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Flocking for Multi-agent Systems with Switching Topology in a Noisy Environment

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Abstract— **This paper considers the flocking of a class of multi-agent systems with switching topology in a noisy environment. We have shown that although the information is contaminated by variable noises, all agents can form and maintain flocking if the gradient of the environment is bounded and the interaction graphs are jointly connected. The proposed results reveal that the discussed multi-agent system with switching topology can maintain the flocking in the considered noisy environment.**

I. INTRODUCTION

Flocking is a group behavior of a large number of agents with a common objective. In nature, flocking behavior provides more chance for the group of birds, fishes or other animals to evade an attack from predators and find food. From the engineering point of view, such a collective behavior has broad applications in the fields like mobile robots, surveillance and search systems, unmanned air vehicles(UAVs) and autonomous underwater vehicles(AUVs), etc. Hence, the mechanism of flocking has attracted lots of researchers from different disciplines and many results have been obtained(See [3]−[12] and references therein). More specifically, Y. Liu and K. M. Passino [5] propose a discrete model and obtain the convergent results for the aggregation of the swarm under total asynchronism(i.e., asynchronism with time delays). In [6], V. Gazi and K. M. Passino propose a continuous first-order dynamic model with attraction/repulsion terms and environment potential function, and then the convergence of the swarm is proved. Thereafter, Y. F. Liu and K. M. Passino [7] extend the results in [6] to a second order model, in which they consider the noise influence on the aggregation of the swarm. A common limitation for the models in [6]∼[7] is that the interaction topology is a complete graph, i.e., each agent knows all information about the group and thus the algorithm is centralized. Recently, Li and Jia [8] extend the works in [7] to the fixed topology case, where the local information is used. The other important works about the flocking can see [10]−[12]. Although the switching topology case are discussed in the literatures [10]−[12], the noises are not considered by them.

Considering that the interaction graph in [7] is not consistent with the practical situation, we extend it to a more general case, in which both switching topology and noises are considered. In this paper, the global information is represented by the potential function because it can reflect the change of the environment which is usually unknown or known little by us. Thus, it is necessary for the agents to detect such a change by themselves. Meanwhile, for each agent, the local information is employed to update its states although these information has been contaminated by the noises. The obtained results give the sufficient conditions for the considered system to achieve flocking in a noisy environment.

This paper is organized as follows. In section II, some background knowledge and the model of the system are given. In section III, the error system is derived and the control law for each agent is designed. The ultimately bounded analysis of the system is presented in section IV. In section V, we conclude our work.

II. PROBLEM FORMULATION

To solve coordinated control problems, graph theory has become a basic tool which provides a natural method to describe the relations between the coordinated variables. A undirected graph of order n is always denoted as $G = (V,E,A)$, where $V = \{v_1, v_2, \cdots, v_n\}$ denotes the set of nodes, E is a finite set of sets. Each element of E is a set that is comprised of exactly two(distinct) nodes, which is denoted as (v_i, v_j). The elements of E are called the edges of G. Because the graph is undirected, $(v_i, v_j) \equiv (v_j, v_i)$. And the symmetrical matrix $A = [a_{ij}]$ denotes the weights of the edge between nodes v_i and v_j. In general, the elements $a_{ij} = a_{ji} \geq 0$ for all $i \neq j (i, j \in I)$ and $a_{ii} = 0$ for all $i \in I$, where $I = \{1, 2, \cdots, N\}$ is an index set. If $a_{ij} = a_{ji} > 0$, it means that (v_i, v_j) is an edge of the graph G. A *path* from a node v_i to a node v_j in graph G is a sequence of nodes $v_{i_1}, v_{i_2}, \cdots, v_{i_{k-1}}, v_{i_k}$ such that $v_i = v_{i_1}$, $v_j = v_{i_k}$, and $(v_{i_{m-1}}, v_{i_m}) \in E$ for $m = 1, \cdots, k$. If there exists a path between any two nodes of G, then G is connected, otherwise, disconnected. For a series of graphs $G_{j_1}, \cdots, G_{j_n}$ with the same nodes, we say graph G_u is the *union* of them if the nodes of G_u are the same as any one of these graphs and the weight $\bar{a}_{ij}$ of edge $(v_i,\ v_j)$ of G_u is defined as follows.

$$\bar{a}_{ij} = \frac{1}{n} \sum_{k=1}^{n} a_{ij}^k \ ,$$

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where a_{ij}^k is the weight of edge (v_i, v_j) of graph G_k for $k = j_1, \cdots, j_n$. As $\bar{a}_{ij} > 0$, it means that (v_i, v_j) is an edge of G_u. If G_u is connected, we say that graphs $G_{j_1}, \cdots, G_{j_n}$ is *jointly connected*.

In what follows, we consider a multi-agent system consisting of N agents. Here, each agent is modeled as a mass point and its dimension is ignored. The dynamics of agent i is given as follows,

$$\dot{x}_i = v_i, \qquad \dot{v}_i = \frac{1}{m_i} u_i, \tag{1}$$

where $i \in I$, $x_i, v_i, u_i \in R^n$ and $m_i \in R$ respectively denote the position, velocity, control input and mass of agent i. In this paper, we assume that $m_i \neq 0$ for all i and each agent can obtain the position and velocity information of its neighbors with some sensing errors. The agent to agent interactions considered here are an attractive/repulsive function where each agent seeks to be in a position that is comfortable relative to its neighbors. Attraction indicates that each agent wants to be close to every other agent and it provides the mechanism for achieving flocking. Repulsion provides the mechanism where each agent does not want to be too close to any other agent(e.g., for agents to avoid collision). In general, there are many ways to define attraction and repulsion. In our model, attraction is represented by the term in u_i like $-k_x(x_i - x_j)$ where $k_x > 0$ is the scalar representing the strength of the attraction. If the neighbors are far apart, then there is a large attraction between them; and if they are close there is a small attraction. For repulsion, we let the two-norm $\|z\| = \sqrt{z^\top z}$ and use a repulsion term in u_i of the form

$$f(x_i - x_j) = k_r (x_i - x_j) \exp\left(-\frac{\|x_i - x_j\|^2}{2c^2}\right), \tag{2}$$

where $k_r > 0$ denotes the magnitude of the repulsion. If $f(x_i - x_j) = k_x(x_i - x_j)$, i.e., the attraction and the repulsion are balanced between two neighbors. Then, $\|x_i - x_j\| = 0$ or $\|x_i - x_j\| = c\sqrt{2\ln\frac{k_r}{k_x}} = \delta$, it is clear that when $\|x_i - x_j\| < \delta$, the repulsion takes effect; otherwise, the attraction dominates. So the parameter c reflects the repulsion range between neighbors. Meanwhile, $f(y)$ has a unique maximum value of $k_r c \exp(-\frac{1}{2})$ at $\|y\| = c$.

For the considered multi-agent system, all agents move in an unknown environment which is represented by an artificial potential function $F(\cdot)$. We assume that $F(\cdot)$ is continuous with finite slope at all points, meanwhile, its gradient satisfies the following assumption.

Assumption 1: Let the environment be represented by a potential function $F(\cdot)$ and its gradient at x_i be $\nabla_{x_i} F(x_i)$, then there exists a constant $\overline{\sigma} > 0$ such that

$$\|\nabla_{x_i} F(x_i)\| \leq \overline{\sigma}, \quad \forall x_i \in R^n. \tag{3}$$

We assume that each agent can sense the negative gradient of the environment at its position and try to follow it. But the sensing value of agent i always has some sufficiently smooth error d_{f_i} so agent i actually senses

$$\nabla_{x_i} F(x_i) - d_{f_i} \tag{4}$$

To proceed, for the considered system (1), we give a definition about neighbors and an assumption about the position topology of all agents as follows.

Definition 1: (Neighbor) Any two agents i and j are called neighbors each other if $d_{ij} = \|x_i(t) - x_j(t)\| \leq d$, where $d > 0$ is a given scalar value, $x_i(t)$ and $x_j(t)$ are the positions of the agents i and j at time t, respectively. All the neighbors of agent i at time t are denoted by a set $N_i(t)$,

$$N_i(t) \triangleq \{j : \|d_{ij}\| \leq d, j \in I, j \neq i\},$$

meanwhile, let $|N_i(t)|$ denote the number of the neighbors of agent i at time t. Furthermore, as agent i has no neighbors, we stipulate $|N_i(t)| = 1$. In general, the time t is omitted, i.e., $N_i(t) = N_i$ and $|N_i(t)| = |N_i|$.

Assumption 2: For the system (1), there exists a constant $T > 0$ and an infinite sequence of uniformly bounded, non-overlapping, continuous time intervals $[t_i, t_i + T), i = 0, 1, \cdots$, starting at $t_0 = 0$, such that the union of position graphs of all agents is jointly connected during each time interval i.

III. Control Laws and Error Dynamics

In this section, we will derive the error system and propose the control law for each agent. To proceed, let $\overline{x}_i/\overline{v}_i$ denote the average position/velocity of the neighbors of agent i as follows.

$$\overline{x}_i = \frac{1}{|N_i|}\sum_{j\in N_i} x_j\,, \quad \overline{v}_i = \frac{1}{|N_i|}\sum_{j\in N_i} v_j\,, \quad i \in I. \tag{5}$$

And $\overline{x}/\overline{v}$ denote the position/velocity of the swarm center.

$$\overline{x} = \frac{1}{N}\sum_{i=1}^{N} x_i\,, \quad \overline{v} = \frac{1}{N}\sum_{i=1}^{N} v_i\,, \qquad i \in I. \tag{6}$$

Now, for a given agent $i \in I$, the local error is defined as $e_{x_i} = x_i - \overline{x}_i$ and $e_{v_i} = v_i - \overline{v}_i$; and the global error is defined as $\overline{e}_{x_i} = x_i - \overline{x}$ and $\overline{e}_{v_i} = v_i - \overline{v}$. Clearly, the following relations can be derived.

$$\begin{aligned} e_{x_i} &= x_i - \overline{x}_i = \overline{e}_{x_i} + \overline{x} - \frac{1}{|N_i|}\sum_{j\in N_i} x_j \\ &= \overline{e}_{x_i} + \overline{x} - \frac{1}{|N_i|}\sum_{j\in N_i}(\overline{e}_{x_j} + \overline{x}) \\ &= \overline{e}_{x_i} + \overline{x} - \frac{1}{|N_i|}\sum_{j\in N_i}\overline{e}_{x_j} - \overline{x} = \overline{e}_{x_i} - \frac{1}{|N_i|}\sum_{j\in N_i}\overline{e}_{x_j}, \end{aligned} \tag{7}$$

and,

$$\begin{aligned} e_{v_i} &= v_i - \overline{v}_i = \overline{e}_{v_i} + \overline{v} - \frac{1}{|N_i|}\sum_{j\in N_i} v_j \\ &= \overline{e}_{v_i} + \overline{v} - \frac{1}{|N_i|}\sum_{j\in N_i}(\overline{e}_{v_j} + \overline{v}) \\ &= \overline{e}_{v_i} + \overline{v} - \frac{1}{|N_i|}\sum_{j\in N_i}\overline{e}_{v_j} - \overline{v} = \overline{e}_{v_i} - \frac{1}{|N_i|}\sum_{j\in N_i}\overline{e}_{v_j}. \end{aligned} \tag{8}$$

Thus, the error system can be defined as follows.

$$\dot{\overline{e}}_{x_i} = \overline{e}_{v_i}, \quad \dot{\overline{e}}_{v_i} = \dot{v}_i - \dot{\overline{v}}. \tag{9}$$

We assume that each agent can get information about the positions and velocities of its neighbors, and then calculate its position/velocity relative to the local position/velocity center $\bar{x}_i$ and $\bar{v}_i$, but with some errors. Particularly, let $d_{x_i} \in R^n$ and $d_{v_i} \in R^n$ be these errors for agent i, respectively. We assume that $d_{x_i}(t)$ and $d_{v_i}(t)$ are any trajectories that are sufficiently smooth and fixed a priority for all time, meanwhile, these terms are uniformly called as "noise". Thus, agent i actually sense

$$\hat{e}_{x_i} = e_{x_i} - d_{x_i}, \quad \hat{e}_{v_i} = e_{v_i} - d_{v_i}, \tag{10}$$

Here, we further assume that each agent can accurately obtain its position and velocity. Meanwhile, no agent can obtain the states of swarm center $\bar{x}$ and $\bar{v}$. Thus, substituting equations (7) and (8) into equation (10), we have

$$\hat{e}_{x_i} = \bar{e}_{x_i} - \frac{1}{|N_i|}\sum_{j\in N_i}\bar{e}_{x_j} - d_{x_i}, \tag{11}$$

and,

$$\hat{e}_{v_i} = \bar{e}_{v_i} - \frac{1}{|N_i|}\sum_{j\in N_i}\bar{e}_{v_j} - d_{v_i}. \tag{12}$$

In what follows, the control input for agent i is proposed as

$$\begin{aligned} u_i = & - m_i k_p(\nabla_{x_i}F(x_i) - d_{f_i}) - m_i k_x \hat{e}_{x_i} - m_i k_v \hat{e}_{v_i} \\ & - m_i k v_i + m_i \sum_{j\in N_i} f(x_i - \hat{x}_j), \;\; i \in I, \end{aligned} \tag{13}$$

where $k_p > 0$ denotes the desire of agent i to follow the negative gradient of the environment, $k_x > 0$ and $k_v > 0$ represent the magnitude of attraction between neighbors, k is the coefficient of velocity damping term, and the last term exactly denotes the repulsion which is exerted to agent i by its neighbors. It is noted that parameters k_p, k_x, k_v and k are the same for all agents. This means that all agents are identical. In fact, different agent can have different k_p, k_x, k_v and k, and the below analysis can also be derived. One reason we take them as same is to simplify the analysis. On the other hand, because different agent has different error terms, this implies that the systems are robust to the models and thus we assume that the parameter uncertainty can be partly reflected. For the last term in u_i, we assume that any two agents i and j are disturbed by the same noises as they are neighbors and $\|x_i - \hat{x}_j\| = \|x_j - \hat{x}_i\| = \|x_i - x_j\| + \hat{d}_{ij}$ with the upper bounded constant $\hat{d}_{ij}$ as the sensing error. In addition, the control law (13) is non-smooth because the neighbor set N_i is changed along the time. In what follows, by substituting equation (13) into $\bar{v}$ of equation (6), we have

$$\begin{aligned} \dot{\bar{v}} = & \frac{1}{N}\sum_{i=1}^{N}[-k_p(\nabla_{x_i}F(x_i) - d_{f_i}) - kv_i + \sum_{j\in N_i} f(x_i - \hat{x}_j)] \\ & - k_x\hat{e}_{x_i} - k_v\hat{e}_{v_i} \\ = & \frac{1}{N}\sum_{i=1}^{N}[-k_p(\nabla_{x_i}F(x_i) - d_{f_i}) - k_x(e_{x_i} - d_{x_i}) \\ & - k_v(e_{v_i} - d_{v_i}) - kv_i + \sum_{j\in N_i} f(x_i - \hat{x}_j)] \\ = & \frac{1}{N}\sum_{i=1}^{N}[-k_p\nabla_{x_i}F(x_i) - k_x e_{x_i} - k_v e_{v_i} - kv_i \\ & + \sum_{j\in N_i} f(x_i - \hat{x}_j)] + \frac{1}{N}\sum_{i=1}^{N}[k_p d_{f_i} + k_x d_{x_i} + k_v d_{v_i}] \\ = & \frac{1}{N}\sum_{i=1}^{N}[-k_p\nabla_{x_i}F(x_i) - k_x(\bar{e}_{x_i} - \frac{1}{|N_i|}\sum_{j\in N_i}\bar{e}_{x_j}) \\ & - k_v(\bar{e}_{v_i} - \frac{1}{|N_i|}\sum_{j\in N_i}\bar{e}_{v_j}) - kv_i + \sum_{j\in N_i} f(x_i - \hat{x}_j)] \\ & + \frac{1}{N}\sum_{i=1}^{N}[k_p d_{f_i} + k_x d_{x_i} + k_v d_{v_i}] \\ = & - k\bar{v} - \frac{1}{N}\sum_{i=1}^{N}k_p\nabla_{x_i}F(x_i) + \frac{1}{N}\sum_{l=1}^{N}(\frac{k_x}{|N_l|}\sum_{j\in N_l}\bar{e}_{x_j}) \\ & + \frac{1}{N}\sum_{l=1}^{N}(\frac{k_v}{|N_l|}\sum_{j\in N_l}\bar{e}_{v_j}) + \frac{1}{N}\sum_{i=1}^{N}[k_p d_{f_i} + k_x d_{x_i} + k_v d_{v_i}], \end{aligned} \tag{14}$$

where we use the fact that $\sum_{i=1}^{N}[\sum_{j\in N_i} f(x_i - \hat{x}_j)] = 0$; $\frac{k_x}{N}\sum_{i=1}^{N}\bar{e}_{x_i} = \frac{k_x}{N}\sum_{i=1}^{N}(x_i - \bar{x}) = 0$; $\frac{k_v}{N}\sum_{i=1}^{N}\bar{e}_{v_i} = \frac{k_v}{N}\sum_{i=1}^{N}(v_i - \bar{v}) = 0$. Let $E_i = [\bar{e}_{x_i}^{\top}, \bar{e}_{v_i}^{\top}]^{\top}$ and $E = [E_1^{\top}, E_2^{\top}, \cdots, E_N^{\top}]^{\top}$, and note that $kv_i - k\bar{v} = k\bar{e}_{v_i}$. Then, substituting equations (11)-(14) into (9), we have

$$\begin{aligned} \dot{\bar{e}}_{v_i} = & \dot{v}_i - \dot{\bar{v}} = -k_p(\nabla_{x_i}F(x_i) - d_{f_i}) - k_x\hat{e}_{x_i} - k_v\hat{e}_{v_i} - kv_i \\ & + \sum_{j\in N_i} f(x_i - \hat{x}_j) + k\bar{v} + \frac{1}{N}\sum_{i=1}^{N}k_p\nabla_{x_i}F(x_i) \\ & - \frac{1}{N}\sum_{l=1}^{N}(\frac{k_x}{|N_l|}\sum_{j\in N_l}\bar{e}_{x_j}) - \frac{1}{N}\sum_{l=1}^{N}(\frac{k_v}{|N_l|}\sum_{j\in N_l}\bar{e}_{v_j}) \\ & - \frac{1}{N}\sum_{i=1}^{N}[k_p d_{f_i} + k_x d_{x_i} + k_v d_{v_i}] \\ = & - k_p(\nabla_{x_i}F(x_i) - d_{f_i}) - k_x(\bar{e}_{x_i} - \frac{1}{|N_i|}\sum_{j\in N_i}\bar{e}_{x_j} - d_{x_i}) \\ & - k_v(\bar{e}_{v_i} - \frac{1}{|N_i|}\sum_{j\in N_i}\bar{e}_{v_j} - d_{v_i}) - kv_i + \sum_{j\in N_i} f(x_i - \hat{x}_j) \\ & + k\bar{v} + \frac{1}{N}\sum_{i=1}^{N}k_p\nabla_{x_i}F(x_i) - \frac{1}{N}\sum_{l=1}^{N}(\frac{k_x}{|N_l|}\sum_{j\in N_l}\bar{e}_{x_j}) \\ & - \frac{1}{N}\sum_{l=1}^{N}(\frac{k_v}{|N_l|}\sum_{j\in N_l}\bar{e}_{v_j}) - \frac{1}{N}\sum_{i=1}^{N}[k_p d_{f_i} + k_x d_{x_i} + k_v d_{v_i}] \\ = & - k_x\bar{e}_{x_i} - (k_v + k)\bar{e}_{v_i} + f_i(E) + g_i(E) + \phi_i(E) + \psi_i(E), \end{aligned} \tag{15}$$

where

$$\begin{aligned} f_i(E) = & - k_p\nabla_{x_i}F(x_i) + \frac{k_p}{N}\sum_{l=1}^{N}\nabla_{x_l}F(x_l); \\ g_i(E) = & \sum_{j\in N_i} f(x_i - \hat{x}_j); \\ \phi_i(E) = & \frac{k_x}{|N_i|}\sum_{j\in N_i}\bar{e}_{x_j} + \frac{k_v}{|N_i|}\sum_{j\in N_i}\bar{e}_{v_j} - \\ & \frac{k_x}{N}\sum_{l=1}^{N}(\frac{1}{|N_l|}\sum_{j\in N_l}\bar{e}_{x_j}) - \frac{k_v}{N}\sum_{l=1}^{N}(\frac{1}{|N_l|}\sum_{j\in N_l}\bar{e}_{v_j}); \\ \psi_i(E) = & - \frac{1}{N}\sum_{i=1}^{N}[k_p d_{f_i} + k_x d_{x_i} + k_v d_{v_i}] + k_x d_{x_i} \\ & + k_v d_{v_i} + k_p d_{f_i}. \end{aligned} \tag{16}$$

Let I_n be an $n \times n$ identity matrix, then substituting (15) into (9), error dynamics of agent i can be written in a compact form as

$$\dot{E}_i = \underbrace{\begin{bmatrix} 0 & I_n \\ -k_x I_n & -(k_v+k)I_n \end{bmatrix}}_{A} E_i + \underbrace{\begin{bmatrix} 0 \\ I_n \end{bmatrix}}_{B} (f_i(E)+g_i(E)+\phi_i(E)+\psi_i(E)),\ i \in I. \tag{17}$$

Note that matrix A with $k_x > 0$, $k_v + k > 0$ has eigenvalues given by the roots of $[s^2 + (k_v+k)s + k_x]^n$, which are in the strict left half plane. Thus, A is Hurwitz. In addition, we can see from (17) that the noise and the switch of the information topology can only affect the second terms, and $\|E_i\|$ is a deterministic term. Thus the error systems can be analysis as a perturbed systems [14].

IV. Ultimately Uniformly Bounded Analysis of Flocking

In this section, we study the ultimate boundedness of inter-agents trajectories. Our analysis methodology involves viewing the error dynamics (17) as generating $E_i(t)$ trajectories for a given $E_i(0)$ and the fixed error trajectories $d_{x_i}(t), d_{v_i}(t)$ and $d_{f_i}(t)$, $t \geq 0$. Here, most of error trajectories are not considered except ones satisfying the bounded conditions as follows [7].

$$\begin{aligned} \|d_{f_i}(t)\| \leq & D_{f_i};\ \ \|d_{x_i}(t)\| \leq D^i_{x_1}\|E_i\| + D^i_{x_2}; \\ \|d_{v_i}(t)\| \leq & D^i_{v_1}\|E_i\| + D^i_{v_2}. \end{aligned} \tag{18}$$

where $D_{f_i}, D^i_{x_1}, D^i_{x_2}, D^i_{v_1}$ and $D^i_{v_2}$ are known nonnegative constants for all $i \in I$. It is noted that we do the same assumption as [7]. On the one hand, from the practical point of view, the agents located far away from the swarm center has little chance to share the groups averaging influence, and thus they have more possibility to be distracted by the noises, just as the things always happen in the sheepherd. In this sense, the above assumptions are suitable. But on the other hand, in the considered system, each agent has the limited sensing ability, and thus it only can receive the information from its neighbors which are located within a limited range around it. In this case, the sensing noise level around each agent can be seen as a bounded term. So the terms $D^i_{x_1}$ and $D^i_{v_1}$ should be little enough in order to match such a situation. In addition, the noise d_{f_i} is unaffected by the position of an agent. By considering only these class of fixed sensing error trajectories, we prune the set of possibilities for $\|E_i\|$ trajectories and it is only for theses pruned set our analysis holds. To proceed, a key lemma is given as follows.

Lemma 1: Suppose that each agent of system (1) has the control input u_i defined in (13). Assume that the assumption 2 is satisfied, then there exists a scalar constant $R > 0$ such that $\|E_i(t)\| < R$ for all time t, where $i \in I$.

Proof: At time t, let $G(t)$ be the position graph of system and $x(t) = [x_1(t)^\top, x_2(t)^\top, \cdots, x_N(t)^\top]^\top$ the positions of all agents. Due to system (1) satisfying the assumption 2, then for an arbitrary time interval $\tau \leq t < \tau + T$, $\tau \geq 0$, the union of position graphs is jointly connected. Let $G_u(\tau) = \bigcup_{t=\tau}^{\tau+T-1} G(t)$ denote the union of position graphs during time interval τ, $x_u = [x(\tau)^\top, x(\tau+1)^\top, \cdots, x(\tau+T-1)^\top]^\top$ denote the positions of all agents during time interval τ. Then $G_u(\tau)$ can be seen as a connected graph with NT vertices. The position center of $G_u(\tau)$ is

$$\begin{aligned} \bar{x}_u(\tau) = & \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=\tau}^{\tau+T-1} x_i(t) = \frac{1}{T}\sum_{t=\tau}^{\tau+T-1}\left(\frac{1}{N}\sum_{i=1}^{N} x_i(t)\right) \\ = & \frac{1}{T}\sum_{t=\tau}^{\tau+T-1} \bar{x}(t), \end{aligned} \tag{19}$$

where $\bar{x}(t)$ is the position center of all agents at time t. On the other hand, the value of $e^i_u(t) = x_i(t) - \bar{x}_u(\tau)$ must be finite because $G_u(\tau)$ is connected, i.e., there exists a constant $R^* > 0$ such that $\|x_i(t) - \bar{x}_u(\tau)\| \leq R^*$ holds for all $x_i(t)$, $i \in I$, $t \in [\tau, \tau+T-1)$. Obviously, all $\bar{x}(t)$ are the inner points of the convex combinations of all positions $x_i(t)$ and thus we have $\|x_i(t) - \bar{x}(t)\| \leq R^*$ for all i at time t. Combined with all agents having the limited velocity, there must exist a constant $R > 0$ such that $\|E_i(t)\| < R$ holds at time t for all $i \in I$. Meanwhile, due to $\tau \geq 0$ being arbitrarily selected, the conclusion holds for all time t. ■

Lemma 1 reveals that the error state E_i always has upper bound and thus guarantees the below analysis. In what follows, we give the main results.

Theorem 1: Assume that the system (1) has the control input u_i defined in (13), assumption 1 and 2 are satisfied, and all noises are bounded by (18), let

$$\begin{aligned} \beta_1 = & \frac{1}{2k_x(k_v+k)}\left\{(k_x+1)^2 + (k_v+k)^2 \right. \\ & \left. + \sqrt{[(k_v+k)^2+(k_x+1)^2][(k_v+k)^2+(k_x-1)^2]}\right\}, \end{aligned} \tag{20}$$

if we have

$$k_x D^i_{x_1} + k_v D^i_{v_1} \leq \frac{1}{\beta_1}, \tag{21}$$

and the parameters are such that

$$\beta_1\sqrt{k_x^2+k_v^2}\sum_{i=1}^{N}\frac{2+\sqrt{{D^i_{x_1}}^2+{D^i_{v_1}}^2}}{(1-\theta_i)[1-\beta_1(k_x D^i_{x_1}+k_v D^i_{v_1})]} < 1, \tag{22}$$

where $0 < \theta_i < 1$ for all i, then the trajectories of (17) are uniformly bounded.

Proof: For the error system (17), let Lyapunov function for each agent be chosen as

$$V_i(E_i) = E_i^\top P E_i, \tag{23}$$

where $P = P^\top > 0$ is a positive-definite matrix with order $2n \times 2n$. Then, choose the Lyapunov function for the composite system as

$$V(E) = \sum_{i=1}^{N} E_i^\top P E_i. \tag{24}$$

Then, by equations (17) and (23), we have

$$\begin{aligned} \dot{V}_i(E_i) = & \dot{E}_i^\top P E_i + E_i^\top P E_i = E_i^\top \underbrace{(PA + A^\top P)}_{-Q} E_i \\ & + 2E_i^\top PB(f_i(E)+g_i(E)+\phi_i(E)+\psi_i(E)). \end{aligned} \tag{25}$$

Note that when $Q = Q^\top > 0$, the unique solution P of $A^\top P + PA = -Q$ has $P = P^\top > 0$ as needed. Since for any positive definite matrix $M \in R^{n\times n}$ and vector $x \in R^{n\times 1}$, we have $\lambda_{min}(M)x^\top x \le x^\top Mx \le \lambda_{max}x^\top x$, where $\lambda_{min}(M)$ and $\lambda_{max}(M)$ denote the minimum and maximum eigenvalue of M, respectively. Thus, by equation (24), we directly have

$$\lambda_{min}(P)\sum_{i=1}^{N}\|E_i\| \le V(E) \le \lambda_{max}(P)\sum_{i=1}^{N}\|E_i\|.$$

Then,

$$\begin{aligned}
\dot{V}(E) &= \sum_{i=1}^{N}\dot{V}_i(E_i) = \sum_{i=1}^{N}\Big[-E_i^\top QE_i + 2E_i^\top PB\,(f_i(E) \\
&\quad + g_i(E) + \phi_i(E) + \psi_i(E))] \\
&\le \sum_{i=1}^{N}\Big\{-\lambda_{min}(Q)\|E_i\|^2 + 2\lambda_{max}(P)\|E_i\|\,[(k_p\bar{\sigma} + k_p\bar{\sigma}) \\
&\quad + |N_i|k_r c\exp(-\frac{1}{2}) + \frac{\sqrt{k_x^2+k_v^2}}{|N_i|}\sum_{j\in N_i}\|E_j\| \\
&\quad + \frac{\sqrt{k_x^2+k_v^2}}{N}\sum_{l=1}^{N}\left(\frac{1}{|N_l|}\sum_{j\in N_l}\|E_j\|\right) + k_x\|d_{x_i}\| \\
&\quad + k_v\|d_{v_i}\| + k_p\|d_{f_i}\| \\
&\quad + \frac{1}{N}\sum_{l=1}^{N}\left(k_x\|d_{x_l}\| + k_v\|d_{v_l}\| + k_p\|d_{f_l}\|\right)\Big]\Big\} \\
&\le \sum_{i=1}^{N}\Big\{-\lambda_{min}(Q)\|E_i\|^2 + 2\lambda_{max}(P)\|E_i\|\,[2k_p\bar{\sigma} \\
&\quad + |N_i|k_r c\exp(-\frac{1}{2}) + \frac{\sqrt{k_x^2+k_v^2}}{|N_i|}\sum_{j\in N_i}\|E_j\| \\
&\quad + \frac{\sqrt{k_x^2+k_v^2}}{N}\sum_{l=1}^{N}\left(\frac{1}{|N_l|}\sum_{j\in N_l}\|E_j\|\right) \\
&\quad + k_x\|(D_{x_1}^i\|E_i\| + D_{x_2}^i) + k_v(D_{v_1}^i\|E_i\| + D_{v_2}^i) \\
&\quad + k_p D_{f_i} + \frac{1}{N}\sum_{l=1}^{N}\Big(k_x(D_{x_1}^l\|E_i\| + D_{x_2}^l) \\
&\quad + k_v(D_{v_1}^l\|E_i\| + D_{v_2}^l) + k_p D_{f_l}\Big)\Big]\Big\} \\
&= \sum_{i=1}^{N}(-c_1^i\|E_i\| + c_2^i\|E_i\| + \|E_i\|\sum_{j=1}^{N}a_{ij}\|E_j\|)
\end{aligned} \tag{26}$$

where $\lambda_{min}(Q) > 0$ is the minimum eigenvalue of Q and $\lambda_{max}(P) > 0$ is the maximum eigenvalue of P, c_1^i, c_2^i and a_{ij} are as follows

$$a_{ij} = \begin{cases} \dfrac{2\lambda_{max}(P)}{N}\,[\sqrt{k_x^2+k_v^2}\displaystyle\sum_{l\in N_j}\frac{1}{|N_l|} + k_x D_{x_1}^j \\ \qquad + k_v D_{v_1}^j\,], \qquad j\overline{\in} N_i; \\ 2\lambda_{max}(P)\,[\sqrt{k_x^2+k_v^2}\left(\dfrac{1}{|N_i|} + \dfrac{1}{N}\displaystyle\sum_{l\in N_j}\frac{1}{|N_l|}\right) \\ \qquad + \dfrac{1}{N}\left(k_x D_{x_1}^j + k_v D_{v_1}^j\right)\Big], \quad j\in N_i. \end{cases} \tag{27}$$

$$c_1^i = \lambda_{min}(Q)[1 - \frac{2\lambda_{max}(P)}{\lambda_{min}(Q)}(k_x D_{x_1}^i + k_v D_{v_1}^i)]$$

$$\begin{aligned} c_2^i =& 2\lambda_{max}(P)[2k_p\bar{\sigma} + k_x D_{x_1}^i + k_v D_{v_1}^i + k_p D_{f_i} \\ &+ \frac{1}{N}\sum_{l=1}^{N}(k_p d_{f_l} + k_x d_{x_l} + k_v d_{v_l})] + |N_i|k_r c\exp(-\frac{1}{2}), \end{aligned}$$

Clearly, if

$$k_x D_{x_1}^i + k_v D_{v_1}^i \le \frac{1}{\beta_0}, \tag{28}$$

where

$$\beta_0 = \frac{2\lambda_{max}(P)}{\lambda_{min}(Q)}.$$

then $c_1^i > 0$, so the first term in (26) gives a negative contribution to $\dot{V}(E)$. To proceed, it is noted that as β_0 small as possible, the system may tolerate noise with the largest possible bounds($D_{x_1}^i$ and $D_{v_1}^i$) while keeping stability. Just as [7] and [14] state, β_0 is minimized by take $Q = I_n$ and it is easy to calculate that $\min(\beta_0) = \frac{2\lambda_{max}(P)}{\lambda_{min}(Q)}|_{Q=I_n} = \beta_1$. Substituting β_1 into equation (27), we can get the new form of c_1^i, c_2^i and a_{ij}. Although they are omitted here, we will directly use them in the below analysis.

Now, let us return to (26) and note that for any $0 < \theta_i < 1$, we have

$$\begin{aligned} &-c_1^i\|E_i\|^2 + c_2^i\|E_i\| \\ &= -(1-\theta_i)c_1^i\|E_i\|^2 - \theta_i c_1^i\|E_i\|^2 + c_2^i\|E_i\| \\ &\le -(1-\theta_i)c_1^i\|E_i\|^2 = \sigma_i\|E_i\|^2, \ \forall\ \|E_i\| \ge r_i. \end{aligned} \tag{29}$$

where $\sigma_i = -c_1^i(1-\theta_i)$ and $r_i = \frac{c_2^i}{\theta_i c_1^i}$. Then, if $\|E_i\| \ge r_i$, the first two term in (26) combined will give a negative contribution to $\dot{V}(E)$. In what follows, we seek conditions under which $\dot{V}(E) < 0$. By definition 1, we know $|N_i| \ge 1$ for all i and thus $\frac{1}{|N_i|} + \frac{1}{N}\sum_{l\in N_j}\frac{1}{|N_l|} \le 2$ for agent i. So, by (27), we have

$$\begin{aligned} a_{ij} \le& \beta_1\left[2\sqrt{k_x^2+k_v^2} + \frac{1}{N}(k_x D_{x_1}^j + k_v D_{v_1}^j)\right] \\ \le& \beta_1\sqrt{k_x^2+k_v^2}\left(2 + \frac{1}{N}\sqrt{{D_{x_1}^j}^2 + {D_{v_1}^j}^2}\right) = a_{ij}^* \end{aligned}$$

Next, just as [7] does, the agents can be divided into two sets according to whether $\|E_i\| \le r_i$ or not. One is

$$\Pi_o = \{i : \|E_i\| \ge r_i, i \in 1,2,\cdots,N\} = \{i_o^1, i_o^2, \cdots, i_o^{N_o}\},$$

and the other is

$$\Pi_I = \{i : \|E_i\| < r_i, i \in 1,2,\cdots,N\} = \{i_I^1, i_I^2, \cdots, i_I^{N_I}\},$$

where $\Pi_o \bigcup \Pi_I = \{1,2,\cdots,N\}$, $\Pi_o \bigcap \Pi_I = \emptyset$. Let the size of Π_o and Π_I are N_o and N_I,respectively, then $N_o + N_I = N$. Although we do not explicitly know the sets Π_o and Π_I, they must exist. Meanwhile, both Π_o and Π_I depend on the time t but we will allow that time to be arbitrary so the analysis below will be for all time. From now on, we assume $N_o > 0$ and for the case $N_o = 0$ will be discussed in the later. Then, by lemma 1 and equations (26)-(29), we have

$$\begin{aligned} \dot{V}(E) \le& \sum_{i\in\Pi_o}\sigma_i\|E_i\|^2 + \sum_{i\in\Pi_o}\left(\|E_i\|\sum_{j\in\Pi_o}a_{ij}^*\|E_j\|\right) \\ &+ \sum_{i\in\Pi_o}(K_1 + K_3 a_{ii}^*)\|E_i\| + K_2 + K_4. \end{aligned}$$

where we used the fact that any $\|E_i\|$ has the upper bound by lemma 1. Thus, for each fixed N_o, there exists positive constants $K_1(N_o)$, $K_2(N_o)$, $K_3(N_o)$, and $K_4(N_o)$ such that

$$K_1(N_o) \geq \sum_{j\in\Pi_I} a^*_{ij}\|E_j\|; \;\; K_2(N_o) \geq \sum_{j\in\Pi_I} [-c_1^i\|E_i\|^2 + c_2^i\|E_i\|];$$
$$K_3(N_o) \geq \sum_{i\in\Pi_I} \|E_i\|; \;\; K_4(N_o) \geq \sum_{j\in\Pi_I}\left(\|E_i\|\sum_{j\in\Pi_I} a^*_{ij}\|E_j\|\right). \quad (30)$$

Let $\omega = \left[\|E_{i_o^1}\|, \|E_{i_o^2}, \cdots, \|E_{i_o^{N_o}}\|\right]^\top$, and the $N_o\times N_o$ matrix S be specified by

$$s_{jn} = \begin{cases} -(\sigma_{i_o^j} + a^*_{ii_o^j}), & \text{for } j = n; \\ -a^*_{ii_o^j}, & \text{for } j \neq n. \end{cases} \quad (31)$$

Then, we have

$$\dot V(E) \leq -\omega^\top S\omega + \sum_{i\in\Pi_o}(K_1 + K_3 a^*_{ii})\|E_i\| + K_2 + K_4.$$

Clearly, if the matrix S is positive definite, i.e., $\lambda_{min}(S) > 0$, we have

$$\dot V(E) \leq -\lambda_{min}(S)\sum_{i\in\Pi_o}\|E_i\|^2 + \sum_{i\in\Pi_o}(K_1 + K_3 a^*_{ii})\|E_i\| + K_2 + K_4. \quad (32)$$

Thus, by (31), when the $\|E_i\|$ for $i\in\Pi_o$ is sufficiently large, the sign of the $\dot V(E)$ is determined by the first term in (31). This is valid for any value of N_o, $1 \leq N_o \leq N$. Hence, for any $N_o \neq 0$, the system is uniformly ultimately bounded if S is positive definite, i.e., $S > 0$. In what follows, we seek the conditions for $S > 0$.

By (31), S is a real symmetric matrix. So a necessary and sufficient condition for $S > 0$ is that its successive principal minors are all positive. Define $|s_m|$ as the determinants of the principal minors of S, $m = 1, \cdots, N_o$. Then, we can show that

$$|s_m| = \left\{1 + \sum_{j=1}^m \frac{a^*_{i,i_o^j}}{\sigma_{i_o^j}}\right\}\prod_{k=1}^m(-\sigma_{i_o^k}).$$

Due to $-\sigma_{i_o^k} > 0$ for all $k = 1, \cdots, m$, to have all the previous determinants positive, we need

$$\sum_{j=1}^m \frac{a^*_{ii_o^j}}{\sigma_{i_o^j}} > -1,$$

that is,

$$\sum_{j=1}^m \frac{\beta_1\sqrt{k_x^2 + k_v^2}\left(2 + \frac{1}{N}\sqrt{D_{x_1}^{i_o^j\,2} + D_{v_1}^{i_o^j\,2}}\right)}{(1-\theta_{i_o^j})[1 - \beta_1(k_x D_{x_1}^{i_o^j} + k_v D_{v_1}^{i_o^j})]} < 1. \quad (33)$$

for all $m = 1, \cdots, N_o$. Due to $1 \leq m \leq N_o \leq N$, (33) is satisfied when (22) is satisfied and thus, $S > 0$ for all $N_o \neq 0$. Hence, when $\|E_i\|$ is sufficiently large, $\dot V(E) < 0$ and the uniform boundedness of the trajectories of the error system is achieved.

To complete the proof, we need to consider the case $N_o = 0$. Note that when $N_o = 0$, $\|E_i\| < r_i$ for all i. If we have $N_o = 0$ for all time, then we could simply take $max_i(r_i)$ as the uniform ultimate bound. Otherwise, at certain time, the system changes such that some $\|E_i\| \geq max_i(r_i)$, then we have $N_o \geq 1$ immediately, then the analysis follows the above. Thus, in either case we obtain the uniform ultimate boundedness. ∎

So far, we have proven that the error system (17) is uniformly ultimately bounded when certain conditions are satisfied.

Remark 1: Comparison theorem 1 with the corresponding result in [7], the main difference between them is that our result is a distributed algorithm but the results in [7] is a centralized algorithm. As a result, it is remarkably reduced that the communication costs for a multi-agent system to maintain a flocking.

Remark 2: The conclusions are available for the case of any dimension of Euclidean space because the dimension of space is not used in the above analysis.

V. Conclusions

This paper has extended the results in [7] to a more general case with switching topology and noises, which remarkably reduces the communication cost for all agents to achieve flocking in a noisy environment because the proposed results only require the position graphs being jointly connected. In future work, the time-delay influence on the convergence of the system will be considered.

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