denote the nonlinear vectors. In this paper, every nonlinear term is assumed to be of the following form

$$\begin{aligned} & f_j(\bullet) \in K_j[0, k_j] = \{f_j(\sigma_j) | f_j(0) = 0, \\ & \quad 0 < \sigma_j f_j(\sigma_j) \leq k_j \sigma_j^2 (\sigma_j \neq 0), j = 1, 2, \cdots, n\}. \end{aligned} \tag{4}$$

For the control input $u(t)$, let $u^f(t)$ be the signal from the actuator that has failed. Then the following actuator failure model, as introduced in [12] and [9], is adopted here

$$u^f(t) = M_1 u(t) + M_u \delta_1(t) \tag{5}$$

where $M_1 = \text{diag}\{m_{11}, m_{12}, \cdots, m_{1m}\}$ is the actuator failure matrix, $M_u = \text{diag}\{m_{u1}, m_{u2}, \cdots, m_{um}\}$ is the actuator failure disturbance matrix, $\delta_1(t) = [\delta_{11}(t)\ \ \delta_{12}(t)\ \ \cdots\ \ \delta_{1m}(t)] \in \mathbb{R}^m$ is the square-integrable actuator failure disturbance vector. Furthermore, m_{1i} and m_{ui} satisfy

$$0 \leq \underline{m}_{1i} \leq m_{1i} \leq \bar{m}_{1i},\ \bar{m}_{1i} \geq 1,\ i = 1, 2, \cdots, m, \tag{6a}$$

$$0 \leq m_{ui} \leq \bar{m}_{ui},\ i = 1, 2, \cdots, m. \tag{6b}$$

Denote

$$\begin{aligned} M_{10} &= \text{diag}\{\tilde{m}_{11}, \tilde{m}_{12}, \cdots, \tilde{m}_{1m}\}, \\ J_1 &= \text{diag}\{j_{11}, j_{12}, \cdots, j_{1m}\}, \\ L_1 &= \text{diag}\{l_{11}, l_{12}, \cdots, l_{1m}\} \end{aligned} \tag{7}$$

where $\tilde{m}_{1i} = \frac{1}{2}(\bar{m}_{1i} + \underline{m}_{1i})$, $j_{1i} = \frac{\bar{m}_{1i} - \underline{m}_{1i}}{\bar{m}_{1i} + \underline{m}_{1i}}$, $l_{1i} = \frac{m_{1i} - \tilde{m}_{1i}}{\tilde{m}_{1i}}$.

Then from (7), we can easily obtain

$$M_1 = M_{10}(I + L_1),\ |L_1| \leq J_1 \leq I. \tag{8}$$

Similarly, let $y^f(t)$ be the signal from the sensor that has failed. The sensor failure model adopted is

$$y^f(t) = M_2 y(t) + M_y \delta_2(t), \tag{9}$$

where $M_2 = \text{diag}\{m_{21}, m_{22}, \cdots, m_{2q}\}$ is the sensor failure matrix, $M_y = \text{diag}\{m_{y1}, m_{y2}, \cdots, m_{yq}\}$ is the sensor failure disturbance matrix, and $\delta_2(t) = [\delta_{21}(t)\ \ \delta_{22}(t)\ \ \cdots\ \ \delta_{2q}(t)] \in \mathbb{R}^q$ is the square-integrable sensor failure disturbance vector. Furthermore, m_{2i} and m_{yi} satisfy

$$0 \leq \underline{m}_{2i} \leq m_{2i} \leq \bar{m}_{2i},\ \bar{m}_{2i} \geq 1,\ i = 1, 2, \cdots, q, \tag{10a}$$

$$0 \leq m_{yi} \leq \bar{m}_{yi},\ i = 1, 2, \cdots, q. \tag{10b}$$

Remark 1: When $m_{1i} = 0,\ m_{ui} = 0$ or $m_{2i} = 0,\ m_{yi} = 0$, it covers the complete failure of $u_i(t)$ or $y_i(t)$ respectively. When $m_{1i} = 1,\ m_{ui} = 0$ or $m_{2i} = 1,\ m_{yi} = 0$, which corresponds to the case of no failure of $u_i(t)$ or $y_i(t)$ respectively. If $m_{1i} > 0$ or $m_{2i} > 0$, it covers the partial failure of $u_i(t)$ or $y_i(t)$ respectively.

The unforced singular system of (1) can be written as

$$\begin{cases} E\dot{x}(t) = A(\theta)x(t) + D_1(\theta)\omega(t) + E_1 f(\sigma(t)) \\ \quad z(t) = C_2(\theta)x(t) + D_3(\theta)\omega(t). \end{cases} \tag{11}$$

Definition 1: [2], [15] 1) The pair $(E, A(\theta))$ is said to be regular if $det(sE - A(\theta))$ is not identically zero.
2) The pair $(E, A(\theta))$ is said to be impulse-free if $deg(det(sE - A(\theta))) = \text{rank } E$.

Lemma 1: [2] Suppose the pair $(E, A(\theta))$ is regular and impulse free, then the solution to (11) exists and is impulse free and unique on $[0, \infty)$.

Definition 2: [2], [15] 1) The singular system (11) is said to be regular and impulse free if the pair $(E, A(\theta))$ is regular and impulse free.
2) The singular system (11) is said to stable if for any compatible initial conditions $x(0) \in \mathbb{R}^n$, there exists scalars $\alpha > 0$ and $\beta > 0$ such that $\|x(t)\|_2 \leq \alpha e^{-\beta t}\|x(0)\|_2$.
3) The singular system (11) is said to robustly stable if system (11) is regular, impulse-free and stable for all admissible uncertainties (2) and (3).

Definition 3: Let the constant $\gamma > 0$ be given. Singular system (11) is said to be robustly stable with an H_∞ norm bound γ if for all admissible uncertainties (2) and (3), singular system (11) satisfies
1) Singular system (11) (with $\omega(t) = 0$) is robustly stable;
2) For the zero initial condition of $x(t)$ and non-zero $\omega(t)$, the following condition holds

$$J = \int_0^\infty z^T(t)z(t)dt - \gamma^2 \int_0^\infty \omega^T(t)\omega(t)dt < 0.$$

The reliable robust H_∞ tracking problem is to design a state-feedback controller such that
1) The resultant closed-loop system is robustly stable with an H_∞ norm bound γ when all control components are operational, but also in case of some sensor and actuator failures by (5) and (9), respectively.
2) The output $y(t)$ tracks the reference signal $r(t)$ without steady-state error, that is

$$\lim_{t \to \infty} e(t) = 0,\ e(t) = r(t) - y(t). \tag{12}$$

Throughout this paper, we shall use the following concepts and introduce the following some useful lemmas.

Lemma 2: [2], [5] The singular system $E\dot{x}(t) = Ax(t)$ or the pair $(E,\ A)$ is regular, impulse-free and stable if and only if there exists a matrix P such that

$$EP^T = PE^T \geq 0, \quad AP^T + PA^T < 0.$$

Lemma 3: [16] Let A, D, E and $F(t)$ be real matrices of appropriate dimensions with $\|F(t)\| \leq U$, where U is an known real constant matrix, there exists a scalar $\epsilon > 0$ such that $DF(t)E + E^T F^T(t) D^T \leq \epsilon DUD^T + \epsilon^{-1} E^T U E$.

III. MAIN RESULTS

A. Bounded Real Lemma

In this section, we will present the stability criteria of system (11). According to Definition 3, two issues must be dealt with: i) robust stability of system (11) with $\omega(t) = 0$; and ii)$J = \int_0^\infty z^T(t)z(t)dt - \gamma^2 \int_0^\infty \omega^T(t)\omega(t)dt < 0$ for a prescribed constant $\gamma > 0$.

Theorem 1: Suppose that all the pairs $(E, A_i, C_{2i}), i = 1, \cdots, k$ is observable, for a given constant $\gamma > 0$, the singular system (11) is robustly stable with an H_∞ norm bound γ if there exist matrix $P > 0$ and scalar $\varepsilon > 0$, for all $i = 1, \cdots, k$, such that

$$EP^T = PE^T \geq 0 \tag{13}$$

$$\begin{bmatrix} \Gamma_{11} & D_{1i} & PC_{2i}^T & PC^T K^T \\ * & -\gamma^2 I & D_{3i}^T & 0 \\ * & * & -I & 0 \\ * & * & * & -\varepsilon I \end{bmatrix} < 0 \tag{14}$$

where $\Gamma_{11} = A_i P^T + PA_i^T + \varepsilon E_1 E_1^T$, $K = \mathrm{diag}\{k_1, \cdots, k_n\}$.

Proof: First, we prove robustly stability of the system (11) with $\omega(t) = 0$. Suppose both (13) an (14) hold, then according to [17], if (14) is satisfied, one has

$$\begin{bmatrix} \tilde{\Gamma}_{11} & D_1(\theta) & PC_2^T(\theta) & PC^T K^T \\ * & -\gamma^2 I & D_3^T(\theta) & 0 \\ * & * & -I & 0 \\ * & * & * & -\varepsilon I \end{bmatrix} < 0 \tag{15}$$

where $\tilde{\Gamma}_{11} = A(\theta)P^T + PA^T(\theta) + \varepsilon E_1 E_1^T$.

From (15), we have that

$$A(\theta)P^T + PA^T(\theta) < 0 \tag{16}$$

By Lemma 2, it follows from (16) and (13) that the pair $(E, A(\theta))$ is regular and impulse free. So by Definition 2, we can derive that the singular system (11) is regular and impulse free.

Then we shall show the stability of the singular system (11). From (14), by Schur complement argument, it is easy to see that

$$A_i P^T + PA_i^T < 0 \tag{17}$$

By Lemma 2, it follows that from (17) and (13) the pair (E, A_i) is regular, impulse free and stable.

The regularity and the absence of impulses of the pair (E, A_i) implies that there exist two invertible matrices G and $H \in \mathbb{R}^{n\times n}$ such that [18]

$$\begin{aligned} \bar{E} &:= GEH = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \\ \bar{A}_i &:= GA_iH = \begin{bmatrix} A_{ir} & 0 \\ 0 & I_{n-r} \end{bmatrix} (i = 1, \cdots, k) \end{aligned} \tag{18}$$

where $I_r \in \mathbb{R}^{r\times r}$, $I_{n-r} \in \mathbb{R}^{(n-r)\times(n-r)}$ are identity matrices, $A_{ir} \in \mathbb{R}^{r\times r}$. According to (16), let

$$\begin{aligned} \bar{P} &:= GPH^{-T} = \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{21} & \bar{P}_{22} \end{bmatrix},\ \bar{E}_1 := GE_1, \\ \bar{C} &:= CH,\ \bar{D}_{1i} := GD_{1i}, \\ \bar{C}_{2i} &:= C_{2i}H,\ \bar{K} := K,\ \bar{D}_{3i} := D_{3i} \end{aligned} \tag{19}$$

Then, from (13), we have

$$\bar{E}\bar{P}^T = \bar{P}\bar{E}^T \geq 0 \tag{20}$$

Pre-multiplying $\mathrm{diag}\{G, I, I, I\}$ and post-multiplying $\mathrm{diag}\{G^T, I, I, I\}$ to the left and right side of (14) yields

$$\begin{bmatrix} \bar{\Gamma}_{11} & \bar{D}_{1i} & \bar{P}\bar{C}_{2i}^T & \bar{P}\bar{C}^T\bar{K}^T \\ * & -\gamma^2 I & \bar{D}_{3i}^T & 0 \\ * & * & -I & 0 \\ * & * & * & -\varepsilon I \end{bmatrix} < 0 \tag{21}$$

where $\bar{\Gamma}_{11} = \bar{A}_i\bar{P}^T + \bar{P}\bar{A}_i^T + \varepsilon\bar{E}_1\bar{E}_1^T$.

Noting the expression of $\bar{E}$ in (18) and using (20), we can deduce that $\bar{P}_{11} = \bar{P}_{11}^T \geq 0$ and $\bar{P}_{21} = 0$, therefore $\bar{P}$ reduces to

$$\bar{P} = \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \\ 0 & \bar{P}_{22} \end{bmatrix}. \tag{22}$$

Substituting (18) and (22) into (21), one eventually gets

$$\begin{bmatrix} A_{ir}\bar{P}_{11} + \bar{P}_{11}A_{ir}^T & \bar{P}_{12} \\ \bar{P}_{12}^T & \bar{P}_{22} + \bar{P}_{22}^T \end{bmatrix} < 0 \tag{23}$$

Since the inequality (23) holds, we have that $\bar{P}_{22}$ is invertible.

Now, let $\xi(t) = H^{-1}x(t) = \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix}$, where $\xi_1(t) \in \mathbb{R}^r$, $\xi_2(t) \in \mathbb{R}^{n-r}$. Using the expressions in (18) and (19), the singular system (11) can be described as

$$\begin{cases} \bar{E}\dot{\xi}(t) = \bar{A}(\theta)\xi(t) + \bar{D}_1(\theta)\omega(t) + \bar{E}_1 f(\eta(t)) \\ z(t) = \bar{C}_2(\theta)\xi(t) + \bar{D}_3(\theta)\omega(t). \end{cases} \tag{24}$$

where $\eta(t) = \bar{C}\xi(t)$. It is easy to see that the stability of the singular system (11) $\omega(t) = 0$ is equivalent to that of the system (24). In view of this, next we shall prove that the system (24) with $\omega(t) = 0$ is stable. Since $\bar{P}_{11} = \bar{P}_{11}^T \geq 0$ and $A_{ir}\bar{P}_{11} + \bar{P}_{11}A_{ir}^T < 0$, as (22) shows, it follows that $\bar{P}_{11} > 0$. Define

$$V(\xi(t)) = \xi_1^T(t)\bar{P}_{11}^{-1}\xi_1(t) = \xi^T(t)\bar{P}^{-1}\bar{E}\xi(t) \tag{25}$$

The expression in (4) is equivalent to be described as

$$\begin{aligned} &\sigma_j f_j(\sigma_j)(\sigma_j f_j(\sigma_j) - k_j\sigma_j^2) \leq 0 \quad (j = 1, 2, \cdots, n) \\ &\Rightarrow f_j^2(\sigma_j) \leq k_j^2\sigma_j^2 \Rightarrow \|f(\sigma_j)\|^2 \leq \|KCx(t)\|^2 \end{aligned}$$

Therefore

$$f^T f \leq \xi^T(t)\bar{C}^T\bar{K}^T\bar{K}\bar{C}\xi(t) \tag{26}$$

Then, taking time derivative of $V(\xi(t))$ and adopting the multiconvexity concept in [17] yield:

$$\begin{aligned}\dot{V}(\xi(t)) =&\dot{\xi}(t)\bar{E}^T\bar{P}^{-T}\xi(t) + \xi^T(t)\bar{P}^{-1}\bar{E}\dot{\xi}(t)\\ =&\xi^T(t)\bar{P}^{-1}(\bar{A}_i\bar{P}^T + \bar{P}\bar{A}_i^T)\bar{P}^{-T}\xi(t)\\ &+ f^T\bar{E}_1^T\bar{P}^{-T}\xi(t) + \xi^T(t)\bar{P}^{-1}\bar{E}_1 f\end{aligned}$$

According to Lemma 3, we have $\dot{V}(\xi(t)) \leq \xi^T(t)\bar{P}^{-1}\bar{\Omega}\bar{P}^{-T}\xi(t)$, where

$$\bar{\Omega} = \begin{bmatrix} \bar{A}_i\bar{P}^T + \bar{P}\bar{A}_i^T + \varepsilon\bar{E}_1\bar{E}_1^T & \bar{P}\bar{C}^T\bar{K}^T \\ * & -\varepsilon I \end{bmatrix}.$$

From (21), by Schur complement argument, we know that $\bar{\Omega} < 0$. which implies $\dot{V}(\xi(t)) < 0$. So there exist $\beta > 0$ such that

$$\dot{V}(\xi(t)) \leq -\beta V(\xi(t)). \tag{27}$$

From (27), we have $V(\xi(t)) \leq e^{-\beta t}V(\xi(0))$, which gives

$$\|\xi(t)\|_2 \leq \alpha e^{-\beta t}\|\xi(0)\|_2, \quad \alpha = \frac{\lambda_{max}(\bar{P}^{-1}\bar{E})}{\lambda_{min}(\bar{P}^{-1}\bar{E})}.$$

According to Definition 2, we know that the system (11) with $\omega(t) = 0$ is robustly stable.

Considering the effect of the external disturbance on the system, we can show by taking the time derivative of $V(\xi(t))$ and using condition (21) that

$$\dot{V}(\xi(t)) \leq -z^T(t)z(t) + \gamma^2\omega^T(t)\omega(t) \tag{28}$$

Integrating both side of (28) from zero to ∞ and noting that the assumption of zero initial condition, we obtain

$$\begin{aligned}J =& \int_0^\infty z^T(t)z(t)dt - \gamma^2\int_0^\infty \omega^T(t)\omega(t)dt\\ \leq& \int_0^\infty z^T(t)z(t)dt - \gamma^2\int_0^\infty \omega^T(t)\omega(t)dt\\ &+ V(\infty) - V(0) < 0\end{aligned}$$

Consequently, from Definition 3, the singular system (11) is robustly stable with an H_∞ norm bound γ. ■

B. Reliable Robust H_∞ Tracking Controller Design

In this section, we present the solution to reliable robust H_∞ tracking controller design problem for the singular system (1) with respect to actuator failures or sensor failures, respectively.

1) Actuator Failures Case: It is well known that the tracking error integral action of a controller can effectively eliminate the steady-state tracking error. In order to obtain a robust H_∞ tracking controller with state feedback plus tracking error integral, and we define the augmented state vector $\varsigma(t) = \left[x^T(t), (\int_0^t e(t)dt)^T\right]^T$, disturbance vector $\tilde{\omega}(t) = \left[\omega^T(t), \eta^T(t), \delta_1^T(t), r^T(t)\right]^T$, and introduce the following augmented description of singular system (1) with actuator fault model (5),

$$\begin{cases}\tilde{E}\dot{\varsigma}(t) =\tilde{A}(\theta)\varsigma(t) + \tilde{B}_1(\theta)M_1u(t) + \tilde{D}_1(\theta)\tilde{\omega}(t)\\ \qquad + \tilde{E}_1\tilde{f}(\tilde{\sigma}(t)), \quad \tilde{\sigma}(t) = \tilde{C}\varsigma(t),\\ z =\tilde{C}_2(\theta)\varsigma(t) + B_2(\theta)M_1u(t) + \tilde{D}_3(\theta)\tilde{\omega}(t)\end{cases} \tag{29}$$

where

$$\tilde{E} = \begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix},\ \tilde{A}(\theta) = \begin{bmatrix} A(\theta) & 0 \\ -C_1(\theta) & 0 \end{bmatrix},$$
$$\tilde{B}_1(\theta) = \begin{bmatrix} B_1(\theta) \\ 0 \end{bmatrix},\ \tilde{E}_1 = \begin{bmatrix} E_1 & 0 \\ 0 & 0 \end{bmatrix},$$
$$\tilde{D}_1(\theta) = \begin{bmatrix} D_1(\theta) & 0 & B_1(\theta)M_u & 0 \\ 0 & -D_2(\theta) & 0 & 1 \end{bmatrix},$$
$$\tilde{f}(\tilde{\sigma}(t)) = \begin{bmatrix} f(\tilde{\sigma}) \\ 0 \end{bmatrix},\ \tilde{C} = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix},$$
$$\tilde{C}_2(\theta) = \begin{bmatrix} C_2(\theta) & 0 \end{bmatrix},$$
$$\tilde{D}_3(\theta) = \begin{bmatrix} D_3(\theta) & 0 & B_2(\theta)M_u & 0 \end{bmatrix}$$

and $\tilde{A}(\theta) = \sum_{i=1}^k \tilde{A}_i\theta_i,\ \tilde{B}_1(\theta) = \sum_{i=1}^k \tilde{B}_{1i}\theta_i,\ \tilde{D}_1(\theta) = \sum_{i=1}^k \tilde{D}_{1i}\theta_i,\ \tilde{C}_2(\theta) = \sum_{i=1}^k \tilde{C}_{2i}\theta_i,\ \tilde{D}_3(\theta) = \sum_{i=1}^k \tilde{D}_{3i}\theta_i$, where

$$\tilde{A}_i = \begin{bmatrix} A_i & 0 \\ -C_{1i} & 0 \end{bmatrix},\ \tilde{B}_{1i} = \begin{bmatrix} B_{1i} \\ 0 \end{bmatrix},$$
$$\tilde{D}_{1i} = \begin{bmatrix} D_{1i} & 0 & B_{1i}M_u & 0 \\ 0 & -D_{2i} & 0 & 1 \end{bmatrix},\ \tilde{C}_{2i} = \begin{bmatrix} C_{2i} & 0 \end{bmatrix},$$
$$\tilde{D}_{3i} = \begin{bmatrix} D_{3i} & 0 & B_{2i}M_u & 0 \end{bmatrix},\ (i = 1, \cdots, k).$$

Consider the augmented system (29) with the following state feedback tracking controller

$$u(t) = L\varsigma(t) \tag{30}$$

then the resultant closed-loop augmented system is

$$\begin{cases}\tilde{E}\dot{\varsigma}(t) =(\tilde{A}(\theta) + \tilde{B}_1(\theta)M_1L)\varsigma(t) + \tilde{D}_1(\theta)\tilde{\omega}(t)\\ \qquad + \tilde{E}_1\tilde{f}(\tilde{\sigma}(t)),\\ z =(\tilde{C}_2(\theta) + B_2(\theta)M_1L)\varsigma(t) + \tilde{D}_3(\theta)\tilde{\omega}(t).\end{cases} \tag{31}$$

Lemma 4: [13], [14] For the system (29), if there exists a controller (30) such that the resultant closed-loop system (31) is robustly stable, then the measured output $y(t)$ tracks the reference signal $r(t)$ without steady-state error, that is $\lim_{t\to\infty} e(t) = 0$.

According to Theorem 1 and Lemma 4, we give the following sufficient condition for designing the reliable robust H_∞ tracking controller.

Theorem 2: Suppose that all the pairs $(\tilde{E}, \tilde{A}_i, \tilde{C}_{2i}), i = 1, \cdots, k$ is observable, for a given constant $\gamma > 0$, if there exist matrices $P > 0$, Y and scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ for all $i = 1, \cdots, k$, such that

$$\tilde{E}P^T = P\tilde{E}^T \geq 0 \tag{32}$$

$$\begin{bmatrix} (1,1) & \tilde{D}_{1i} & (1,3) & P\tilde{C}^T\tilde{K}^T & Y^TJ_1^{1/2} \\ * & -\gamma^2 I & \tilde{D}_{3i}^T & 0 & 0 \\ * & * & (3,3) & 0 & 0 \\ * & * & * & -\varepsilon_1 I & 0 \\ * & * & * & * & -\varepsilon_2 I \end{bmatrix} < 0 \quad (33)$$

then, we can construct a reliable robust H_∞ tracking control law $u(t) = YP^{-T}\varsigma(t)$ such that the resultant closed-loop singular system (31) is robustly stable with an H_∞ norm bound γ, and the measured output $y(t)$ tracks the reference signal $r(t)$ without steady-state error. where

$$\begin{aligned} (1,1) =& \tilde{A}_iP^T + P\tilde{A}_i^T + \tilde{B}_{1i}M_{10}Y + Y^TM_{10}^T\tilde{B}_{1i}^T \\ &+ \varepsilon_1\tilde{E}_1\tilde{E}_1^T + \varepsilon_2\tilde{B}_{1i}M_{10}J_1M_{10}^T\tilde{B}_{1i}^T, \\ (1,3) =& P\tilde{C}_{2i}^T + Y^TM_{10}^TB_{2i}^T + \varepsilon_2\tilde{B}_{1i}M_{10}J_1M_{10}^TB_{2i}^T, \\ (3,3) =& -I + \varepsilon_2B_{2i}M_{10}J_1M_{10}^TB_{2i}^T, \\ \tilde{K} =& \operatorname{diag}\{K, 0\}. \end{aligned}$$

Proof: By Theorem 1, the system (29) is robustly stable with an H_∞ norm bound γ if there exist matrix $P > 0$ and scalar $\varepsilon > 0$ for all $i = 1, \cdots, k$, the following inequalities holds,

$$\tilde{E}P^T = P\tilde{E}^T \geq 0 \quad (34)$$

$$\begin{bmatrix} \Upsilon_{11} & \tilde{D}_{1i} & \Upsilon_{13} & P\tilde{C}^T\tilde{K}^T \\ * & -\gamma^2 I & \tilde{D}_{3i}^T & 0 \\ * & * & -I & 0 \\ * & * & * & -\varepsilon I \end{bmatrix} < 0 \quad (35)$$

where $\Upsilon_{11} = (\tilde{A}_i + \tilde{B}_{1i}M_1L)P^T + P(\tilde{A}_i + \tilde{B}_{1i}M_1L)^T + \varepsilon\tilde{E}_1\tilde{E}_1^T$, $\Upsilon_{13} = P(\tilde{C}_{2i} + B_{2i}M_1L)^T$.

From (8) and Lemma 3, and letting $L = YP^{-T}$, we know that (35) is equivalent to (33).

Then according to Lemma 4, the singular system (31) is robustly stable with an H_∞ norm bound γ, and the measured output $y(t)$ tracks the reference signal $r(t)$ without steady-state error. ■

Remark 2: Since $\operatorname{rank} E = r < n$, there exists a matrix $\Phi \in \mathbb{R}^{(n+q)\times(n-r)}$ with $\operatorname{rank}\Phi = n - r$ such that $\tilde{E}\Phi = 0$. Define $P = \tilde{E}\Theta + \Psi\Phi^T$, where Θ is positive definite matrix, Ψ is matrix with appropriate dimension respectively. Obviously, $\tilde{E}P^T = P\tilde{E}^T \geq 0$ holds. Substituting $P = \tilde{E}\Theta + \Psi\Phi^T$ into (33), we obtain a new ***LMI*** denoted by (33)'. Solving Θ and Ψ from (33)' and using the relation $P = \tilde{E}\Theta + \Psi\Phi^T$, we can finally design a reliable robust H_∞ tracking controller with respect to actuator failures as

$$u(t) = Y(\tilde{E}\Theta + \Psi\Phi^T)^{-T}\varsigma(t). \quad (36)$$

Remark 3: The solution of reliable robust tracking control problem is given in Theorem 2. If the disturbance attenuation level γ is given, we can directly solve the feasibility problem of ***LMI*** (33)' to obtain a suitable state feedback control law. Otherwise, we can solve the following optimization problem

$$\begin{aligned} &\min \gamma \\ &s.t.\ LMI\ (33)', \Theta > 0, \varepsilon_1 > 0, \varepsilon_2 > 0 \end{aligned}$$

to obtain a minimal disturbance attenuation level.

Remark 4: According to Remark 1, when there are no failures of $u(t)$, the standard robust H_∞ tracking controller also can be designed by letting $m_{1i} = 1$, $m_{ui} = 0$ in Theorem 2.

2) Sensor Failures Case: Consider the sensor faults model (9), following the same philosophy as that in actuator faults, and define disturbance vector $\hat{\omega}(t) = \left[\omega^T(t), \eta^T(t), \delta_2^T(t), r^T(t)\right]^T$, we can construct the augmented system as follows,

$$\begin{cases} \tilde{E}\varsigma(t) = \hat{A}(\theta)\varsigma(t) + \tilde{B}_1(\theta)u(t) + \hat{D}_1(\theta)\hat{\omega}(t) \\ \qquad + \tilde{E}_1\tilde{f}(\tilde{\sigma}(t)), \quad \tilde{\sigma}(t) = \tilde{C}\varsigma(t), \\ z = \tilde{C}_2(\theta)\varsigma(t) + B_2(\theta)u(t) + \hat{D}_3(\theta)\hat{\omega}(t) \end{cases} \quad (37)$$

where $\varsigma(t)$, $\tilde{E}$, $\tilde{B}_1(\theta)$, $\tilde{E}_1$, $\tilde{C}_2(\theta)$, $\tilde{f}(\tilde{\sigma}(t))$ are the same as those in (29), $\hat{A}(\theta) = \begin{bmatrix} A(\theta) & 0 \\ -M_2C_1(\theta) & 0 \end{bmatrix}$, $\hat{D}_1(\theta) = \begin{bmatrix} D_1(\theta) & 0 & 0 & 0 \\ 0 & -M_2D_2(\theta) & -M_y & 1 \end{bmatrix}$, $\hat{D}_3(\theta) = \begin{bmatrix} D_3(\theta) & 0 & 0 & 0 \end{bmatrix}$ and $\hat{A}(\theta) = \sum_{i=1}^{k}\hat{A}_i\theta_i$, $\hat{D}_1(\theta) = \sum_{i=1}^{k}\hat{D}_{1i}\theta_i$, $\hat{D}_3(\theta) = \sum_{i=1}^{k}\hat{D}_{3i}\theta_i$, where

$$\begin{aligned} \hat{A}_i &= \begin{bmatrix} A_i & 0 \\ -M_2C_{1i} & 0 \end{bmatrix}, \\ \hat{D}_{1i} &= \begin{bmatrix} D_{1i} & 0 & 0 & 0 \\ 0 & -M_2D_{2i} & -M_y & 1 \end{bmatrix}, \\ \hat{D}_{3i} &= \begin{bmatrix} D_{3i} & 0 & 0 & 0 \end{bmatrix}, \ (i = 1, \cdots, k). \end{aligned}$$

Consider the augmented system (37) with the state feedback tracking controller (30), then the resultant closed-loop augmented system is

$$\begin{cases} \tilde{E}\dot{\varsigma}(t) = (\hat{A}(\theta) + \tilde{B}_1(\theta)L)\varsigma(t) + \hat{D}_1(\theta)\hat{\omega}(t) \\ \qquad + \tilde{E}_1\tilde{f}(\tilde{\sigma}(t)), \\ z = (\tilde{C}_2(\theta) + B_2(\theta)L)\varsigma(t) + \hat{D}_3(\theta)\hat{\omega}(t) \end{cases} \quad (38)$$

According to Theorem 1 and Lemma 4, the following theorem is easily derived.

Theorem 3: Suppose that all the pairs $(\tilde{E}, \hat{A}_i, \hat{C}_{2i}), i = 1, \cdots, k$ is observable, for a given constant $\gamma > 0$, if there exist matrices $P > 0$, V and scalar $\varepsilon > 0$ for all $i = 1, \cdots, k$, such that

$$\tilde{E}P^T = P\tilde{E}^T \geq 0 \quad (39)$$

$$\begin{bmatrix} \Theta_{11} & \hat{D}_{1i} & P\tilde{C}_{2i}^T + V^TB_{2i}^T & P\tilde{C}^T\tilde{K}^T \\ * & -\gamma^2 I & \hat{D}_{3i}^T & 0 \\ * & * & -I & 0 \\ * & * & * & -\varepsilon I \end{bmatrix} < 0 \quad (40)$$

then, we can construct a reliable robust H_∞ tracking control law $u(t) = VP^{-T}\varsigma(t)$ such that the resultant closed-loop singular system (38) is robustly stable with an H_∞ norm bound γ, and the measured output $y(t)$ tracks the reference signal $r(t)$ without steady-state error. where $\Theta_{11} = \hat{A}_iP^T + P\hat{A}_i^T + \tilde{B}_{1i}V + V^T\tilde{B}_{1i}^T + \varepsilon\tilde{E}_1\tilde{E}_1^T$, $\tilde{K} = \operatorname{diag}\{K, 0\}$.

Proof: Following the same line as that in the proof of Theorem 2, Theorem 3 follows immediately. ■

Remark 5: Remark 2 and 3 is also suitable for sensor failure case.

Remark 6: According to Remark 1, when there are no failures of $y(t)$, the standard robust H_∞ tracking controller also can be designed by letting $m_{2i} = 1$, $m_{yi} = 0$ in Theorem 3.

IV. NUMERICAL EXAMPLE

In this section, we consider the system (1) with the following parameters to demonstrate the applicability of the proposed design algorithm.

$$\begin{aligned}
&E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ A_1 = \begin{bmatrix} 0.9 & 0 \\ 1 & -5 \end{bmatrix}, \\
&A_2 = \begin{bmatrix} 0.3 & 1 \\ 0.2 & -0.5 \end{bmatrix}, \ B_{11} = B_{12} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \\
&D_{11} = D_{12} = \begin{bmatrix} 0.2 \\ 0.5 \end{bmatrix}, E_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \\
&C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \ K = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.07 \end{bmatrix}, \\
&C_{11} = C_{12} = \begin{bmatrix} -1 & 0.2 \end{bmatrix}, \ D_{21} = D_{22} = 0.01, \\
&C_{21} = C_{22} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \ B_{21} = B_{22} = 1, \ r(t) = 0 \\
&D_{31} = D_{32} = 0.7, \ \theta_1 = \frac{1-\sin t}{2}, \ \theta_2 = \frac{1+\sin t}{2}.
\end{aligned} \tag{41}$$

In the case of actuator failure (5), we assume that actuator have a failure of a 96% reduction in signal strength. That is, $\underline{m}_{11} = 0.04$, $\bar{m}_{11} = 1$, $M_u = 0.47$. $w(t)$, $\eta(t)$ and $\delta_1(t)$ are square-integrable stochastic noise with variance 0.02, 0.01 and 0.01 respectively. Obviously, the pairs $(\tilde{E}, \tilde{A}_i, \tilde{C}_{2i}), i = 1, 2$ is observable. The eigenvalues of system (41) are 0, 0.6873 and -2.8373, so the system (41) is unstable.

According to Theorem 2 and Remark 2, for a given $\gamma = 2.5$, setting $\Phi = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$ and employing *Matlab LMI Toolbox* to solve the feasibility problem of *LMI* (34)' gives the following reliable robust H_∞ tracking controller,

$$L = \begin{bmatrix} -206.0819 & -29.2161 & -0.3366 \end{bmatrix}. \tag{42}$$

At the same time, according to Remark 4, by setting $M_1 = 1$ and $M_u = 0$, we also can obtain the standard robust H_∞ tracking controller

$$L_s = \begin{bmatrix} -7.8696 & -1.2838 & -0.0005 \end{bmatrix}. \tag{43}$$

Then, using the reliable robust H_∞ tracking controller (42), the eigenvalues of resultant closed-loop system are -16.9617, -2.8418 and -0.0016, so the resultant closed-loop system is stable. On the other hand, using the standard robust H_∞ tracking controller (43), the eigenvalues of resultant closed-loop system are -2.8409, $0.005 + 0.0042i$ and $0.005 - 0.0042i$, so the resultant closed-loop system is unstable.

Moreover, when the disturbance attenuation level γ is unknown, according to Remark 3, the minimal attenuation level γ for reliable robust H_∞ tracking control is $\gamma_{min} = 0.975$, whereas the minimal attenuation level γ for standard robust H_∞ tracking control $\gamma_{min} = 1.24$.

V. CONCLUSION

This paper dealt with the reliable robust H_∞ tracking control problem for a class of uncertain *Lur'e* singular systems. The uncertainties are assumed to be convex polytopic. Based on *LMIs*, a sufficient condition was presented to design the reliable robust H_∞ tracking controller, which guaranteed the closed-loop system regular, impulse-free, stable with H_∞ performance, and the output of closed-loop system tracking the reference signal without steady-state error when all control components are operational as well as when some control components experience failures. Finally, a numerical example is given to show the effectiveness of the proposed methods.

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