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Nonlinear Predictive GMV Control

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Abstract—***A Nonlinear Predictive Generalized Minimum Variance* (*NPGMV*) control algorithm is introduced for the control of *nonlinear multivariable systems.* The plant model is represented by a series combination of a nonlinear operator, which is assumed finite-gain stable, and a linear state-space model, which can include time delays and unstable modes. The main contribution is to incorporate predictive action into the recently introduced Nonlinear GMV controller by defining a multi-step cost index and using a minimum-variance form of the usual GPC cost function. The solution is very different to traditional nonlinear model predictive control, providing a solution which is similar to fixed model based controllers. This does not provide the same constrained optimization features but it does give a controller which is very simple to implement.**

I. Introduction

THE objective in this paper is to derive a relatively simple controller for nonlinear systems and one that has some of the advantages of the popular *Generalized Predictive Control* (*GPC*) algorithms.

The *Model Based Predictive Control* (*MBPC*) approach based on linear systems theory has been very successful in applications, particularly in the process industries. The most popular predictive control algorithms have been *Dynamic Matrix Control* (*DMC*) [1] and *Generalized Predictive Control* (*GPC*) [2,3]. Richalet [4,5] has been very influential in the development of predictive control and has applied the technique successfully in a wide range of applications. The *GPC* controller was originally obtained in a polynomial systems form and a state-space version, suitable for large systems, was derived later [6].

Most systems, however, are inherently nonlinear, and in many cases cannot be adequately described by linear models, especially when the operating regimes change frequently. Moreover, the increasingly more stringent product quality and energy cost requirements make it necessary to develop and use nonlinear models to achieve optimal performance.

There has been a rich history of research in the field of nonlinear predictive control. Some of the approaches include the use of Lyapunov functions [7], quadratic Hammerstein and Volterra models [8], observer-based nonlinear quadratic dynamic matrix control [9], and linear quadratic feasible predictive control [10]. Excellent reviews of existing NMPC techniques can also found in [11,12].

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The control strategy introduced here builds upon previous results on *Nonlinear Generalized Minimum Variance* (*NGMV)* control, which was derived recently for nonlinear model-based multivariable systems [13-15]. The present major development over the basic *NGMV* control law involves an extension of the cost-index to include future tracking error and control costing terms in a *GPC* type of problem where the linear plant subsystem model is represented in state equation form. When the system is linear, the results revert to those for a *GPC* controller.

The main advantage over the other nonlinear MPC approaches lies in the simplicity of the control law: the controller, although it incorporates the nonlinear model of the system, is nevertheless *fixed*, i.e. its component blocks can be pre-computed off-line and no on-line optimization is needed. Of course, the controller performance may not be as good as the one potentially achievable by computationally intensive on-line schemes, however this is compensated by generality and low complexity of the solution, which may be an important issue in fast real-time applications. Due to that simplicity, no explicit optimal constraint handling is possible. However, the input limits can be taken into account implicitly by defining appropriate penalty functions on control signals.

Compared with adaptive methods based on multiple linear models, no switching or controller blending is required since the internal nonlinear model of the system is supposed to be valid for the whole operating range of interest.

II. System Description

The system model considered in this paper is separated into a nonlinear and linear part, connected in series. Such a structure may for example represent plant dynamics which is essentially linear but with nonlinear actuators, Hammerstein models or nonlinear systems with output linear dynamics and/or transport delays.

The nonlinear part of the system can be grossly nonlinear, dynamic and may have a very general form but the remaining system description is chosen so that relatively simple results are obtained. The disturbance signal is assumed to have a linear time-invariant model representation. This is not very restrictive, since in many applications the disturbance model is only a linear time-invariant (*LTI)* approximation. The system in Fig. 1 includes the plant model together with the reference, measurement noise and disturbance signals. The white measurement noise $v(t)$ is assumed to have a covariance: $R_f = R_f^T \geq 0$ and the zero-mean white noise source $\xi(t)$ has an identity covariance matrix.

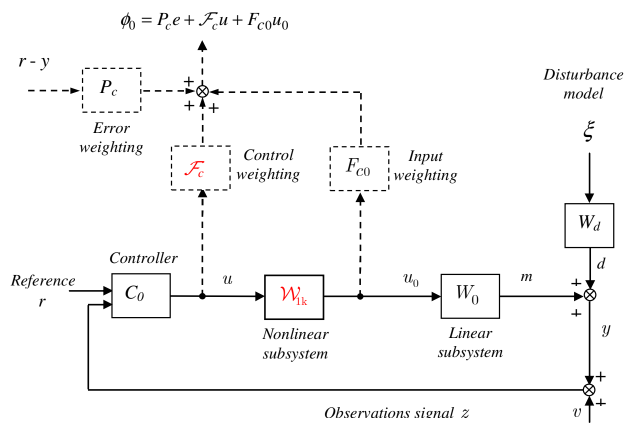

Fig.1 Two Degrees of Freedom Feedback Control for Nonlinear Plant

The nonlinear plant model is denoted as:

$$(\mathcal{W}_1 u)(t) = z^{-k}(\mathcal{W}_{1k} u)(t) \tag{1}$$

where $z^{-k}I$ is a diagonal matrix of common delay elements in the output signal paths. The generalization to different delays in different signal paths complicates the solution but is straightforward [16]. The output of the non-linear subsystem $\mathcal{W}_{1k}$ will be denoted as: $u_0(t) = (\mathcal{W}_{1k}u)(t)$. The nonlinear subsystem: $\mathcal{W}_1$ is assumed to be *finite gain stable* but the linear subsystem, denoted: $W_0 = z^{-k}W_{0k}$, introduced in more detail below, can contain any unstable modes. For stability analysis the time sequences can be considered to be contained in extensions of the discrete Marcinkiewicz space $m_2(R_+,R_n)$ [17]. This is the space of time sequences with averaged square summable signals, which have finite power.

A. Linear State-Space Subsystem Models

The linear state-space model including the linear plant and disturbance states may be represented as:

$$x(t+1) = Ax(t) + Bu_0(t-k) + D\xi(t) \tag{2}$$

$$y(t) = Cx(t) + Eu_0(t-k) \tag{3}$$

The input signal channels are assumed to include a *k-steps delay*. The delay-free plant transfer of the linear sub-system, referred to above, may be written as: $W_{0k} = C\Phi B + E$ where the *resolvent matrix*: $\Phi = (zI - A)^{-1}$.

Future Outputs and States: The predictive action of the controller uses future values of the linear states, which may be obtained as:

$$x(t+i+k) = A^i x(t+k) + \sum_{j=1}^{i} A^{i-j}\left(Bu_0(t+j-1) + D\xi(t+j+k-1)\right) \tag{4}$$

The *weighted output equation* can include any cost-function weighting, such as: $y_p(t) = P_c(z^{-1})y(t)$, by augmenting the state model. The new weighted output $y_p(t)$ to be regulated at future times then has the form:

$$y_p(t) = C_p x(t) + E_p u_0(t-k) \tag{5}$$

B. Prediction Model

The *i-steps* ahead prediction of the output signal may be calculated by noting the above result (5) and assuming for the present that the future values of the control action are known. Thus let: $\hat{y}_p(t+i+k \mid t) = E\{y_p(t+i+k) \mid t\}$ then,

$$\hat{y}_p(t+i+k \mid t) = CA^i\hat{x}(t+k \mid t) + \sum_{j=1}^{i} CA^{i-j}Bu_0(t+j-1) + Eu_0(t+i) \tag{6}$$

Collecting together the results for the prediction horizon of N steps, the vector of predicted outputs: $\hat{Y}_{t+k,N}$ may be obtained:

$$\underbrace{\begin{bmatrix} \hat{y}_p(t+k \mid t) \\ \hat{y}_p(t+1+k \mid t) \\ \hat{y}_p(t+2+k \mid t) \\ \vdots \\ \hat{y}_p(t+N+k \mid t) \end{bmatrix}}_{} = \underbrace{\begin{bmatrix} CI \\ CA \\ CA^2 \\ \vdots \\ CA^N \end{bmatrix}}_{C_N A_N} \hat{x}(t+k \mid t) + \underbrace{\begin{bmatrix} E & 0 & \cdots & 0 & 0 \\ CB & E & \cdots & \vdots & 0 \\ \vdots & CB & \ddots & & \vdots \\ & \vdots & \ddots & E & 0 \\ CA^{N-1}B & CA^{N-2}B & \cdots & CB & E \end{bmatrix}}_{V_N = C_N B_N + E_N} \underbrace{\begin{bmatrix} u_0(t) \\ u_0(t+1) \\ \vdots \\ u_0(t+N) \end{bmatrix}}_{U^0_{t,N}} \tag{7}$$

with the following definition of terms:
$C_N = diag\{C, C, \ldots, C\}$, $E_N = diag\{E, E, \ldots, E\}$

$$A_N = \begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^N \end{bmatrix} \quad B_N = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ B & 0 & \cdots & & 0 \\ \vdots & B & \ddots & & \vdots \\ & & \ddots & & 0 \\ A^{N-1}B & A^{N-2}B & \cdots & B & 0 \end{bmatrix} \quad D_N = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ D & 0 & \ddots & 0 \\ AD & D & & \\ \vdots & & \ddots & 0 \\ A^{N-1}D & A^{N-2}D & \cdots & D \end{bmatrix}$$

$$W_{t,N} = \begin{bmatrix} \xi(t) \\ \xi(t+1) \\ \vdots \\ \xi(t+N-1) \end{bmatrix} \quad U^0_{t,N} = \begin{bmatrix} u_0(t) \\ u_0(t+1) \\ \vdots \\ u_0(t+N) \end{bmatrix} \quad R_{t,N} = \begin{bmatrix} r_p(t) \\ r_p(t+1) \\ \vdots \\ r_p(t+N) \end{bmatrix} \tag{8}$$

The state estimate $\hat{x}(t+k \mid t)$ may be obtained in a computationally efficient form from a *Kalman filter* [18] where the number of states in the filter is not increased by the number of the delays k. The prediction equation:

$$\hat{x}(t+k \mid t) = A^k\hat{x}(t \mid t) + T_0(k, z^{-1})Bu_0(t) \tag{9}$$

where $T_0(k, z^{-1})$ denotes a *finite impulse response block*:

$$T_0(k, z^{-1}) = z^{-1}\left(I + z^{-1}A + z^{-2}A^2 + \ \ldots\ + z^{-k+1}A^{k-1}\right) \tag{10}$$

The *N-step ahead* prediction in (7) can be written as:

$$\hat{Y}_{t+k,N} = C_N A_N \hat{x}(t+k \mid t) + V_N U^0_{t,N} \tag{11}$$

Output prediction error: $\tilde{Y}_{t+k,N} = Y_{t+k,N} - \hat{Y}_{t+k,N}$

$$= C_N A_N x(t+k) + V_N U^0_{t,N} + C_N D_N W_{t+k,N} - (C_N A_N \hat{x}(t+k \mid t) + V_N U^0_{t,N}) \tag{12}$$

Thence, the *inferred output* estimation error:

$$\tilde{Y}_{t+k,N} = C_N A_N \tilde{x}(t+k|t) + C_N D_N W_{t+k,N}$$

where the k-steps-ahead *state estimation error*: $\tilde{x}(t+k|t) = x(t+k) - \hat{x}(t+k \mid t)$.

Note for later use that the *state estimation error* is independent of the choice of control action. Also recall that the optimal $\hat{x}(t+k \mid t)$ and $\tilde{x}(t+k \mid t)$ are orthogonal and the expectation of the product of the future values of the control action (assumed known in deriving the prediction equation), and the zero mean white noise driving signals, is null. It follows that the vector of predicted signals: $\hat{Y}_{t+k,N}$ in (11) and the prediction error: $\tilde{Y}_{t,N}$ are orthogonal.

III. GENERALISED PREDICTIVE CONTROL

A brief review of the derivation of the *GPC* controller is provided below where for the moment the input will be taken to be that for the linear subsystem (u_0), since it also provides results needed for the definition of the nonlinear problem of interest. The following *GPC* performance index with equal prediction and control horizons is considered:

$$J = E\{\sum_{j=0}^{N} e_p(t+j+k)^T e_p(t+j+k) + \lambda_j^2 u_0(t+j)^T u_0(t+j)) \Big| t\} \quad (13)$$

where the vector of future weighted reference signal values is denoted by $r_p(t+j+k)$ and the weighted error signal: $e_p(t) = r_p(t) - y_p(t)$.

A. Optimal Solution for State Estimate Feedback: Review

Assume that the states are not available, then an optimal state estimator must be introduced and the cost function must be written in terms of the state estimate and the state estimation error.

The multi-step cost function may be written in a concise vector form by introducing:

$$J = E\{(R_{t+k,N} - Y_{t+k,N})^T (R_{t+k,N} - Y_{t+k,N}) + U_{t,N}^{0T} \Lambda_N^2 U_{t,N}^0 \mid t\} \quad (14)$$

If it is assumed that a Kalman filter is introduced for state estimation and prediction, then from (14) obtain,

$$J = E\{(R_{t+k,N} - (\hat{Y}_{t+k,N} + \tilde{Y}_{t+k,N}))^T (R_{t+k,N} - (\hat{Y}_{t+k,N} + \tilde{Y}_{t+k,N}))$$
$$+ U_{t,N}^{0T} \Lambda_N^2 U_{t,N}^0 \mid t\} \quad (15)$$

where the cost weightings: $\Lambda_N^2 = diag\{\lambda_0^2, \lambda_1^2, ..., \lambda_N^2\}$.

The terms in the cost-index can then be simplified, first by noting the optimal estimate $\hat{Y}_{t+k,N}$ is orthogonal to the estimation error $\tilde{Y}_{t+k,N}$ and second by recalling the future reference or set-point trajectory $R_{t+k,N}$ is assumed to be a known signal over the *N+1* steps. Simplifying, obtain:

$$J = (R_{t+k,N} - \hat{Y}_{t+k,N})^T (R_{t+k,N} - \hat{Y}_{t+k,N}) + U_{t,N}^{0T} \Lambda_N^2 U_{t,N}^0 + J_0 \quad (16)$$

where $J_0 = E\{\tilde{Y}_{t+k,N}^T \tilde{Y}_{t+k,N} \mid t\}$. Substituting from (11) for $\hat{Y}_{t+k,N}$ and writing:

$$\tilde{R}_{t+k,N} = R_{t+k,N} - C_N A_N \hat{x}(t+k \mid t), \quad (17)$$

the cost function may be expanded as:

$$J = (\tilde{R}_{t+k,N} - V_N U_{t,N}^0)^T \ (\tilde{R}_{t+k,N} - V_N U_{t,N}^0) + U_{t,N}^{0T} \Lambda_N^2 U_{t,N}^0 + J_0$$
$$= \tilde{R}_{t+k,N}^T \tilde{R}_{t+k,N} - U_{t,N}^{0T} V_N^T \tilde{R}_{t+k,N} - \tilde{R}_{t+k,N}^T V_N U_{t,N}^0$$
$$+ U_{t,N}^{0T} \left(V_N^T V_N + \Lambda_N^2\right) U_{t,N}^0 + J_0 \quad (18)$$

The procedure for minimizing this cost term, if the signals are deterministic, is similar to that when the conditional cost function is considered. That is, the gradient of the cost-function must be set to zero, to obtain the vector of future optimal control signals. From a perturbation and gradient calculation [18], noting J_0 term is independent of the control action, the *GPC future optimal control* becomes:

$$U_{t,N}^0 = \left(V_N^T V_N + \Lambda_N^2\right)^{-1} V_N^T \left(R_{t+k,N} - C_N A_N \hat{x}(t+k \mid t)\right) \quad (19)$$

The *GPC* optimal control at time *t* is defined from this vector based on the *receding horizon* principle [19], i.e. the optimal control is taken as the first element in the vector of future controls: $U_{t,N}^0$.

B. Equivalent Cost Optimization Problem

It is now shown that the above problem is equivalent to a special cost minimization control problem which is needed to motivate the *NPGMV* problem introduced later. Let the constant positive definite, real symmetric matrix: $V_N^T V_N + \Lambda_N^2$ that enters the above solution, be factorized:

$$Y^T Y = V_N^T V_N + \Lambda_N^2 \quad (20)$$

Then observe that by *completing the squares* in equation (18) the cost-function may be written as:

$$J = \left(\tilde{R}_{t+k,N}^T V_N Y^{-1} - U_{t+k,N}^{0T} Y^T\right)\left(Y^{-T} V_N^T \tilde{R}_{t+k,N} - Y U_{t,N}^0\right)$$
$$+ \tilde{R}_{t+k,N}^T (I - V_N Y^{-1} Y^{-T} V_N^T) \tilde{R}_{t+k,N} + J_0$$

That is, the cost-function may be written as:

$$J = \hat{\Phi}_{t+k,N}^T \hat{\Phi}_{t+k,N} + J_{10}(t) \quad (21)$$

and

$$\hat{\Phi}_{t+k,N} = Y^{-T} V_N^T \left(R_{t+k,N} - C_N A_N \hat{x}(t+k \mid t)\right) - Y U_{t,N}^0 \quad (22)$$

The terms that are independent of the control action may be written as: $J_{10}(t) = J_0 + J_1(t)$ where

$$J_1(t) = \tilde{R}_{t+k,N}^T \left(I - V_N Y^{-1} Y^{-T} V_N^T\right) \tilde{R}_{t+k,N} \quad (23)$$

Since the last term $J_{10}(t)$ in equation (21) does not depend upon the control action, the optimal control is found by setting the first term to zero, giving (19). It follows the *GPC* optimal controller for the above linear system is the same as the controller to minimize the norm of the *signal* $\hat{\Phi}_{t+k,N}$.

Theorem 3.1: Equivalent Minimum Variance Problem

Consider the minimization of the *GPC* cost index (13) for the system and assumptions introduced in §2. If the cost index is redefined to have a multi-step minimum variance form:

$$\tilde{J}(t) = E\{\Phi_{t+k,N}^T \Phi_{t+k,N} \mid t\},$$

where

$$\Phi_{t+k,N} = P_{CN}(R_{t+k,N} - Y_{t+k,N}) + F_{CN}^0 U_{t,N}^0 \quad (24)$$

and the cost-function weightings: $P_{CN} = Y^{-T} V_N^T$ and $F_{CN}^0 = -Y^{-T} \Lambda_N^2$, then the vector of future optimal controls is identical to the *GPC* controls in (19). ■

Solution: The proof follows by collecting results above. ■

IV. NONLINEAR PREDICTIVE GMV CONTROL

Recall that the actual input to the system is the control signal $u(t)$ shown in Fig. 1, rather than the input to the linear subsystem $u_0(t)$. The cost function for the nonlinear control problem of interest must therefore include an additional control signal costing term, although the costing on the intermediate signal $u_0(t)$ can be retained to examine limiting cases and to provide a useful actuator output costing. If the smallest delay in each output channel of the plant is of magnitude k steps, this implies that the control signal affects the output at least k steps later. For this reason the control signal costing is defined as:

$$(\mathcal{F}_{c} u)(t) = z^{-k} (\mathcal{F}_{ck} u)(t) \tag{25}$$

Typically this weighting on the nonlinear sub-system input will be a linear dynamic operator but it may also be chosen to be nonlinear to cancel the plant input nonlinearities in appropriate cases and it may also be used to introduce an anti-windup capability [15]. The control weighting operator $\mathcal{F}_{ck}$ will be assumed to be invertible. Thus, consider a new signal whose variance is to be minimized:

$$\phi_0(t) = P_c e(t) + F_{c0} u_0(t) + (\mathcal{F}_c u)(t) \tag{26}$$

In analogy with previous *GPC* problem a *multi-step cost index* may be defined:

Extended Multi-Step Cost Index

$$J_p = E\{\Phi_{t+k,N}^{0T} \Phi_{t+k,N}^0 \mid t\} \tag{27}$$

The signal $\Phi_{t+k,N}^0$ is defined to include the future control signal costing terms:

$$\Phi_{t+k,N}^0 = P_{CN} E_{t+k,N} + F_{CN}^0 U_{t,N}^0 + (\mathcal{F}_{ck,N} U_{t,N})$$

$$= P_{CN}(R_{t+k,N} - Y_{t+k,N}) + F_{CN}^0 U_{t,N}^0 + (\mathcal{F}_{ck,N} U_{t,N}) \tag{28}$$

where the non-linear function $\mathcal{F}_{ck,N} U_{t,N}$ will normally be defined to have a simple diagonal form:

$$(\mathcal{F}_{ck,N} U_{t,N}) = diag\{(\mathcal{F}_{ck} u)(t), \ldots, (\mathcal{F}_{ck} u)(t+N)\} \tag{29}$$

and the vector of inputs: $U_{t,N}^0 = (\mathcal{W}_{1k,N} U_{t,N})$, where $\mathcal{W}_{1k,N}$ also has a block diagonal matrix form:

$$(\mathcal{W}_{1k,N} U_{t,N}) = [(\mathcal{W}_{1k} u)(t)^T, \ldots, (\mathcal{W}_{1k} u)(t+N)^T]^T \tag{30}$$

A. The NPGMV Control Solution

Observe from (24) that $\Phi_{t,N}^0 = \Phi_{t,N} + z^{-k}(\mathcal{F}_{ck,N} U_{t,N})$ and $\Phi_{t+k,N}^0 = \hat{\Phi}_{t+k,N}^0 + \tilde{\Phi}_{t+k,N}^0$ with

$$\hat{\Phi}_{t+k,N}^0 = \hat{\Phi}_{t+k,N} + (\mathcal{F}_{ck,N} U_{t,N})$$

$$= P_{CN}(R_{t+k,N} - \hat{Y}_{t+k,N}) + F_{CN}^0 U_{t,N}^0 + (\mathcal{F}_{ck,N} U_{t,N}) \tag{31}$$

The future predicted values in the signal: $\hat{\Phi}_{t+k,N}^0$ involve the estimated vector of weighted outputs $\hat{Y}_{t+k,N}$ and these are orthogonal to $\tilde{Y}_{t+k,N}$. Therefore:

$$\tilde{J}(t) = \hat{\Phi}_{t+k,N}^{0T} \hat{\Phi}_{t+k,N}^0 + \tilde{J}_1(t) \tag{32}$$

where the optimal control sets: $\hat{\Phi}_{t+k,N}^0 = 0$. The condition for optimality has the form:

$$P_{CN}(R_{t+k,N} - \hat{Y}_{t+k,N}) + (\mathcal{F}_{ck,N} + F_{CN}^0 \mathcal{W}_{1k,N}) U_{t,N} = 0 \tag{33}$$

The vector of future optimal control signals, to minimize the cost-index (32), follows from the condition for optimality in equation (33) and satisfies:

$$U_{t,N} = -(\mathcal{F}_{ck,N} - Y^{-T} \Lambda_N^2 \mathcal{W}_{1k,N})^{-1} P_{CN}(R_{t+k,N} - \hat{Y}_{t+k,N}) \tag{34}$$

The *optimal predictive control law* is clearly nonlinear, since it involves the term: $\mathcal{F}_{ck,N}$ and the model for the plant: $\mathcal{W}_{1k,N}$. Substituting from (11) for $\hat{Y}_{t+k,N}$, equation (33) may be written as:

$$P_{CN}(R_{t+k,N} - C_N A_N \hat{x}(t+k \mid t)) + (\mathcal{F}_{ck,N} - Y \mathcal{W}_{1k,N}) U_{t,N} = 0 \tag{35}$$

B. Optimal Nonlinear Predictive Control Signal

These expressions can be simplified by substituting for the predicted state, given (9) and $u_0(t) = \mathcal{W}_{1k} u(t)$, and writing:

$$\tilde{R}_{t+k,N}^0 = R_{t+k,N} - C_N A_N A^k \hat{x}(t \mid t) \tag{36}$$

Thence from (35) the condition for optimality becomes:

$$P_{CN}\left(\tilde{R}_{t+k,N}^0 - C_N A_N T_0(k, z^{-1}) B(\mathcal{W}_{1k} u)(t)\right) + \\ + (\mathcal{F}_{ck,N} - Y \mathcal{W}_{1k,N}) U_{t,N} = 0 \tag{37}$$

giving

$$U_{t,N} = -(\mathcal{F}_{ck,N} - Y \mathcal{W}_{1k,N})^{-1} P_{CN} \\ \times \left(\tilde{R}_{t+k,N}^0 - C_N A_N T_0(k, z^{-1}) B(\mathcal{W}_{1k} u)(t)\right) \tag{38}$$

To simplify the equations also introduce the matrix:

$$C_\phi = P_{CN} C_N A_N = Y^{-T} V_N^T C_N A_N \tag{39}$$

The condition for optimality in (37) may then be written:

$$P_{CN} \tilde{R}_{t+k,N}^0 + \left(\mathcal{F}_{ck,N} - Y \mathcal{W}_{1k,N} - C_\phi T_0(k, z^{-1}) B C_{I0} \mathcal{W}_{1k,N}\right) U_{t,N} = 0 \tag{40}$$

Theorem 4.1: NPGMV Optimal Control Law

Consider the linear components of the plant, disturbance and output weighting models in augmented state equation form (2), (3) with input from the nonlinear plant dynamics $\mathcal{W}_{1k}$. The nonlinear plant operator: $\mathcal{W}_{1k}$ is assumed to be open-loop *finite gain stable* and the operator: $\left(C_{I0}(Y_1 + C_\phi \Phi B)\mathcal{W}_{1k} - \mathcal{F}_{ck}\right)$ is also assumed to have a stable causal inverse, due to the choice of weighting operators: P_{CN}, F_{CN}^0 and $\mathcal{F}_{ck,N}$. The *multi-step predictive controls* cost-function to be minimized, involving a sum of future cost terms, is defined in vector form as:

$$J_p = E\{\Phi_{t+k,N}^{0T} \Phi_{t+k,N}^0 \mid t\} \tag{41}$$

where $\Phi^0_{t+k,N}$ includes future error, input and control terms:

$$\Phi^0_{t+k,N} = P_{CN} E_{t+k,N} + F^0_{CN} U^0_{t,N} + (\mathcal{F}_{ck,N} U_{t,N}) \quad (42)$$

The error and input cost-function weightings: $P_{CN} = Y^{-T} V_N^T$ and $F^0_{CN} = -Y^{-T}\Lambda_N^2$, and the control signal cost-function weighting has the form: $(\mathcal{F}_c u)(t) = (\mathcal{F}_{ck} u)(t-k)$, where k represents the transport delay and $\mathcal{F}_{ck}$ is full rank and invertible. Define the constant matrix factor Y to satisfy: $Y^T Y = V_N^T V_N + \Lambda_N^2$, then the *NPGMV* optimal control law to minimize the variance (41) is given as:

$$U_{t,N} = -\left(\mathcal{F}_{ck,N} - (Y + C_\phi T_0(k,z^{-1})B)\mathcal{W}_{1k,N}\right)^{-1} \times P_{CN}\left(R_{t+k,N} - C_N A_N A^k \hat{x}(t \mid t)\right) \quad (43)$$

where $C_\phi = P_{CN} C_N A_N$ ■

C. Implementation of the Predictive Optimal Control

It is useful to separate the vector of controls into that to be applied at time t and the vector of future controls. This partition also enables the algorithm to be considerably simplified. The control at time t is computed for $N > 0$ by introducing $C_{I0} = [I, 0, ..., 0]$. This enables the control at time t to be found as

$$u(t) = [I, 0, ..., 0]\, U_{t,N} \quad (44)$$

Because of the block diagonal structure of the control signal costing $\mathcal{F}_{ck,N}$, $C_{I0}\mathcal{F}^{-1}_{ck,N} = [\mathcal{F}^{-1}_{ck}, 0, ... , 0] = \mathcal{F}^{-1}_{ck} C_{I0}$. The *optimal control at time t* can be computed, using (40), as:

$$u(t) = -\mathcal{F}^{-1}_{ck} C_{I0}\left(P_{CN}\tilde{R}^0_{t+k,N} - (Y + C_\phi T_0(k,z^{-1})BC_{I0})\mathcal{W}_{1k,N} U_{t,N}\right) \quad (45)$$

Note from equation (30) that the vector $\mathcal{W}_{1k,N} U_{t,N}$ may be written as $(\mathcal{W}_{1k,N} U_{t,N}) == [(\mathcal{W}_{1k} u)(t)^T,\ (\mathcal{W}_{1k,N-1} U^f_{t,N})^T]^T$

Using a related partition, write the factor Y in the form: $Y = [Y_1 \ \ Y_2]$, where the number of columns in Y_1 equals the dimension of the signal $u_0(t)$, denoted as m. The *optimal control* may now be expressed, using (40) and (45), as:

$$u(t) = -\mathcal{F}^{-1}_{ck} C_{I0}[P_{CN}\tilde{R}^0_{t+k,N} - (Y_1 + C_\phi T_0(k,z^{-1})B)(\mathcal{W}_{1k} u)(t) - Y_2(\mathcal{W}_{1k,N-1} U^f_{t,N})] \quad (46)$$

where $U^f_{t,N}$ is a vector of future control actions.

For the proposed nonlinear predictive control it may be shown that a nonlinear operator ($C_{I0}(P_{CN} C_N A_N \Phi B + Y_1)\mathcal{W}_{1k} - \mathcal{F}_{ck}$) must have a stable inverse where the measure of stability, such as finite gain, depends upon the assumption on the nonlinear plant sub-system: $\mathcal{W}_{1k}$. These stability results are discussed in [14].

V. SIMULATION EXAMPLE

One of the potential applications of the NPGMV control is in robotics. Consider for example a planar, two-link manipulator shown in Fig. 5. The objective is to control the vector of joint angular positions $\boldsymbol{q}$ with the vector of torques $\boldsymbol{\tau}$ applied at the manipulator joints so that they follow a desired trajectory: $\boldsymbol{q_d}$. This problem was analyzed in detail in [20] where it was shown that a multi-loop PD controller can be used to move the links to required fixed positions.

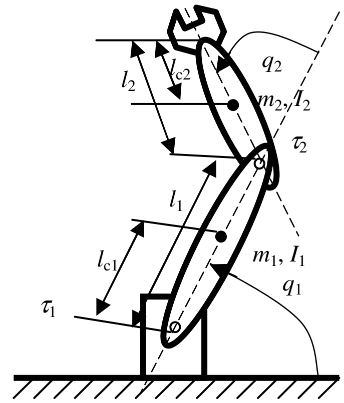

Fig. 5: Two-link robotic manipulator

The dynamics of the system is highly nonlinear and may be described, using the Lagrangian equations of motion, by the following differential equation [20]:

$$H(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau \quad (47)$$

where

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix},\ C(q) = \begin{bmatrix} -h\dot{q}_2 & -h(\dot{q}_1 + \dot{q}_2) \\ h\dot{q}_1 & 0 \end{bmatrix}$$

with

$$H_{11} = m_1 l_{c1}^2 + I_1 + m_2[l_1^2 + l_{c2}^2 + 2 l_1 l_{c2}\cos q_2] + I_2$$

$$H_{12} = H_{21} = m_2 l_1 l_{c2}\cos q_2 + m_2 l_{c2}^2 + I_2$$

$$H_{22} = m_2 l_{c2}^2 + I_2, \quad h = m_2 l_1 l_{c2}\sin q_2$$

$H(q)$ is the inertia matrix, $C(q,\dot{q})\dot{q}$ is a vector of centripetal and Coriolis torques, and $g(q)$ is a vector of torques due to gravity. For the purpose of the example, we assume that the manipulator is operating in the horizontal plane, and hence $g(q) \equiv 0$.

For the purpose of the example, we assume the following numerical values for the parameters:
$m_1 = 1$, $l_1 = 1$, $m_2 = 2$, $I_1 = 0.12$, $l_{c1} = 0.5$, $I_2 = 0.25$, $l_{c2} = 0.6$.

The NPGMV controller design was performed, based on the nominal stabilizing PD controller, i.e. the dynamic weightings were initially defined as $P_c = C_{PD}$ and $\mathcal{F}_{ck} = -I$, with $K_D = 100I$ and $K_P = 20K_D$. The sampling time was $T_s = 2ms$. The control weighting (penalty on control action) was then decreased to improve the tracking performance. The position control results for both the PD and NPGMV controller with increasing prediction horizon N are shown in Fig. 6 and 7, where in the simulations the maximum applicable torque were set to 10^4 Nm. The effect of predictive action is clearly seen in the plots and generally results in improved tracking performance.

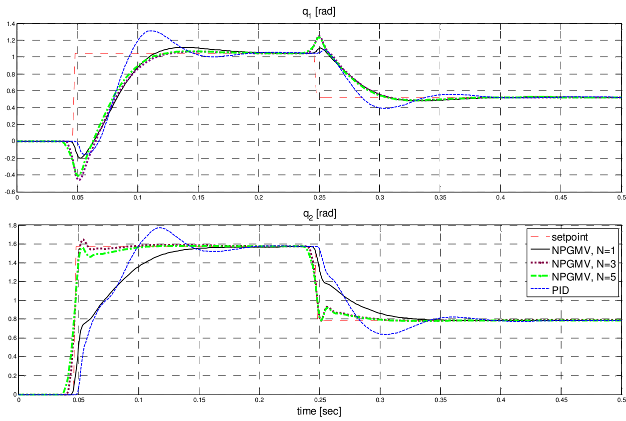

Fig. 6: Position control for varying *N*: angles

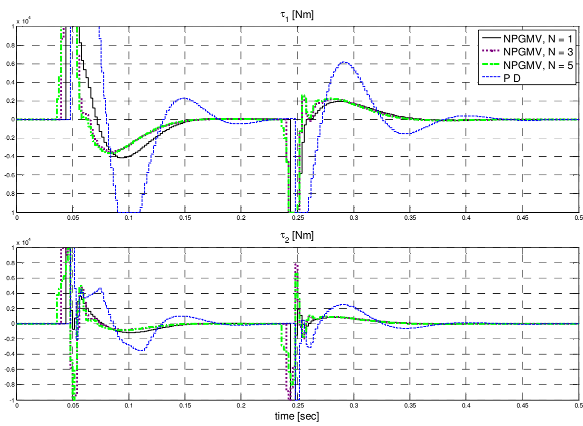

Fig. 7: Position control for varying *N*: torques

VI. CONCLUDING REMARKS

The nonlinear predictive controller introduced in this paper is remarkably different from traditional nonlinear model predictive control design methods. The solution is similar to a fixed model based controller. This does not provide equivalent constrained optimization features but it does give a controller which is very simple to implement. The predictive controls strategy described is a development of the *NGMV* design method, which has been shown to be particularly easy to understand.

The controller has the very nice property that if the system is linear then the controller reverts to the linear generalized predictive control design method, which is well known and accepted in industry. It is well known that the robustness of MPC controllers seems to improve as the number of steps increases. This was one of the motivations for introducing the multi-step cost index in the NGMV family of designs.

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