2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Conditions for Feedback Stabilizability in Switched Linear Systems

Federico Najson
Instituto de Ingeniería Eléctrica – Facultad de Ingeniería
Universidad de la República
Montevideo, Uruguay

Abstract— **This communication is concerned with state-feedback stabilizability of discrete-time switched linear systems. Necessary and sufficient conditions for state-feedback exponential stabilizability are presented. It is shown that, a switched linear system is state-feedback exponentially stabilizable if and only if an associated sequence converges to zero. Equivalently, a switched linear system is state-feedback exponentially stabilizable if and only if a dynamic programming equation admits a solution of some kind. We also address the issue of testing the stabilizability of a given switched system by computing the elements of a new associated sequence of upper bounds for the elements of the previously mentioned sequence. These computations involve the solution of convex programming problems. The elements of both associated sequences are shown to be related via Lagrange duality. Numerical examples illustrate some of the results reported in the paper.**

I. INTRODUCTION

In the present article, we will use the term switched system to refer to a class of dynamical system described by a differential or difference equation whose right hand side is dynamically selected from a given finite set of (right hand side) functions, and this selection is governed by a function (of the time) usually termed switching signal.

Switched systems are useful in modeling processes exhibiting significatively different behaviors depending on a state (or status, or mode) that takes discrete finite values and which describes the mode of operation of the process. Such processes naturally appear, in numerous and different manners, in engineering. For instance: In power electronics, it is well-known [12], [11] that various types of power converters present the aforementioned characteristic behavior. In control systems, control schemes have been considered [3], [4] in which a master (supervisor) controller has the task of switching or selecting between a given finite set of available controllers to close the loop with a given plant. Thus, the overall (closed-loop) system is a switched system. Switched systems are also being considered [7] to model the complex behavior of processes that are subject to the occurrence of faults. Such scenario is modeled by considering a finite number of possible faults (that may suffer the process) each associated with a different dynamic behavior described by a significatively simpler model.

Switched systems have lately attracted the attention of researchers in the control systems community. The practical relevance of devising methods for analysis of switched systems and for the design of such systems in order to achieve certain specifications is a motivating factor for studying these systems. The quest for devising such methods have encourage research in many directions. Some of the research in the area is documented in various survey papers [5], [2], [9] and monographs [11], [8], [6], [10]. Some of the important issues that are being studied concern with certain stability problems related to these systems. As stated in [5], it is possible to recognize three basic problems concerning the stability of switched systems: (1) The problem of finding conditions for stability, for arbitrary switching signals. (2) The problem of finding conditions for stability, for switching signals belonging to some given set. (3) The problem of finding conditions for the existence of a switching signal that stabilizes the system. The topic of the present article is related to (but different from) problem (3). It is concerned with the problem of finding conditions for the existence of a state-feedback that stabilizes the system. More specifically, the present work is devoted to the problem of finding conditions to determine the state-feedback exponential stabilizability in discrete-time switched linear systems.

The present article is organized as follows. In section II mathematical preliminaries and definitions are introduced. Necessary and sufficient conditions for state-feedback exponential stabilizability of switched linear systems are presented and proved in section III. In this section, a sequence is associated to each switched linear system. We then prove that the state-feedback exponential stabilizability of the switched system is equivalent to the property of convergency (to zero) of the associated sequence. Other conditions, which are equivalent to the state-feedback exponential stabilizability of the switched system, are also presented. Section IV is devoted to the problem of testing the state-feedback stabilizability of a given switched system by computing upper bounds for the elements of the associated sequence. In that manner, a new associated sequence (of upper bounds) is introduced. The computation of the elements of this new associated sequence involves solving convex programming problems. It is also shown that the elements of both associated sequences are related by Lagrange duality. Two numerical examples, included in section V, illustrate some of the results reported in this work. Summary and concluding remarks are in section VI.

Most of the notation used through the paper is standard. $\mathbb{Z}^+$ denote the non-negative integers. For $k \in \mathbb{Z}^+$, we use $\mathbb{Z}^{[0,k]}$ to also denote the set $\mathbb{Z}^{[0,k]} = \{0, \dots, k\}$. We use l_+^n to denote the set of all the sequences $\{x_k\} \subset \mathbb{R}^n$, $k \in \mathbb{Z}^+$.

E-mail address: fnajson@fing.edu.uy

978-1-4244-2079-7/08/$25.00 ©2008 AACC.

For a matrix $X \in \mathbb{R}^{n \times n}$, $\rho(X)$ is its spectral radius.

II. Preliminaries

Let $N \in \mathbb{Z}^+$, $N > 0$, be given. We will denote by $\mathcal{Q}$ the set $\mathcal{Q} = \{1, \ldots, N\}$. Let us introduce the following sets of control functions (or switching signals)

$$\begin{aligned} \mathcal{Q}_k &= \{q : q : \mathbb{Z}^{[0,k-1]} \longrightarrow \mathcal{Q}\}, \ k \in \mathbb{Z}^+, \ k > 0, \\ \mathcal{Q}_\infty &= \{q : q : \mathbb{Z}^+ \longrightarrow \mathcal{Q}\}. \end{aligned}$$

Let the matrices $A_i \in \mathbb{R}^{n \times n}$, $i \in \{1, \ldots, N\}$, be given. The present article is concerned with the dynamical system described by

$$x(k+1) = A_{q(k)} x(k), \ k \in \mathbb{Z}^+, \ x(0) = x_0 \in \mathbb{R}^n, \ q \in \mathcal{Q}_\infty. \tag{1}$$

The motion of this controlled dynamical system will be denoted by $x(\cdot\,; x_0, q)$.

To each mapping $\kappa : \mathbb{R}^n \longrightarrow \mathcal{Q}$ we associate the diagonal (or static) operator $\mathcal{F}_\kappa : l_+^n \longrightarrow \mathcal{Q}_\infty$ defined by

$$\mathcal{F}_\kappa(x)(k) = \kappa(x(k)), \ k \in \mathbb{Z}^+.$$

It is clear that if we also associate to each mapping $\kappa : \mathbb{R}^n \longrightarrow \mathcal{Q}$ the (closed-loop) dynamical system described by

$$x_{cl}(k+1) = A_{\kappa(x_{cl}(k))} x_{cl}(k), \ k \in \mathbb{Z}^+, \ x_{cl}(0) = x_0 \in \mathbb{R}^n, \tag{2}$$

then, it follows that $x(\cdot\,; x_0, \mathcal{F}_\kappa(x)) = x_{cl}(\cdot\,; x_0)$.

In this work we will adopt the following definitions.

Definition 1: The switched system (1) is feedback exponentially stabilizable whenever there exist a mapping $\kappa : \mathbb{R}^n \longrightarrow \mathcal{Q}$ and scalars $\alpha \geq 1$ and $0 < \beta < 1$ such that the motions of the associated (closed-loop) dynamical system (2) satisfy

$$\|x_{cl}(k; x_0)\| \leq \alpha \beta^k \|x_0\|, \ k \in \mathbb{Z}^+, \ x_0 \in \mathbb{R}^n.$$

Definition 2: The switched system (1) is uniformly exponentially convergent whenever there exist scalars $\alpha \geq 1$ and $0 < \beta < 1$ that obey the following property:
For each $x_0 \in \mathbb{R}^n$ there exists $q_{x_0} \in \mathcal{Q}_\infty$ such that the corresponding motion of (1) satisfies

$$\|x(k; x_0, q_{x_0})\| \leq \alpha \beta^k \|x_0\|, \ k \in \mathbb{Z}^+.$$

III. Feedback Stabilizability of the Switched System

It is convenient to associate to the sets $\mathcal{Q}_k$, $k \in \mathbb{Z}^+$, $k > 0$, of control functions the following sets $\mathcal{S}_k$, $k \in \mathbb{Z}^+$, $k > 0$, of matrices:

$$\mathcal{S}_k = \{S \in \mathbb{R}^{n \times n} \, : \, S = A_{q(k-1)} \ldots A_{q(0)}, q \in \mathcal{Q}_k\}.$$

We associate, also, to the switched system (1), the sequence of functions $V_k : \mathbb{R}^n \longrightarrow \mathbb{R}^+$, $k \in \mathbb{Z}^+$, $k > 0$, defined by

$$V_k(x) = \min_{q \in \mathcal{Q}_k} \|x(k; x, q)\|^2 = \min_{S \in \mathcal{S}_k} x^* S^* S x, \tag{3}$$

and the sequence $\mu_k \in \mathbb{R}^+$, $k \in \mathbb{Z}^+$, $k > 0$, defined by

$$\mu_k = \max_{\|x\| \leq 1} V_k(x) = \max_{\|x\| \leq 1} \min_{q \in \mathcal{Q}_k} \|x(k; x, q)\|^2. \tag{4}$$

Some simple properties of $\{V_k\}$ and $\{\mu_k\}$ are collected in the next Fact.

Fact 1: For each given $k \in \mathbb{Z}^+$, $k > 0$, it follows that

(1) V_k is locally Lipschitz.
(2) $V_k(\lambda x) = \lambda^2 V_k(x), \ \lambda \in \mathbb{R}, \ x \in \mathbb{R}^n$.
(3) $\mu_k = \max_{\|x\|=1} V_k(x)$.
(4) $\mu_k \leq \min_{S \in \mathcal{S}_k} \|S\|^2$.
(5) For each given $h \in \mathbb{Z}^+$, $h > 0$, it follows that

$$\mu_{hk} \leq \big(\mu_k\big)^h.$$

Next, we present the main result of this section.

Theorem 1: The switched system (1) is feedback exponentially stabilizable if and only if, the associated sequence $\{\mu_k\}$ is such that, any (and then all) of the following equivalent conditions are satisfied:

(i) There exists $k_0 \in \mathbb{Z}^+$, $k_0 > 0$, such that $\mu_{k_0} < 1$.
(ii) $\lim_{k \to +\infty} \mu_k = 0$.

Proof: The proof is organized as follows. Next, we prove that condition (i) is equivalent to the feedback exponential stabilizability of the switched system (1). In order to prove the equivalence between conditions (i) and (ii), we just have to prove that (i) implies (ii). That proof is included (below) in the Sufficiency part.

(Necessity.) By assumption there exist a mapping $\kappa : \mathbb{R}^n \longrightarrow \mathcal{Q}$ and scalars $\alpha \geq 1$ and $0 < \beta < 1$ such that the motions of the associated system (2) satisfy

$$\|x_{cl}(k; x_0)\| \leq \alpha \beta^k \|x_0\|, \ k \in \mathbb{Z}^+, \ x_0 \in \mathbb{R}^n.$$

Choose $k_0 \in \mathbb{Z}^+$, $k_0 > 0$, such that $\alpha^2 \beta^{2k_0} < 1$. And define the following family of control functions:

$$q_{x_0} = \mathcal{F}_\kappa(x_{cl}(\cdot\,; x_0)), \ x_0 \in \mathbb{R}^n, \ \|x_0\| \leq 1.$$

Then, using the definition of V_{k_0}, we have that

$$\begin{aligned} & V_{k_0}(x_0) = \min_{q \in \mathcal{Q}_{k_0}} \|x(k_0; x_0, q)\|^2 \leq \\ & \|x(k_0; x_0, q_{x_0})\|^2 = \|x_{cl}(k_0; x_0)\|^2 \leq \\ & \alpha^2 \beta^{2k_0}, \ x_0 \in \mathbb{R}^n, \ \|x_0\| \leq 1. \end{aligned}$$

Therefore,

$$\mu_{k_0} = \max_{\|x_0\| \leq 1} V_{k_0}(x_0) \leq \alpha^2 \beta^{2k_0} < 1.$$

(Sufficiency.) By assumption there exists $k_0 \in \mathbb{Z}^+$, $k_0 > 0$, such that $\mu_{k_0} < 1$. It will be assumed, without loss of generality, that $k_0 > 1$. (Notice that in case that $k_0 = 1$ we can appeal to Fact 1 (property (5)) to define a new $k_0^{\text{new}} = hk_0$ with $h \in \mathbb{Z}^+$, $h > 1$. Thus, $k_0^{\text{new}} > 1$, and moreover $\mu_{k_0^{\text{new}}} \leq (\mu_{k_0})^h < 1$.) We choose $\gamma \in \mathbb{R}^+$ such that $\mu_{k_0} < \gamma < 1$.

For a given $x_0 \in \mathbb{R}^n$, we will consider the optimal control problem

$$\min_{q \in \mathcal{Q}_{k_0}} \|x(k_0; x_0, q)\|^2, \tag{5}$$

and we will use $\hat{q}_{k_0, x_0}$ to denote a solution for that problem. Therefore, for any given $x_0 \in \mathbb{R}^n$

$$\begin{aligned} & \|x(k_0; x_0, \hat{q}_{k_0, x_0})\|^2 = \min_{q \in \mathcal{Q}_{k_0}} \|x(k_0; x_0, q)\|^2 = \\ & V_{k_0}(x_0) \leq \mu_{k_0} \|x_0\|^2 \leq \gamma \|x_0\|^2. \end{aligned}$$

Let us define

$$M = \max\{1\,,\ \max_{S\in\bigcup_{j=1}^{k_0-1}\mathcal{S}_j} \|S\|^2\}\,.$$

Let $x_0 \in \mathbb{R}^n$ be given, and let $h \in \mathbb{Z}^+$, $h > 0$, be given. Let $\tilde{q}_{hk_0,x_0} \in \mathcal{Q}_{hk_0}$ be a control function made up by concatenating solutions of the optimal control problem (5) with the following initial conditions:

$$\hat{x}_0 = x_0\,,\ \hat{x}_1 = x(k_0;\hat{x}_0,\hat{q}_{k_0,\hat{x}_0})\,,\ldots\,,\\ \hat{x}_{h-1} = x(k_0;\hat{x}_{h-2},\hat{q}_{k_0,\hat{x}_{h-2}})\,.$$

That is, using the above notation, the control function $\tilde{q}_{hk_0,x_0}$ is defined by

$$\tilde{q}_{hk_0,x_0}(jk_0+i) = \hat{q}_{k_0,\hat{x}_j}(i)\,,\\ i\in\{0,\ldots,(k_0-1)\}\,,\ j\in\{0,\ldots,(h-1)\}\,.$$

Now, it is easy to see that, with the above defined control function $\tilde{q}_{hk_0,x_0}$ the following inequalities are satisfied:

$$\|x(k;x_0,\tilde{q}_{hk_0,x_0})\|^2 \le M\gamma^j\|x_0\|^2\,,\\ k\in\mathbb{Z}^+\,,\ k\in\{jk_0,\ldots,jk_0+(k_0-1)\}\,,\\ j\in\{0,\ldots,(h-1)\}\,.$$

It is then clear that the above expression implies that $\lim_{k\to+\infty}\mu_k = 0$. In effect, given $\epsilon > 0$ arbitrary, we choose $j_0 \in \mathbb{Z}^+$, $j_0 > 0$, such that $M\gamma^{j_0} < \epsilon$. Then, for any $k \in \mathbb{Z}^+$, $k \ge j_0k_0$, it is verified that (where we have chosen $h \in \mathbb{Z}^+$, $h > 0$, such that $k \le hk_0$; thus $j \ge j_0$)

$$\mu_k = \max_{\|z\|\le 1} V_k(z) = V_k(x_0) = \min_{q\in\mathcal{Q}_k}\|x(k;x_0,q)\|^2 \le\\ \|x(k;x_0,\tilde{q}_{hk_0,x_0})\|^2 \le M\gamma^j \le M\gamma^{j_0} < \epsilon\,,$$

where x_0 denotes an optimal solution of the problem $\max_{\|z\|\le1} V_k(z)$.

For each $k \in \mathbb{Z}^+$, $k > 0$, we now define the cost functional $J_k : \mathbb{R}^n \times \mathcal{Q}_k \longrightarrow \mathbb{R}^+$ by

$$J_k(x_0,q) = \sum_{i=0}^{k}\|x(i;x_0,q)\|^2\,.$$

For any given $x_0 \in \mathbb{R}^n$ we will consider the following family of optimal control problems (parameterized by $k \in \mathbb{Z}^+$, $k > 0$):

$$\min_{q\in\mathcal{Q}_k} J_k(x_0,q)\,,$$

and we will denote by $U_k(x_0)$ the optimal value of that problem. Therefore, for any given $x_0 \in \mathbb{R}^n$, we have that (for $h \in \mathbb{Z}^+$, $h > 0$, such that $k \le hk_0$)

$$U_k(x_0) = \min_{q\in\mathcal{Q}_k} J_k(x_0,q) \le \min_{q\in\mathcal{Q}_{hk_0}} J_{hk_0}(x_0,q) \le\\ J_{hk_0}(x_0,\tilde{q}_{hk_0,x_0}) = \sum_{i=0}^{hk_0}\|x(i;x_0,\tilde{q}_{hk_0,x_0})\|^2 \le\\ \sum_{j=0}^{h} k_0M\gamma^j\|x_0\|^2 \le k_0M\frac{1}{(1-\gamma)}\|x_0\|^2\,.$$

It was then proved that

$$\|x_0\|^2 \le U_k(x_0) \le k_0M\frac{1}{(1-\gamma)}\|x_0\|^2\,,\\ x_0\in\mathbb{R}^n\,,\ k\in\mathbb{Z}^+\,,\ k>0\,.$$

It is also easy to see that the following property is verified:

$$U_{k+1}(x_0) \ge U_k(x_0)\,,\ x_0\in\mathbb{R}^n\,,\ k\in\mathbb{Z}^+\,,\ k>0\,.$$

It then follows that, for each $x_0 \in \mathbb{R}^n$, the limite $\lim_{k\to+\infty} U_k(x_0)$ exists. That lead us to the introduction of the function $W : \mathbb{R}^n \longrightarrow \mathbb{R}^+$ defined by

$$W(x_0) = \lim_{k\to+\infty} U_k(x_0)$$

which satisfies

$$\|x_0\|^2 \le W(x_0) \le k_0M\frac{1}{(1-\gamma)}\|x_0\|^2\,,\ x_0\in\mathbb{R}^n\,,$$

and furthermore, since

$$\begin{aligned} U_{k+1}(x_0) &= \left(\|x_0\|^2 + \min_{q\in\mathcal{Q}} U_k(x(1;x_0,q))\right)\\ &= \left(\|x_0\|^2 + \min_{q\in\mathcal{Q}} U_k(A_qx_0)\right)\,,\\ &\quad x_0\in\mathbb{R}^n\,,\ k\in\mathbb{Z}^+\,,\ k>0\,,\end{aligned}$$

it then follows that W is a solution of the following dynamic programming equation:

$$W(x_0) = \|x_0\|^2 + \min_{q\in\mathcal{Q}} W(A_qx_0)\,,\ x_0\in\mathbb{R}^n\,.$$

Hence, there exists a mapping $\kappa : \mathbb{R}^n \longrightarrow \mathcal{Q}$ satisfying

$$\kappa(x_0) \in \arg\min_{q\in\mathcal{Q}} W(A_qx_0)\,,\ x_0\in\mathbb{R}^n\,.$$

Thus, it is verified that

$$W(A_{\kappa(x_0)}x_0) - W(x_0) = -\|x_0\|^2\,,\ x_0\in\mathbb{R}^n\,,$$

which means that W is a Lyapunov function for the exponential stability of the trivial solution of the associated closed-loop dynamical system (2). In effect, it is easy to verify that (with that feedback mapping) the motions of the associated closed-loop dynamical system (2) satisfy

$$\|x_{cl}(k;x_0)\| \le \alpha\beta^k\|x_0\|\,,\ k\in\mathbb{Z}^+\,,\ x_0\in\mathbb{R}^n\,,$$

with

$$\alpha = \sqrt{k_0M\frac{1}{(1-\gamma)}}\,,\ \beta = \sqrt{\frac{k_0M-(1-\gamma)}{k_0M}}\,,$$

which completes the proof of the Theorem. ■

The constructive proof of the previous Theorem provides us also with the following conclusion.

Proposition 1: For the switched system (1), under consideration, the following assertions are equivalent:

- (i) There exists $k_0 \in \mathbb{Z}^+$, $k_0 > 0$, such that $\mu_{k_0} < 1$.
- (ii) $\lim_{k\to+\infty}\mu_k = 0$.
- (iii) There exists a function $W : \mathbb{R}^n \longrightarrow \mathbb{R}^+$ satisfying
    - $W(\lambda x_0) = \lambda^2W(x_0)\,,\ \lambda\in\mathbb{R}\,,\ x_0\in\mathbb{R}^n$,
    - $\|x_0\|^2 \le W(x_0) \le \overline{\alpha}\|x_0\|^2\,,\ x_0\in\mathbb{R}^n$, for some $\overline{\alpha} > 1$,

which solves the following dynamic programming equation:

$$W(x_0) = \|x_0\|^2 + \min_{q\in\mathcal{Q}} W(A_q x_0)\,,\ x_0 \in \mathbb{R}^n\,. \quad (6)$$

Moreover, a function W as in (iii) defines a feedback mapping $\kappa : \mathbb{R}^n \longrightarrow \mathcal{Q}$ by

$$\kappa(x_0) \in \arg\min_{q\in\mathcal{Q}} W(A_q x_0)\,,\ x_0 \in \mathbb{R}^n\,.$$

Such a feedback exponentially stabilizes the switched system (1). And the function W, is a Lyapunov function for the exponential stability of the trivial solution of the associated closed-loop dynamical system (2).

Remark 1: It follows from the previous result that (in case the switched system (1) is feedback exponentially stabilizable) a exponentially stabilizing feedback mapping $\kappa : \mathbb{R}^n \longrightarrow \mathcal{Q}$ can always be chosen with the following property:

$$\kappa(\lambda x_0) = \kappa(x_0)\,,\ \lambda \in \mathbb{R}\,,\ x_0 \in \mathbb{R}^n\,.$$

We observe that our proof of Theorem 1 also proves that: The switched system (1) is uniformly exponentially convergent if and only if there exists $k_0 \in \mathbb{Z}^+$, $k_0 > 0$, such that $\mu_{k_0} < 1$. (That particular result has already been reported in [10].) We have therefore the following conclusion from Theorem 1.

Corollary 1: The switched system (1) is feedback exponentially stabilizable if and only if it is uniformly exponentially convergent.

IV. Determination of the Feedback Exponential Stabilizability of a Switched System

This section is devoted to the problem of testing the property, of being feedback exponentially stabilizable, in the switched system (1) under consideration. With that purpose in mind, instead of focusing on the solvability of the dynamic programming equation (6), in the present work we concentrate our attention on the convergency property of the associated sequence $\{\mu_k\}$. Thus, in order to determine the feedback stabilizability of the switched system, we have to address the issue of computing μ_k, $k \in \mathbb{Z}^+$, $k > 0$, or that of computing an upper bound for μ_k. Regarding that issue we first have the following result.

Theorem 2: Consider the switched system (1) and its associated sequence $\{\mu_k\}$. Let $k \in \mathbb{Z}^+$, $k > 0$, be given. Assume that, for this given k, the set $\mathcal{S}_k$ is such that $0 \notin \mathcal{S}_k$.

Then, under that condition, the scalars $\nu, \lambda \in \mathbb{R}^+$, are related by

$$\lambda = \max_{\|x\|\le\sqrt{\nu}} V_k(x) = \mu_k\,\nu\,, \quad (7)$$

if and only if

$$\nu = \min_{X\in\widehat{\mathcal{P}}_k(\lambda)} \mathrm{trace}(X)\,, \quad (8)$$

where

$$\widehat{\mathcal{P}}_k(\lambda) = \{X \in \mathbb{R}^{n\times n} : X = X^*,\ X \ge 0,\ \mathrm{rank}(X) \le 1,\ \mathrm{trace}(S^*SX) \ge \lambda\,\forall S \in \mathcal{S}_k\}\,.$$

Proof: (Necessity.) Consider the optimization problem in (8):

$$\min_{X\in\widehat{\mathcal{P}}_k(\lambda)} \mathrm{trace}(X)\,.$$

It is clearly equivalent to the following one:

$$\min_{x\in\{x\in\mathbb{R}^n:\,\|Sx\|^2\ge\lambda\,\forall S\in\mathcal{S}_k\}} \|x\|^2\,.$$

Regarding that optimization problem we first make the following two important remarks.

- Since by assumption $0 \notin \mathcal{S}_k$ then,
$$\{x \in \mathbb{R}^n : \|Sx\|^2 \ge \lambda\,\forall S \in \mathcal{S}_k\} \ne \emptyset\,,\ \forall\lambda \in \mathbb{R}^+\,.$$
- It is clear that, in the above optimization problem, the minimization is indeed achieved. That explains the use of $\min$ instead of $\inf$.

Next, in this part of the proof, we will consider two cases.

<u>Case $\lambda = 0$.</u> When $\lambda = 0$ then

$$0 \in \{x \in \mathbb{R}^n : \|Sx\|^2 \ge \lambda\,\forall S \in \mathcal{S}_k\}\,,$$

which means that

$$\nu = \min_{x\in\{x\in\mathbb{R}^n:\,\|Sx\|^2\ge\lambda\,\forall S\in\mathcal{S}_k\}} \|x\|^2 = 0\,.$$

Hence, in this case, the relationship $\lambda = \mu_k\,\nu$ is verified.

<u>Case $\lambda > 0$.</u> Since ν is defined by

$$\nu = \min_{x\in\{x\in\mathbb{R}^n:\,\|Sx\|^2\ge\lambda\,\forall S\in\mathcal{S}_k\}} \|x\|^2 \quad (9)$$

then it follows that $\nu > 0$. Thus, by definition of ν, we have that:

$$\text{If } x \in \{x \in \mathbb{R}^n : \|Sx\|^2 \ge \lambda\,\forall S \in \mathcal{S}_k\} \implies \|x\|^2 \ge \nu\,.$$

Therefore,

$$\text{If } \|x\|^2 < \nu \implies \exists S \in \mathcal{S}_k : \|Sx\|^2 < \lambda \implies V_k(x) < \lambda\,.$$

Hence, it follows that

$$\max_{\|x\|\le\sqrt{\nu}} V_k(x) \le \lambda\,.$$

From the optimization problem (9) we have that

$$\exists x_0 \in \mathbb{R}^n : \nu = \|x_0\|^2, \text{and } \|Sx_0\|^2 \ge \lambda\,\forall S \in \mathcal{S}_k\,.$$

Moreover, it is easy to see that

$$\exists S \in \mathcal{S}_k : \|Sx_0\|^2 = \lambda\,.$$

Then, it is concluded that $V_k(x_0) = \lambda$. It was therefore proved that

$$\max_{\|x\|\le\sqrt{\nu}} V_k(x) = \lambda\,.$$

(Sufficiency.) In this part of the proof, we will again consider two cases.

<u>Case $\nu = 0$.</u> In this case it is obvious that

$$\lambda = \max_{\|x\|\le\sqrt{\nu}} V_k(x) = V_k(0) = 0\,.$$

Then, it is clear that in this case, the following relationship holds:

$$\nu = \min_{x\in\{x\in\mathbb{R}^n:\,\|Sx\|^2\ge\lambda\,\forall S\in\mathcal{S}_k\}} \|x\|^2\,.$$

Case $\nu > 0$. Since λ is defined via

$$\lambda = \max_{\|x\| \le \sqrt{\nu}} V_k(x) \tag{10}$$

then it follows that $\lambda > 0$. (Otherwise, if it were $\lambda = 0$, then $V_k(x) = 0 \, \forall x : \|x\|^2 \le \nu$, implying that $V_k(x) = 0 \, \forall x \in \mathbb{R}^n$. But, that would be a contradiction since by assumption $0 \notin \mathcal{S}_k$.) By definition of λ we have that

$$\text{If } \|x\|^2 \le \nu \implies \exists S \in \mathcal{S}_k : \|Sx\|^2 \le \lambda \, .$$

Hence,

$$\text{If } x \in \{x \in \mathbb{R}^n : \|Sx\|^2 > \lambda \, \forall S \in \mathcal{S}_k\} \implies \|x\|^2 > \nu \, .$$

Therefore,

$$\min_{x \in \{x \in \mathbb{R}^n : \|Sx\|^2 \ge \lambda \, \forall S \in \mathcal{S}_k\}} \|x\|^2 \ge \nu \, .$$

Now, from the optimization problem (10), we conclude that

$$\exists x_0 \in \mathbb{R}^n : \|x_0\|^2 \le \nu, \text{ and } V_k(x_0) = \lambda \, ,$$

implying that

$$\|x_0\|^2 \le \nu \text{ and } x_0 \in \{x \in \mathbb{R}^n : \|Sx\|^2 \ge \lambda \, \forall S \in \mathcal{S}_k\} \, .$$

Hence, it is concluded that

$$\min_{x \in \{x \in \mathbb{R}^n : \|Sx\|^2 \ge \lambda \, \forall S \in \mathcal{S}_k\}} \|x\|^2 = \nu \, ,$$

which completes the proof of the Theorem. ■

Thus, suggested by Theorem 2, we associate to the switched system (1), the sequence $\nu_k \in (\mathbb{R}^{(\text{ext})})^+$, $k \in \mathbb{Z}^+$, $k > 0$, defined by

$$\nu_k = \begin{cases} +\infty & , 0 \in \mathcal{S}_k \\ \min_{X \in \widehat{\mathcal{P}}_k(1)} \operatorname{trace}(X) & , 0 \notin \mathcal{S}_k \end{cases} \, . \tag{11}$$

We include, in the next two corollaries, some straightforward conclusions that follow from Theorem 2 and Theorem 1.

Corollary 2: Consider the switched system (1) and their associated sequences $\{\mu_k\}$ and $\{\nu_k\}$. These sequences are related by

$$\nu_k = \frac{1}{\mu_k} \, , \; k \in \mathbb{Z}^+ \, , \; k > 0 \, .$$

Corollary 3: The switched system (1) is feedback exponentially stabilizable if and only if, the associated sequence $\{\nu_k\}$ is such that, any (and then all) of the following equivalent conditions are satisfied:

(i) There exists $k_0 \in \mathbb{Z}^+$, $k_0 > 0$, such that $\nu_{k_0} > 1$.

(ii) $\lim_{k \to +\infty} \nu_k = +\infty$.

Next, we consider some convex optimization problems which provide with lower bounds for the elements of the sequence $\{\nu_k\}$ and therefore convey to sufficient conditions for the feedback exponential stabilization of the switched system (1). Let us introduce the sequences $\omega_k \in (\mathbb{R}^{(\text{ext})})^+$, $\underline{\omega}_k \in (\mathbb{R}^{(\text{ext})})^+$, $k \in \mathbb{Z}^+$, $k > 0$, associated to the switched system (1), via the following convex optimization problems:

$$\omega_k = \begin{cases} +\infty & , 0 \in \mathcal{S}_k \\ \min_{X \in \mathcal{P}_k(1)} \operatorname{trace}(X) & , 0 \notin \mathcal{S}_k \end{cases} \, , \tag{12}$$

where

$$\begin{aligned} \mathcal{P}_k(\lambda) &= \{X \in \mathbb{R}^{n \times n} : X = X^*, \, X \ge 0, \\ &\quad \operatorname{trace}(S^* S X) \ge \lambda \, \forall S \in \mathcal{S}_k\} \, , \; \lambda \in \mathbb{R}^+ \, , \end{aligned}$$

$$\underline{\omega}_k = \begin{cases} +\infty & , 0 \in \mathcal{S}_k \\ \max_{v \in \mathcal{R}_k} \sum_{q \in \mathcal{Q}_k} v_q & , 0 \notin \mathcal{S}_k \end{cases} \, , \tag{13}$$

where

$$\mathcal{R}_k = \{v \in (\mathbb{R}^{N^k})^+ : (\sum_{q \in \mathcal{Q}_k} v_q S_q^* S_q) \le I\} \, ,$$

and where, in the last optimization problem, we have used $q \in \mathcal{Q}_k$ to index the corresponding members of $\mathcal{S}_k$ and also the components of the vector $v \in (\mathbb{R}^{N^k})^+$.

Proposition 2: Consider the switched system (1) and their associated sequences $\{\nu_k\}$, $\{\omega_k\}$, and $\{\underline{\omega}_k\}$. The elements of these sequences obey the following relationship:

$$\nu_k \ge \omega_k = \underline{\omega}_k \, , \; k \in \mathbb{Z}^+ \, , \; k > 0 \, .$$

Therefore, if there is a $k_0 \in \mathbb{Z}^+$, $k_0 > 0$, with the property that $\underline{\omega}_{k_0} > 1$, then the switched system (1) is feedback exponentially stabilizable.

Proof: Let $k \in \mathbb{Z}^+$, $k > 0$, be given such that $0 \notin \mathcal{S}_k$. Since

$$\widehat{\mathcal{P}}_k(1) \subset \mathcal{P}_k(1) \, ,$$

then, $\nu_k \ge \omega_k$. Next, let X_0 and v_0 be optimal solutions for the corresponding optimization problems (12) and (13) respectively. Then,

$$\begin{aligned} \omega_k = \operatorname{trace}(X_0) &\ge \operatorname{trace}((\sum_{q \in \mathcal{Q}_k} v_{0,q} S_q^* S_q) X_0) = \\ \sum_{q \in \mathcal{Q}_k} v_{0,q} \operatorname{trace}(S_q^* S_q X_0) &\ge \sum_{q \in \mathcal{Q}_k} v_{0,q} = \underline{\omega}_k \, . \end{aligned}$$

In order to prove that the strict equality, $\omega_k = \underline{\omega}_k$ holds, we use the fact that the optimization problem in (13) is the Lagrange dual of the optimization problem in (12). In effect, the Lagrangian [1] associated to the optimization problem in (12) is the function

$$\mathcal{L}(X, \Lambda, \lambda) = \operatorname{trace}\big((I - \Lambda - \sum_{q \in \mathcal{Q}_k} \lambda_q S_q^* S_q) X\big) + \sum_{q \in \mathcal{Q}_k} \lambda_q \, ,$$

$$\begin{aligned} &X \in \mathbb{R}^{n \times n} : X = X^* \, , \\ &\Lambda \in \mathbb{R}^{n \times n} : \Lambda = \Lambda^* \, , \, \Lambda \ge 0 \, , \; \lambda \in (\mathbb{R}^{N^k})^+ \, , \end{aligned}$$

and the Lagrange dual function is

$$\begin{aligned} &g(\Lambda, \lambda) = \inf_{X \in \{X \in \mathbb{R}^{n \times n} : X = X^*\}} \mathcal{L}(X, \Lambda, \lambda) = \\ &\begin{cases} \sum_{q \in \mathcal{Q}_k} \lambda_q \, , & (I - \Lambda - \sum_{q \in \mathcal{Q}_k} \lambda_q S_q^* S_q) = 0 \\ -\infty \, , & (I - \Lambda - \sum_{q \in \mathcal{Q}_k} \lambda_q S_q^* S_q) \ne 0 \end{cases} \, , \\ &\Lambda \in \mathbb{R}^{n \times n} : \Lambda = \Lambda^* \, , \, \Lambda \ge 0 \, , \; \lambda \in (\mathbb{R}^{N^k})^+ \, . \end{aligned}$$

Therefore, the Lagrange dual optimization problem is

$$\begin{aligned} &\sup_{(\Lambda, \lambda) \in \{(\Lambda, \lambda) : \Lambda \in \mathbb{R}^{n \times n}, \, \Lambda = \Lambda^*, \, \Lambda \ge 0 \, , \; \lambda \in (\mathbb{R}^{N^k})^+\}} g(\Lambda, \lambda) = \\ &\max_{\lambda \in \{\lambda \in (\mathbb{R}^{N^k})^+ : (I - \sum_{q \in \mathcal{Q}_k} \lambda_q S_q^* S_q) \ge 0\}} \sum_{q \in \mathcal{Q}_k} \lambda_q \, , \end{aligned}$$

which is the optimization problem in (13). (From this, it immediately follows that $\omega_k \geq \underline{\omega}_k$ which was already proved using different arguments.) Now, since $0 \notin \mathcal{S}_k$, we claim that

$$\exists X > 0 \,:\, \text{trace}(S^* S X) > 1 \,\forall S \in \mathcal{S}_k \,. \qquad (14)$$

The above claim follows from the fact that the underlying assumption ensures the existence of

$$x_0 \in \mathbb{R}^n \,:\, \|S x_0\| > 1 \,\forall S \in \mathcal{S}_k \,.$$

Notice that, if the last assertion were not true then it would imply that $\cup_{S \in \mathcal{S}_k} \mathcal{N}(S) = \mathbb{R}^n$. (We use $\mathcal{N}(S)$ to denote the null-space of S.) But, this can not be since, by assumption, $\dim(\mathcal{N}(S)) < n \,\forall S \in \mathcal{S}_k$. Since the condition (14), namely the Slater condition for the (primal) optimization problem in (12), holds; and moreover the (primal) optimization problem in (12) is convex, it then follows [1] that strong duality is achieved. That is, we have established that $\omega_k = \underline{\omega}_k$. ■

Remark 2: It is important to remark that the convex optimization problem in (13) is also the Lagrange dual of the (non-convex) optimization problem

$$\min_{x \in \{x \in \mathbb{R}^n : \, \|Sx\|^2 \geq 1 \,\forall S \in \mathcal{S}_k\}} \|x\|^2 \,, \qquad (15)$$

which is equivalent to the optimization problem in (11). In effect, the Lagrangian associated to the optimization problem in (15) is the function

$$\mathcal{L}(x, \lambda) \;=\; x^*(I - \sum_{q \in \mathcal{Q}_k} \lambda_q S_q^* S_q) x + \sum_{q \in \mathcal{Q}_k} \lambda_q \,, \quad x \in \mathbb{R}^n \,,\; \lambda \in (\mathbb{R}^{N^k})^+ \,,$$

and the Lagrange dual function is

$$g(\lambda) = \inf_{x \in \mathbb{R}^n} \mathcal{L}(x, \lambda) = \begin{cases} \sum_{q \in \mathcal{Q}_k} \lambda_q \,, & (I - \sum_{q \in \mathcal{Q}_k} \lambda_q S_q^* S_q) \geq 0 \\ -\infty \,, & (I - \sum_{q \in \mathcal{Q}_k} \lambda_q S_q^* S_q) \ngeq 0 \end{cases} \,, \quad \lambda \in (\mathbb{R}^{N^k})^+ \,.$$

Therefore, the Lagrange dual optimization problem is

$$\sup_{\lambda \in (\mathbb{R}^{N^k})^+} g(\lambda) = \max_{\lambda \in \{\lambda \in (\mathbb{R}^{N^k})^+ : \, (I - \sum_{q \in \mathcal{Q}_k} \lambda_q S_q^* S_q) \geq 0\}} \sum_{q \in \mathcal{Q}_k} \lambda_q \,,$$

which is the optimization problem in (13). Thus, the difference, $(\nu_k - \underline{\omega}_k)$, is the Lagrange duality gap associated to the (non-convex) optimization problem (15).

The next two results are aimed at shredding light on the degree of conservativeness of the test for exponential feedback stabilizability of the switched system (1) by means of the sequence $\{\underline{\omega}_k\}$.

Proposition 3: Consider the switched system (1) and their associated sequence $\{\underline{\omega}_k\}$. If there is $k \in \mathbb{Z}^+$, $k > 0$, such that

$$\min_{S \in \mathcal{S}_k} \rho(S) < 1 \,,$$

then, there exists $k_0 \in \mathbb{Z}^+$, $k_0 > 0$, with the property that $\underline{\omega}_{k_0} > 1$.

Proof: If there is $k \in \mathbb{Z}^+$, $k > 0$, such that $\min_{S \in \mathcal{S}_k} \rho(S) < 1$, then, there exists $k_0 \in \mathbb{Z}^+$, $k_0 > 0$, such that $\min_{S \in \mathcal{S}_{k_0}} \|S\|^2 < 1$. Since, for every $h \in \mathbb{Z}^+$, $h > 0$, it is verified that

$$\min_{v \in \{v \in (\mathbb{R}^{N^h})^+ : \, \sum_{q \in \mathcal{Q}_h} v_q = 1\}} \| \sum_{q \in \mathcal{Q}_h} v_q S_q^* S_q \| \;=\; \frac{1}{\underline{\omega}_h} \,,$$

then,

$$\frac{1}{\underline{\omega}_{k_0}} \;=\; \min_{v \in \{v \in (\mathbb{R}^{N^{k_0}})^+ : \, \sum_{q \in \mathcal{Q}_{k_0}} v_q = 1\}} \| \sum_{q \in \mathcal{Q}_{k_0}} v_q S_q^* S_q \| \;\leq\; \min_{S \in \mathcal{S}_{k_0}} \|S\|^2 < 1 \,.$$

■

Proposition 4: Consider the switched system (1) and their associated sequence $\{\underline{\omega}_k\}$. If there exists $k_0 \in \mathbb{Z}^+$, $k_0 > 0$, with the property that $\underline{\omega}_{k_0} > 1$, then, there exists $q_0 \in \mathcal{Q}$ satisfying $|\det(A_{q_0})| < 1$, implying that

$$\min_{q \in \mathcal{Q}} \min_{i \in \{1, \ldots, n\}} |\lambda_i(A_q)| < 1 \,.$$

Proof: If there exists $k_0 \in \mathbb{Z}^+$, $k_0 > 0$, with the property that $\underline{\omega}_{k_0} > 1$, then, there exists $v \in (\mathbb{R}^{N^{k_0}})^+$: $\sum_{q \in \mathcal{Q}_{k_0}} v_q = 1$ and such that $\sum_{q \in \mathcal{Q}_{k_0}} v_q S_q^* S_q < I$.

We will assume, in what follows, that all the matrices A_q, $q \in \mathcal{Q}$, are nonsingular. Otherwise, the proof is complete. Under that assumption, it follows from the above matrix inequality that

$$\log\det\big(\sum_{q \in \mathcal{Q}_{k_0}} v_q S_q^* S_q \big) < 0 \,.$$

Since the function $\log\det(\cdot)$ is concave (see e.g. [1]) on the positive definite convex cone, we get that

$$\sum_{q \in \mathcal{Q}_{k_0}} v_q \log\det\big(S_q^* S_q\big) \leq \log\det\big(\sum_{q \in \mathcal{Q}_{k_0}} v_q S_q^* S_q \big) < 0 \,,$$

implying that there exits $S \in \mathcal{S}_{k_0}$: $\det(S^* S) < 1$. Hence, there exists $q_0 \in \mathcal{Q}$ satisfying $|\det(A_{q_0})| < 1$. ■

V. Numerical Examples

Two numerical examples are presented in this section in order to illustrate some of the results reported in the paper.

Example 1: Consider the switched system (1) defined by $N = 2$ and $A_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1.25 & 0.45 \\ -0.45 & 0.825 \end{pmatrix}$.

For these matrices, we have that, $|\lambda_1(A_1)| = |\lambda_2(A_1)| = 1$ and $|\lambda_1(A_2)| = |\lambda_2(A_2)| \approx 1.1107$. Therefore it follows from Proposition 4 that the associate sequence $\{\underline{\omega}_k\}$ satisfies $\underline{\omega}_k \leq 1$, $\forall k$. Hence, we cannot conclude on the feedback exponential stabilizability of this switched system by means of $\{\underline{\omega}_k\}$. Due to the low dimensionality of this system, it is possible to perform without difficulty direct numerical computations in order to evaluate some of the elements of the sequence $\{\mu_k\}$. Thus, as a result of these computations, we got that $\mu_8 \approx 0.4952$. Then, using Theorem 1, we arrive to the conclusion that the switched system is feedback exponentially stabilizable. The low dimensionality of this system also allows us to easily compute a solution W for the dynamic

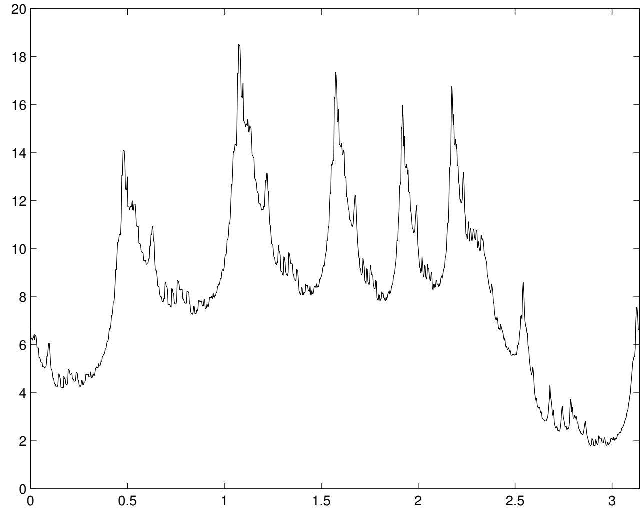

Fig. 1. Graphical representation of W for the system in Example 1.

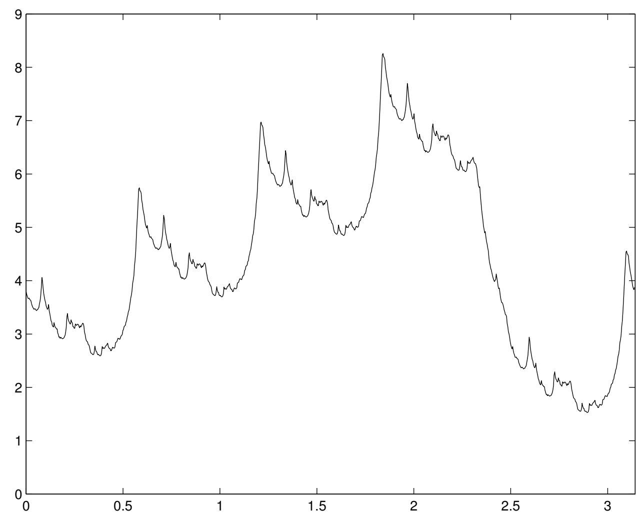

Fig. 2. Graphical representation of W for the system in Example 2.

programming equation (6). The graphical representation of the computed $W(\begin{pmatrix}\cos(\cdot)\\ \sin(\cdot)\end{pmatrix}) : [0,\pi) \longrightarrow \mathbb{R}^+$ is in Figure 1.

Example 2: Assume the switched system (1) is defined by $N = 2$ and $A_1 = \begin{pmatrix}1 & 2\\ 0 & 0.95\end{pmatrix}$, $A_2 = \begin{pmatrix}0.825 & 0.6\\ -0.6 & 0.825\end{pmatrix}$.

Here we have that, $|\lambda_1(A_1)| = 1$, $|\lambda_2(A_1)| = 0.95$ (and $|\lambda_1(A_2)| = |\lambda_2(A_2)| \approx 1.02$). We attempted, in this case, to test the feedback exponential stabilizability of this switched system by computing some of the elements of the associated sequence $\{\underline{\omega}_k\}$. We got that $\underline{\omega}_8 \approx 1.0031$, therefore, invoking Proposition 2, it is concluded that the switched system is feedback exponentially stabilizable. (By direct numerical computation of μ_8 we got $\mu_8 \approx 0.2309$.) As in the previous example, a solution W for the dynamic programming equation (6) was also computed here. The graphical representation of this computed Lyapunov function $W(\begin{pmatrix}\cos(\cdot)\\ \sin(\cdot)\end{pmatrix}) : [0,\pi) \longrightarrow \mathbb{R}^+$ is shown in Figure 2.

VI. Summary and Concluding Remarks

We have proved that a discrete-time switched linear system is state-feedback exponentially stabilizable if and only if its associate sequence $\{\mu_k\}$ converges to zero; or equivalently, if and only if there is $k_0 \in \mathbb{Z}^+$, $k_0 > 0$, with the property that $\mu_{k_0} < 1$. It was also shown that a switched linear system is state-feedback exponentially stabilizable if and only if a dynamic programming equation has a solution W of a particular class. Such a solution, W, defines (that is, provides us with) a stabilizing state-feedback mapping $\kappa : \mathbb{R}^n \longrightarrow \mathcal{Q}$ via $\kappa(x) \in \arg\min_{q\in\mathcal{Q}} W(A_q x)$. That function W, is a Lyapunov function for the exponential stability of the trivial solution of the associated closed-loop switched system. It was also proved that a switched system is state-feedback exponentially stabilizable if and only if it is uniformly exponentially convergent (in the sense of our Definition 2). Instead of focusing on the solvability of the aforementioned dynamic programming equation, we have chosen, in this work, to focus on using the convergency property of the associate sequence $\{\mu_k\}$ in order to determine (or to test) the stabilizability of the switched system. We have therefore studied, and proposed the use of, another associated sequence $\{\underline{\omega}_k\}$ having the following two properties: (1) The computation of each element of that sequence can be performed by solving a convex programming problem. (2) It is verified that $\mu_k \leq \frac{1}{\underline{\omega}_k}$, $k \in \mathbb{Z}^+$. We have also shown, as explained in Remark 2, that the elements of both sequences, $\{\mu_k\}$ and $\{\underline{\omega}_k\}$, are further related via Lagrange duality. Results were presented, in Propositions 3 and 4, which provide with information on the degree of conservativeness of the test for stabilizability of the switched system by means of the sequence $\{\underline{\omega}_k\}$. Further work is still needed in order to fully understand how conservative is that test (based on the sequence $\{\underline{\omega}_k\}$). Two numerical examples were presented to illustrate the results reported in this work.

References

[1] S. Boyd, L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004.

[2] R. DeCarlo, M. Branicky, S. Pettersson, B. Lennartson, Perspectives and Results on the Stability and Stabilizability of Hybrid Systems, *Proceedings of the IEEE*, 88(7):1069-1082, 2000.

[3] J.P. Hespanha, A.S. Morse, Stabilization of Nonholonomic Integrators via Logic-Based Switching, *Automatica*, 35(3):385-393, 1999.

[4] J.P. Hespanha, D. Liberzon, A.S. Morse, Overcoming the Limitations of Adaptive Control by Means of Logic-Based Switching, *Systems and Control Letters*, 49(1):49-65, 2003.

[5] D. Liberzon, A.S. Morse, Basic Problems in Stability and Design of Switched Systems, *IEEE Contr. Syst. Magazine*, 19(5):59-70, 1999.

[6] D. Liberzon, *Switching in Systems and Control*, Birkhäuser, 2003.

[7] F. Najson, A Stochastic Approach for Fault-Tolerant Control of Linear Systems, In *Proc. of the 2006 American Control Conf.*, 2006.

[8] A.V. Savkin, R.J. Evans, *Hybrid Dynamical Systems: Controller and Sensor Switching Problems*, Birkhäuser, 2002.

[9] R.N. Shorten, F. Wirth, O. Mason, K. Wulff, C. King, Stability Criteria for Switched and Hybrid Systems, *SIAM Review*, 49(4):545-592, 2007.

[10] Z. Sun, S.S. Ge, *Switched Linear Systems: Control and Design*, Springer-Verlag, 2005.

[11] A. van der Schaft, H. Schumacher, *An Introduction to Hybrid Dynamical Systems*, Springer-Verlag, 2000.

[12] J.G. Kassakian, M.F. Schlecht, G.C. Verghese *Principles of Power Electronics*, Addison-Wesley, 1991.