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Algebraic Criteria for Consensus Problem of Networked Systems with Continuous-time Dynamics

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Abstract— **This paper addresses algebraic criteria for consensus problem of continuous-time networked systems, in which both fixed and switching topology cases are considered. A special eigenvector ω of Laplacian matrix is first constructed and correlated with the connectivity of digraph. And then, based on this tool, some necessary and/or sufficient algebraic conditions are proposed, which can directly determine whether the consensus problem can be solved or not. Furthermore, it is clearly shown that only the agents corresponding to the positive elements of ω contribute to the group decision value and decide the collective behavior of all agents. Particularly for the fixed topology case, not only the role of each agent is exactly measured by the value of the corresponding element of ω but also the group decision value can be calculated by such a vector and the initial states of all agents.**

I. Introduction

In the last decade, due to the broad applications of networked systems in the fields of mobile robots, unmanned air vehicles(UAVs), autonomous underwater vehicles(AUVs), etc., where the coordination control of all agents is in the central position, the importance of the consensus problem in these fields has been well recognized and many results have been obtained(see [2]−[11]). For example, Jadbabaie *et al.* [2] have shown that the states of all the jointly-connected agents converge to the same value or the value of a given leader's state, where the neighbor-based rule is used and the information flow is bidirectional. Then, Ren and Beard [3] extend the results in [2] to the case in which consensus can be achieved if the union of interaction digraph contains a spanning tree across each bounded time interval. The similar results can be found in [4], where Moreau shows that the conditions for the discrete-time consensus in [3] are also necessary. From the above literatures, what group decision value is reached by all agents is not clear. However, this is not an issue for the average consensus problem since for this special case, the state of each agent converges to the average value of the initial states of all agents [5]−[9]. More specifically, Olfati-Saber and Murray [5] propose a continuous-time update scheme and prove that average consensus can be reached if the interaction digraph is balanced and strongly connected. Recently, Liu *et al.*[9] extend the results in [5] to the switching topology. Obviously, all the above results depend on the structure property of interaction digraphs, for convenience, we uniformly call them as ***geometrical criteria***. It is noted that in order to examine whether the consensus problem can be solved or not, the algorithms are further needed to identify whether the interaction digraphs contain spanning tree or are jointly connected, etc. In addition, it is not clear that agents play the leading role in consensus procedure.

Our main objective in this paper is to develop the ***algebraic criteria*** for consensus problem of continuous-time networked systems. With this in mind, a special nonnegative left eigenvector ω of Laplacian matrix is first constructed and correlated with the connectivity of digraph. It has been proved that the digraph is strongly connected(weakly connected with spanning tree) if and only if the vector ω is positive(nonnegative). Further, if a connected digraph is balanced, then it must be strongly connected and meanwhile, all elements of ω are equal. Since the vector ω has these properties to examine the above digraphs, it is a natural tool to study the considered consensus problem. The obtained results provide a set of algebraic conditions to directly determine whether the consensus problem can be solved or not. In addition, more information about the consensus procedure are reflected. For example, it has been proved that only the agents corresponding to the positive elements of ω decide the collective behavior of all agents and contribute to the group decision value, i.e., they are the leaders of the system. Meanwhile, for the fixed topology case, not only the role of each agent is proportional to the value of the corresponding element of ω but also the group decision value can be calculated by ω and the initial states of all agents. These new facts imply that if the elements of ω are not equal, then different agent plays different role in the consensus procedure.

The remainder of this paper is organized as follows. In section II, some preliminaries and background knowledge are given. Then, some algebraic criteria for connectivity of digraph are presented in section III. Furthermore, algebraic criteria for consensus problem and average consensus problem are proposed in section IV and section V, respectively. Finally, we conclude our work in section VI.

II. Preliminaries and Problem Statement

A. Preliminaries and Background

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In this subsection, we introduce some notations in graph theory and matrix theory which are used throughout this paper.

Let $I=\{0,1,2,\cdots,n\}$ be an index set and $G=(V,E,A)$ be a weighted digraph of order n with the set of nodes $V=\{v_1,v_2,\cdots,v_n\}$, set of edges $E\subseteq V\times V$, and a weighted adjacency matrix $A=[a_{ij}]$ with nonnegative adjacency elements $a_{ij}\geq 0$ for all $i,j\in I$, $i\neq j$ and $a_{ii}=0$ for all $i\in I$. As $a_{ij}>0$, it means that there exists an edge from the node v_j to the node v_i. The set of neighbors of the node v_i is denoted by N_i and defined as $N_i=\{v_j\in V: a_{ij}>0\}$. The in-degree and out-degree of the node v_i are defined as $deg_{in}(v_i)$ and $deg_{out}(v_i)$, respectively, just as follows:

$$deg_{in}(v_i)=\sum_{j=1}^{n}a_{ji},\ deg_{out}(v_i)=\sum_{j=1}^{n}a_{ij}. \tag{1}$$

The *degree matrix* of G is a diagonal matrix denoted by $\triangle=[\triangle_{ij}]$ where $\triangle_{ij}=0$ for all $i\neq j$ and $\triangle_{ii}=deg_{out}(v_i)$. The Laplacian matrix of a weighted digraph G is defined as

$$L(G)=\triangle-A. \tag{2}$$

A *path* in a digraph G is a sequence of edges such that the terminal vertex of one edge is the initial vertex of the next. A digraph G is said to be *strongly connected* if and only if for every pair of distinct vertices v_i and v_j in V, there is a directed path from v_i to v_j. A digraph G is called *weakly connected* if replacing all of its directed edges with undirected edges produces a connected (undirected) graph. A digraph G is called *disconnected* if it is not even weak. Meanwhile, a weighted digraph G is called balanced if its out-degree equal to its in-degree, i.e., $\mathbf{1}^{\top}L=0$ with $1=[1,\cdots,1]^{\top}\in R^{n\times 1}$. For a series of digraphs $G_1,\cdots,G_n$ with the same vertices, we say digraph G_u is the *union* of them if the vertices of G_u are the same as any one of these digraphs and the weight $\overline{a}_{ij}$ of edge $v_i\times v_j$ of G_u is defined as follows.

$$\overline{a}_{ij}=\frac{1}{n}\sum_{k=1}^{n}a_{ij}^{k} \tag{3}$$

where a_{ij}^{k} is the weight of edge $v_i\times v_j$ of G_k, $k\in I$. For convenience let L_u denote the Laplacian matrix of G_u and $\overline{G}$ denote the collection of digraphs $G_1,\cdots,G_n$.

To proceed, a vector $x=[x_1,\cdots,x_n]^{\top}$ is called *positive* if each element of x is positive(i.e., $x_i>0$, $\forall i\in I$). A vector x is called *nonnegative* if each element of x is nonnegative(i.e., $x_i\geq 0$, $\forall i\in I$), meanwhile, there exists at least one nonzero element in x. As all elements in x are zero, the vector x is called *zero vector*. Following the same line, a matrix $A\in R^{m\times n}$ is called *positive* if all its elements are positive and matrix A is called *nonnegative* if all its elements are nonnegative. Now, some lemmas are introduced because they will be used in the below.

Lemma 1: [3] Given a matrix $S=[a_{ij}]\in R^{n\times n}$, where $a_{ii}\geq 0$, $a_{ij}\leq 0, \forall\, i\neq j$, and $\sum_{j=1}^{n}a_{ij}=0$ for each j, then S has at least one zero eigenvalue and all of the nonzero eigenvalues are in the open right half plane. Furthermore, S has exactly one zero eigenvalue if and only if the digraph with S has a spanning tree.

Proof: See the Lemma3.3 in [3]. ■

Lemma 2: [3] If a nonnegative matrix $A=[a_{ij}]\in R^{n\times n}$ has the same positive constant row sums given by $\mu>0$, then μ is an eigenvalue of A with an associated eigenvector $\mathbf{1}$ and $\rho(A)=\mu$, where $\rho(\cdot)$ denotes the spectral radius. In addition, the eigenvalue μ of A has algebraic multiplicity equal to one, if and only if the digraph associated with A has a spanning tree.

Proof: See lemma 3.4 in [3]. ■

B. Problem Statement

In this subsection, we consider a protocol with continuous-time dynamics to solve consensus problem as follows[5]

$$\dot{x}_i=\sum_{v_j\in N_i(t)}a_{ij}(x_j-x_i)\quad i,j\in I, \tag{4}$$

where x_i and $N_i(t)$ denote the state and the neighbor set of the node v_i, respectively.

We say x_i and x_j *agree* if and only if $x_i=x_j$ (component-wise). And we say that system (4) has reached a *consensus* if and only if $x_i=x_j$ for all $i,j\in I$. The common value of consensus variable is called the *group decision value*. Let $\chi: R^{m\times n}\rightarrow R^{m\times m}$ be a consensus function of vectors $x_1^{\top}\cdots,x_n^{\top}$, $x_i\in R^{m\times 1}$, and $x_i(0)$ denote the initial state of agent v_i, then system (4) solves the $\chi-$*consensus problem* if and only if there exists an asymptotically stable equilibrium point $x^*=[x_1^{*\top},\cdots,x_n^{*\top}]^{\top}$ satisfying $x_i^*=\chi(x(0))\in R^{m\times 1}$ for all $i\in I$. The function χ is called *consensus function*. The special cases with $\chi(x)=max_{\|x_i\|}x_i$; $\chi(x_i)=min_{\|x_i\|}x_i$; $\chi(x_i)=\frac{1}{n}\sum_{i=1}^{n}x_i(0)$ are called *max-consensus problem, min-consensus problem* and *average consensus problem*, respectively.

To proceed, we first give the algebraic conditions for connectivity of digraph. These results are the basis of our method.

III. Algebraic Criteria for Connectivity of Digraph

Obviously, all the information about a digraph is reflected by its Laplacian matrix, L. For a given node $v_i (i\in I)$, the row i of L denotes how much the other nodes directly affect the node v_i, meanwhile, the column i of L reflects how much the other nodes are affected by the node v_i. In this sense, the study of Laplacian matrix is helpful for us to learn something about the digraph, especially to understand the structure information of digraph. Here, we first construct a vector ω from the Laplacian matrix L as follows.

$$\begin{aligned}\omega=&[\omega_1,\omega_2,\cdots,\omega_n]^{\top}\\ =&[det(L_{11}),det(L_{22}),\cdots,det(L_{nn})]^{\top},\end{aligned} \tag{5}$$

where $det(L_{ii})$ denotes the determinant of matrix L_{ii}, and $L_{ii}\in R^{(n-1)\times(n-1)}$ $(i\in I)$ is obtained from L by deleting the row i and the column i. In what follows, we give some algebraic results about the connectivity of digraph.

Theorem 1: Suppose that digraph G contains spanning tree, L is its Laplacian matrix, and let the vector ω be defined in (5), then

$$\omega^\top L = 0, \tag{6}$$

meanwhile, ω is nonnegative.

Proof: See Appendix A. ■

Remark 1: For all connected digraphs with spanning tree, theorem 1 gives a uniform formula about the left eigenvector of Laplacian matrix associated with simple zero eigenvalue.

Corollary 1: Suppose that G is a strongly connected weighted digraph and L is its Laplacian matrix, let vector ω be defined in (5), then

$$\omega^\top L = 0,$$

meanwhile, ω is positive.

Proof: Combining Theorem 1 and Theorem 8.4.4. in [13], it is easy to be shown. ■

Corollary 2: Suppose that a weakly connected digraph G has spanning tree and L is its Laplacian matrix, let vector ω be defined in (5), then

$$\omega^\top L = 0,$$

meanwhile, ω is nonnegative and has at least one zero element.

Proof: By Theorem 1 and the fact that G is a weakly connected digraph and contains a spanning tree, this result can be shown easily. Due to the limitation of the space, the detail is omitted. ■

Theorem 2: Suppose that G represents a digraph and L is its Laplacian matrix, then G is strongly connected if and only if the vector ω defined in (5) is positive.

Proof: Necessity: by corollary 1, the conclusion holds obviously. Sufficiency: We show the sufficiency by contradiction. Suppose that the digraph is not strongly connected, then it must be a weakly connected or a disconnected digraph. Firstly, if it is weakly connected and contains spanning tree, then by corollary 1, we know the simple zero eigenvalue of its Laplacian matrix has a nonnegative left eigenvector like (5) and in which there exists at least one zero element. Secondly, if it is weakly connected but does not contain spanning tree, then its Laplacian matrix has at least two zero eigenvalues by lemma 1, so the vector defined in (5) is a zero vector. Thirdly, if it is a disconnected digraph, by lemma 1, its Laplacian matrix also has at least two zero eigenvalues, so the vector defined in (5) is zero, too. Since these three cases are all contradicted with the assumption that such a vector is positive, so the conclusion holds. ■

Theorem 3: Suppose that G represents a digraph and L is its Laplacian matrix, then G is a weakly connected graph with spanning tree if and only if the vector ω defined in (5) is nonnegative and has at least one zero element.

Proof: Necessity: By corollary 2, the conclusion holds obviously. Sufficiency: We show the sufficiency by contradiction. Suppose that G is not a weakly connected digraph with spanning tree. Firstly, if it is a strongly connected digraph, then by corollary 1, the vector ω is positive. Secondly, if it is weakly connected but does not contain spanning tree, then by lemma 1, its Laplacian matrix has at least two zero eigenvalues. So the vector ω must be zero vector. Thirdly, if the digraph G is disconnected, then its Laplacian matrix also has at least two zero eigenvalues, so the vector ω is a zero vector, too. Since all three cases in the above are contradicted with the assumption that the vector ω at least exists one positive element and one zero element, so the conclusion holds. ■

Remark 2: Compared with theorem 2 and theorem 3, the structure difference between a strongly connected digraph and a weakly connected digraph with spanning tree is clearly reflected by the vector ω, i.e., by employing ω, these two type of digraphs can be easily distinguished from each other.

Remark 3: It is noted that the Laplacian matrix of a disconnected digraph may have a nonnegative left eigenvector associated with its one of zero eigenvalues, which does not contradict with the results in theorem 2 and theorem 3.

Corollary 3: Suppose that G is a weakly connected digraph and contains spanning tree, L is its laplacian matrix, let the eigenvector ω of L be defined in (5), then the subgraph induced by the nodes corresponding to nonzero elements of ω is strongly connected.

Proof: Due to the limitation of the space, the detail is omitted. ■

Remark 4: Let set LS denote all nodes which correspond to the positive elements of ω, the above results reveal that there must exist a directed path from any other nodes to the nodes in the set LS, i.e., only the nodes in LS can be the root of the directed spanning tree.

Corollary 4: Suppose that G is a connected digraph and L is its Laplacian matrix, then G is a balanced digraph if and only if the vector ω of L is positive, meanwhile, $\omega_i = \omega_j$ for all $i, j \in I$.

Proof: Combining the definition of the balanced digraph and the Theorem 1, the conclusion is easy to be shown. ■

Remark 5: Corollary 4 is important because it implies that a connected balanced digraph must be strongly connected, i.e., it is impossible that a weakly connected digraph is balanced. In other words, either a strongly connected digraph or a disconnected digraph with strongly connected components has the possibility to be a balanced digraph.

IV. Algebraic Criteria for Consensus Problem

In this section, we study the algebraic criteria for the system (4) to solve consensus problem. It is clear that equation (4) can be rewritten in a compact form as follows.

$$\dot{x}(t) = -L(t)x(t), \tag{7}$$

where $L(t)$ is the laplacian matrix. For the fixed topology, i.e., $L(t)$ is a constant matrix, the solution of (7) is given by

$$x(t) = exp(-Lt)x(0). \tag{8}$$

Here, we are in a position to give the algebraic criterion for such a consensus problem.

Lemma 3: [8] The state x in (7) approaches $\mathbf{span}\{\mathbf{1}\}$ and thus solves an agreement problem for all initial x if and only if the interaction digraph of L contains a spanning directed tree.

Proof: See the theorem 2 in [8]. ■

Theorem 4: Suppose that the networked system is defined in (7), L is Laplacian matrix of its interaction digraph G, then (7) solves consensus problem if and only if the vector ω defined in (5) is nonnegative.

Proof: Necessity: Because system (7) can solve the consensus problem, the digraph G contains a spanning tree by lemma 3. Then, ω is nonnegative by theorem 1.

Sufficiency: Because the vector ω is nonnegative, then the digraph G contains a spanning tree by theorem 1. Thus, (7) can solve the consensus problem by lemma 3. ■

As system (7) can solve consensus problem, it is another important and attractive question that what value is reached by the group and how much each node contributes to final group decision value. Before answer such a problem, suppose that (7) can solve consensus problem, i.e., ω is nonnegative, let

$$\omega_l = \frac{1}{\sum_i \omega_i}\omega, \quad \omega_r = [1, 1, \cdots, 1]^\top, \tag{9}$$

clearly, $\omega_l^\top L = 0$, $L\omega_r = 0$ and $\omega_l^\top \omega_r = 1$. Then the following theorem can be derived.

Theorem 5: Suppose that system (7) can solve consensus problem, L is Laplacian matrix of its interaction digraph G, ω_l and ω_r are defined in (9), then

i):
$$R = \lim_{t\to\infty} exp(-Lt) = \omega_r \omega_l^\top \in R^{n\times n}, \tag{10}$$

furthermore, R is a stochastic matrix.

ii) the group decision value is

$$\alpha = \frac{\sum_i \omega_i x_i(0)}{\sum_i \omega_i}. \tag{11}$$

which belongs to the convex hull of initial states of all agents.

Proof: i). The proof is similar to theorem 3 in [5]. For convenience, we do not omit it. Because system (7) can solve consensus problem, ω_l exists by theorem 4. Let $H = -L$ and J be the Jordan form associated with H, i.e., $H = SJS^{-1}$. We have $exp(Ht) = S\exp(Jt)S^{-1}$ and as $t \to \infty$, $\exp(Jt)$ converges to a matrix $Q = [a_{ij}]$ with a single nonzero element $q_{11} = 1$. The fact that other blocks in the diagonal of $\exp(Jt)$ vanish is due to the property that $Re(\lambda_k(H)) < 0$ for all $k \geq 2$ by lemma 1, where $\lambda_k(H)$ is the k-th largest eigenvalue of H in terms of magnitude $|\lambda_k|$. Notice that $R = SHS^{-1}$. Since $HS = SJ$, the first column of S is ω_r, similarly $S^{-1}H = JS^{-1}$ that means the first row of S^{-1} is $\omega_l^\top$. Due to $SS^{-1} = I_n$, $\omega_l^\top \omega_r = 1$ is satisfied just as equation (9) states. So $R = SQS^{-1} = \omega_r \omega_l^\top$. By a simple calculation, all entries in R are nonnegative and for a given row, the sum of all elements is equal to 1, thus R is a stochastic matrix.

ii). From the above, due to $\omega_l^\top L = 0$, we have $\omega_l^\top \dot{x} = \omega_l^\top u = \omega_l^\top(-Lx) = 0$. So $\omega_l^\top x$ is an invariant quantity. Thus, we have $\omega_l^\top x^* = \omega_l^\top x(0)$, i.e., $x^* = \alpha\mathbf{1}$, where α has the form as (11). ■

Remark 6: Theorem 5 is an extension of the theorem 3 in [5]. Because we have found the vector ω, the importance of this theorem is greatly enhanced.

In what follows, we extend theorem 4 to the switching topology case. To proceed, let Υ be a infinite set of dwell time $\tau_i = t_{i+1} - t_i$, $i \in I$, which is closed under addition and multiplication by integers, let G_u denote the union of the finite interaction digraphs and the elements of its Laplacian matrix L_u be defined by equation (3), then the algebraic characterization of theorem 3.12 in [3] can be given as follows.

Theorem 6: Let $t_1, t_2, \cdots$ be an infinite time sequence at which the interaction digraph for weighting factors switch and $\tau_i = t_{i+1} - t_i \in \Upsilon$, $i \in I$. Let $G(t_i) \in \overline{G}$ be a switching interaction digraph at time $t = t_i$, where $\overline{G}$ is the set of all possible interaction digraphs. Suppose that there exists an infinite sequence of uniformly bounded, non-overlapping time intervals $[t_{i_j}, t_{i_{j+1}})$, $j = 1, 2, \cdots$, starting at $t_{i_1} = t_0$, with the property that interval $[t_{i_j},\ t_{i_{j+1}})$ is uniformly bounded. Let $G_u(j)$ denote the union of the interaction digraphs across interval $[t_{i_j},\ t_{i_{j+1}})$ and $L_u(j)$ denote the Laplacian matrix of $G_u(j)$. Then the continuous-time update scheme (4) achieves consensus asymptotically if the vector $\omega(j)$ of $L_u(j)$ is nonnegative, where $\omega(j)$ is defined by (5). Furthermore, if $\omega(j) = 0$ for all time interval j, then consensus can not be achieved asymptotically.

Proof: Combined the theorem 3 and theorem 3.12 in [3], the conclusion holds obviously. ■

Remark 7: For the switching topology case, we can not directly use the element of ω to estimate the contribution of each agent to the group decision value. However, if the element $\omega_i(j)$ is zero for all time interval j, then agent v_i must make no contribution to the group decision value. This is a new fact that can not be reflected by theorem 3.12 in [3].

V. Algebraic Criteria for Average Consensus Problem

In this section, we study the average consensus problem of networked system with continuous-time dynamics. Just as [5] have reported, the balanced digraph plays crucial role in solving such a problem. Here inspired by theorem 5, an algebraic result about average consensus problem can be given as follows.

Theorem 7: System (7) with invariant topology solves average consensus problem if and only if the vector ω defined in (5) is positive and meanwhile, all elements of ω are equal.

Proof: Combined the theorem 4 and 5, the conclusion holds obviously. ■

Remark 8: Theorem 7 is equivalent to the theorem 4 in [5]. However, we prove it directly by theorem 4 and 5. Following this line, it is clear that the average consensus problem is a special case of the general consensus problem, which is not clarified by the results reported in [5].

Just as the discussion in section 4, let G_u denote the union of finite interaction digraphs and $\overline{G}$ denote the set of all possible interaction digraphs, then we give an algebraic result about average consensus problem with switching topology. This result is the algebraic version of the theorem 4 in [9].

Theorem 8: Assume that $G(t_i) \in \overline{G}(i \in I)$ and L_{t_i} is laplacian matrix of G_{t_i}, suppose that there exists an infinite sequence of uniformly bounded, non-overlapping time intervals $[t_{i_j},\, t_{i_{j+1}})$, $j = 1,2,\cdots$, starting at $t_{i_1} = t_0$. Let $G_u(j)$ denote the union of interaction digraphs during time interval $[t_{i_j},\, t_{i_{j+1}})$ and $L_u(j)$ be its Laplacian matrix. Then continuous-time update scheme (7) asymptotically achieves average consensus if i). $\mathbf{1}^\top L_{t_{i_k}} = 0$, $i_k \in [i_j,\, i_{j+1})$; ii). The vector $\omega(j)$ of $L_u(j)$ is positive, where $\omega(j)$ is defined by (5).

Proof: Combined theorem 2 and theorem 4 in [9], the conclusion holds obviously. ∎

Remark 9: It is clear that the algebraic conditions in theorem 8 are more stricter than that in theorem 6. Further, it is easy to verify that all elements of $\omega(j)$ in theorem 8 are equal.

Remark 10: Combined with corollary 4, theorem 8 and remark 7, we know that each connected component of interaction digraphs is strongly connected and balanced at each time t if system (7) solves average consensus problem. In this sense, all agents make the same contribution to the group decision value if each of them is always in a connected and balanced component.

VI. Conclusions

In this paper, we have established the algebraic criteria for consensus problem of continuous-time networked systems, in which both fixed and switching topology cases are considered. The proposed results not only can algebraically determine whether the consensus problem can be solved or not but also clearly show that the average consensus problem is a special case of the general consensus problem. Furthermore, more information about consensus procedure is revealed. For example, it has been shown that only the agents corresponding to the positive elements of ω decide the collective behavior of all agents and contribute to the group decision value, i.e, the agents corresponding to the zero elements of ω make no any contribution to the group decision value except for converging to it. Particularly for the fixed topology case, the consensus procedure is distinctly clarified. All these new facts give us a deep insight into the consensus procedure.

Appendix

A.Proof of theorem 1

Proof: Because digraph G may be reducible, we prove it directly. The proof is divided into four steps as follows.

Step 1: Because the digraph G has a spanning tree, $rank(L) = n-1$ by lemma 1. Therefore, square matrix L contains at least one nonsingular submatrix of order $n-1$. Without loss of generality, assume that L_{ij} is such a submatrix which is obtained by deleting the row i and the column j of L. So the row vectors of L except for the row i are linearly independent. Now, let L^* denote the submatrix of L obtained by deleting the row i of L as follows.

$$L^* = \begin{bmatrix} a_{11} & -a_{12} & \cdots & -a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ -a_{i-1,1} & -a_{i-1,2} & \cdots & -a_{i-1,n} \\ -a_{i+1,1} & -a_{i+1,2} & \cdots & -a_{i+1,n} \\ \vdots & \vdots & \vdots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

It is clear that $L^* \in R^{(n-1)\times n}$ is a row full rank matrix(i.e., $rank(L^*) = n-1$), meanwhile the sum of each row of L^* is zero because L is a Laplacian matrix. Therefore, any $n-1$ column vectors of L^* must be linearly independent. Thus, the first $(n-1)\times(n-1)$ leading principal submatrix A of L is nonsingular if we renumber the node v_i as the node v_n and the node v_n as the node v_i. To proceed, let $B = [-a_{1n}, -a_{2n}, \cdots, -a_{n-1,n}]^\top$ and $C = [-a_{n1}, -a_{n2}, \cdots, -a_{n,n-1}]$, then there must exist an inverse column permutation matrix $P \in R^{n\times n}$ such that

$$\begin{aligned} LP =& L\begin{bmatrix} A^{-1} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & a_{nn} \end{bmatrix}\begin{bmatrix} A^{-1} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I_{n-1} & B \\ CA^{-1} & a_{nn} \end{bmatrix} \\ =& L^{**}, \end{aligned}$$

where I_{n-1} is identity matrix of order $n-1$.

Step 2: Let

$$\begin{aligned} D =& CA^{-1} = [-a_{n1}, -a_{n2}, \cdots, -a_{n,n-1}]A^{-1} \\ =& [-b_1, -b_2, \cdots, -b_{n-1}], \end{aligned}$$

substitute it into L^{**}, and then perform some elementary column transformations on matrix L^{**}, we have

$$L^{**} \sim \begin{bmatrix} I_{n-1} & 0 \\ D & a_{nn} - DB \end{bmatrix} = L_1^{**}.$$

Since the above operations on L are invertible, $rank(L_1^{**}) = rank(L) = n-1$. Thus, we directly have $a_{nn} - DB = 0$ and get the following two equivalent equations

$$\omega^\top L = 0 \Longleftrightarrow \omega^\top L_1^{**} = 0 \tag{12}$$

In what follows, we calculate the value of b_i for all $i = 1,\cdots,n-1$. Due to $A^{-1} = \frac{1}{det(A)}A^*$, where A^* is the adjoint matrix of A, we have

$$D = \frac{CA^*}{det(A)} = \frac{C}{det(A)}\begin{bmatrix} A_{11}^* & A_{21}^* & \cdots & A_{n-1,1}^* \\ A_{12}^* & A_{22}^* & \cdots & A_{n-1,2}^* \\ \vdots & \vdots & \ddots & \vdots \\ A_{1,n-1}^* & A_{2,n-1}^* & \cdots & A_{n-1,n-1}^* \end{bmatrix},$$

where $A_{ij}^* = (-1)^{i+j}det(A_{ij})$, A_{ij} is the submatrix of A by deleting the row i and the column j. Then combined with the vector C defined in the above, we have

$$\begin{aligned} -b_i =& \frac{1}{det(A)}(-a_{n1}A_{i1}^* - a_{n2}A_{i2}^* - \cdots - a_{n,n-1}A_{i,n-1}^*) \\ =& \frac{(-1)^i}{det(A)}[(-1)^2 a_{n1}det(A_{i,1}) + (-1)^3 a_{n2}det(A_{i,2}) \\ & + \cdots + (-1)^n a_{n,n-1}det(A_{i,n-1})]. \end{aligned} \tag{13}$$

Let $\triangle_i$ denote the $(n-1)\times(n-1)$ submatrix of L as follows

$$\triangle_i = BL\left\{\begin{matrix} 1,2,\cdots,i-1,i+1,\cdots,n \\ 1,2,\cdots,i-1,i,i+1,\cdots,n-1 \end{matrix}\right\}, \tag{14}$$

i.e., $\triangle_i$ is obtained by deleting the row i and the column n of L . Noting that $\triangle_i$ is a square matrix of order $n-1$, we have

$$\begin{aligned} det(\triangle_i) =& (-1)^{n+1}a_{n1}det(\triangle_{n1}) + (-1)^{n+2}a_{n2}det(\triangle_{n2}) + \\ & \cdots + (-1)^{2n-1}a_{n,n-1}det(\triangle_{n,n-1}), \end{aligned} \tag{15}$$

where $\triangle_{ij}$ is submatrix of $\triangle_i$ by deleting the row i and the column j of $\triangle_i$. From equations (13) and (15), we have $det(\triangle_{n1}) = det(A_{i1}), det(\triangle_{n2}) = det(A_{i2}), \cdots, det(\triangle_{n,n-1}) = det(A_{i,n-1})$. Thus, equation (15) becomes

$$\begin{aligned} det(\triangle_i) =& (-1)^{n+1}a_{n1}det(A_{i1}) + (-1)^{n+2}a_{n2}det(A_{i2}) \\ & + \cdots + (-1)^{2n-1}a_{n,n-1}det(A_{i,n-1}) \\ =& (-1)^{n-1}[(-1)^2 a_{n1}det(A_{i1}) + (-1)^3 a_{n2}det(A_{i2}) \\ & + \cdots + (-1)^n a_{n,n-1}det(A_{i,n-1})]. \end{aligned} \tag{16}$$

Then by (13) and (16), we have

$$b_i = \begin{cases} \frac{(-1)^i}{det(A)} det(\triangle_i), & \text{if } n = 2k, k \in N^+; \\ \frac{(-1)^{i-1}}{det(A)} det(\triangle_i), & \text{if } n = 2k+1, k \in N^+. \end{cases} \tag{17}$$

Step 3: In this step, we show that $b_i \geq 0$ for all $i = 1,2,\cdots,n-1$ and give a formula to calculate ω defined in (5) by b_i. In what follows, we only consider the case in which n is even and then show $b_2 \geq 0$ for $i=2$. The others can be proved by the similar way. Under these assumptions, we have

$$\begin{aligned} & det(A)b_2 = det(\triangle_2) \\ &= det \begin{bmatrix} a_{11} & -a_{12} & \dots & -a_{1,n-1} \\ -a_{31} & -a_{32} & \dots & -a_{3,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \dots & -a_{n,n-1} \end{bmatrix} \\ &= (-1)^{n-3} det \begin{bmatrix} a_{11} & -a_{13} & \dots & -a_{12} \\ -a_{31} & a_{33} & \dots & -a_{32} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n3} & \dots & -a_{n2} \end{bmatrix} \\ &= (-1)^{n-3} det \begin{bmatrix} a_{11} & -a_{13} & \dots & a_{11} - \sum_{j=2}^{n-1} a_{1,j} \\ -a_{31} & a_{33} & \dots & a_{33} - \sum_{j=1, j\neq 3}^{n-1} a_{3,j} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n3} & \dots & -\sum_{j=1}^{n-1} a_{n-1,j} \end{bmatrix} \\ &= (-1)^{n-4} det \begin{bmatrix} a_{11} & -a_{13} & \dots & -a_{1n} \\ -a_{31} & a_{33} & \dots & -a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n3} & \dots & a_{nn} \end{bmatrix} = det(L_{22}), \end{aligned} \tag{18}$$

where the third step is obtained by some elementary column transformations; the fourth step is obtained by adding the first $n-2$ columns to the last column; the fifth step used the properties of Laplacian matrix L, $\sum_{j=1}^{n} a_{ij} = 0$ for each i; and the last step is obtained by using the fact that n is even. It is clear that L_{22} is the principal submatrix of L . By *step 2*, we know $det(A) > 0$ and $det(L_{22}) \geq 0$, which directly implies $b_2 \geq 0$. Just as the same analysis, we can show $b_i = det(L_{ii})/det(A) \geq 0$ for $i \in \{1,3,\cdots,n-1\}$. Thus, we have

$$\begin{aligned} & b_1 = \frac{det(L_{11})}{det(A)}; b_2 = \frac{det(L_{22})}{det(A)}; \\ & \cdots; \quad b_{n-1} = \frac{det(L_{n-1,n-1})}{det(A)}. \end{aligned} \tag{19}$$

Substitute (19) into the right equation of (12) and let the free variable $\omega_n = det(A) = det(L_{nn})$, we have

$$\omega_1 = det(L_{11}); \omega_2 = det(L_{22}); \cdots; \omega_n = det(L_{nn}). \tag{20}$$

such that $\omega^\top L = 0$.

Step 4: In this step, we show ω is nonnegative. To proceed, we denote eigenvalues of L_{ii} as λ_i^k, where $i \in I$ and $k \in \{1,2,\cdots,n-1\}$. Because each principal minor L_{ii} of Laplacian matrix L is diagonally dominant and main diagonal element of it is nonnegative, and then by Gerŝgorin disc theorem, each eigenvalue λ_i^k of L_{ii} is in the right open plane, i.e., $\lambda_i^k \geq 0$ for all k. Therefore, $det(L_{ii}) = \lambda_i^1 \lambda_i^2 \cdots \lambda_i^{n-1} \geq 0$ for all $i \in I$. Combined with *Step 1* and *Step 3* ω is nonnegative.

Combining all of the above, the conclusion holds. ■

REFERENCES

[1] T. Vicsek, A. Czirok, E. B. Jacob, I. Cohen, and O. Schochet, Novel Type of Phase Transitions in a System of Self-Driven Particles, *Physics Review Letters*, 75(1995), 6: 1226-1229.

[2] A. Jadbabaie, J. Lin, and A. S. Morse, Coordination of Groups of Mobile Autonomous Agents Using Nearest Neighbour Rules, *IEEE Transactions on Automatic Control*, 48(2003), 6: 988-1001.

[3] W. Ren and R. W. Beard, Consensus Seeking in Multi-agent Systems Under Dynamically Changing Interaction Topologies, *IEEE Transactions on Automatic Control,* 50(2005), 5: 655-660.

[4] L. Moreau, Stability of Multi-agent Systems with Time-dependent Communication Links, *IEEE Transactions on automatic control*, 50(2005), 2: 169-182.

[5] R. Olfati-Saber and R. M. Murray, Consensus Problem in Networks of Agents with Switching Topology and Time-Delays, *IEEE Transactions on Automatic Control*, 49(2004), 9: 1520-1533.

[6] R. Olfati-Saber, J. A. Fax and R. M. Murray, Consensus and Cooperation in Networked Multi-agent Systems, *Proceedings of The IEEE*, 95(2007), 1: 215-233.

[7] J. Cortés, Finite-time Convergent Gradient Flows with Applications to Network Consensus, *Automatica*, 42(2006), 11: 1993-2000.

[8] C. W. Wu, Agreement and Consensus Problems in Groups of Autonomous Agents with Linear Dynamics, *IEEE International Symposium on Circuits and Systems*, May, 2005, 1: 292-295.

[9] Y. Liu, Y. M. Jia, J. P. Du, and F. S. Yu, Average Consensus Problem in Multi-agent Systems, *In Proceedings of the European Control Conference*, July, 2007, Kos, Greece.

[10] P. Lin, Y. M. Jia, J. P. Du, and S. Y. Yuan, Distributed Consensus Control for Networks of Second-order Agents with Switching Topology and Time-delay, *In Peoceedings of the European Control Conference*, July, 2007, Kos, Greece.

[11] H. Q. Lv and Y. M. Jia, Average Consensus Problem for Linear Networked Systems with Double Intergator Dynamics, *Progress in Natural Science*, 17(2007), 8: 204-211, in Chinese.

[12] C. Godsil and G. Royle. Algebraic Graph Theory, *Springer Graduate Texts in Mathematics*, New York, 2001.

[13] R. A. Horn and C. R. Johnson, Matrix Analysis, *Cambridge University Press*, 1985.