2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Event-Triggered Broadcasting across Distributed Networked Control Systems

Xiaofeng Wang and Michael D. Lemmon

Abstract— **This paper examines event-triggered broadcasting of state information in distributed control systems implemented over wireless communication networks. Event-triggering requires a subsystem to only broadcast its state information when the local state error exceeds a given threshold. The paper designs an event triggering scheme that assures asymptotic stability of the entire networked system. The results apply to networks of linear time-invariant systems. We derive a lower bound on the estimated time to next broadcast. Simulation results show that event triggering allows a subsystem to adjust its broadcast periods to the amount of activity in its immediate neighborhood. Simulation results also show that event-triggering's average broadcast period scales well with system size. These results are significant because they show how one might stabilize distributed control systems over ad hoc wireless networks without necessarily requiring a high degree of synchronization within the communication network.**

I. INTRODUCTION

A networked dynamical system consists of numerous loosely coupled systems. These networked systems are found throughout our national infrastructure with specific examples being the electrical power grid and transportation networks. In recent years, it has become popular to refer to such networked systems as *cyber-physical systems* [1]. Increased demands on such infrastructure due to demographic shifts and greater regulatory burdens have made it increasingly difficult to reliably manage these networks in a cost effective manner. There is, therefore, a compelling national need to develop more robust and cost effective methods for controlling such networked systems.

It is impractical to control such large-scale systems in a centralized manner. Centralized control algorithms would require state information from all subsystems before computing the control action. This centralization requires a very powerful communication network to transport state information in a timely manner and it requires extremely detailed models of subsystem interactions. Both of these requirements can greatly limit the scalability of centralized approaches to networked control systems.

An alternative way is to use a distributed approach, where a given subsystem uses its state and the states of its immediate neighbors to determine its control action. In [2] it was shown that optimal controllers with a quadratic objective possess an inherent degree of spatial localization. This suggests that it should be possible to effectively regulate the behavior of distributed systems using local interactions between spatially adjacent subsystems. One approach to distributed control builds upon model predictive control [3], [4]. Significant progress was made toward this goal in an approach that modelled system coupling using linear fractional transformations [5] [6]. More recent work has used integrator backstepping to extend this approach to networks of nonlinear systems [7].

One thing worth mentioning is that in all of this prior work, it is assumed that subsystem controllers can communicate with their neighbors at will. In practice, however, communication (especially wireless communication) takes place over digital networks so that information is transmitted at discrete time instants rather than continuous-time. Moreover such wireless networks have a bandwidth limitation that delay message delivery in a way that may adversely impact overall system stability [8]. The preceding distributed approaches may therefore be inappropriate in controlling distributed systems over real wireless communication networks.

For these reasons, some researchers have begun investigating the timing issue in networked control. Briefly stated, this issue concerns how frequently subsystems should communicate with each other to assure a desired level of system performance. One approach was presented in [9], where one first designs the controllers under the assumption of perfect communication and then determines the maximum allowable transfer interval (MATI) under some assumed communication protocol. This work led to scheduling methods [10] that were able to assure the MATI was not violated. Further work was done in [11], [12] to ensure input-output stability of the system.

In all of this prior work however, the computation of the MATI and the execution of communication protocols must be done in a highly centralized manner. Such centralization is impractical in large-scaled systems. Moreover because MATI is computed before the system is deployed, it must ensure adequate behavior over a wide range of possible input disturbances. As a result, the computed MATI may be conservative. This conservative estimate of the MATI results in the communication network's bandwidth being greater than might be necessary. These limitations suggest a great need for distributed approaches that address this timing issue in way that enables the networked control system to use network bandwidth in a extremely frugal manner.

This paper presents such a distributed approach through the use of a decentralized event-triggered feedback scheme, where a subsystem broadcasts its state information only when "needed". In this case "needed" means that some measure of the agent's local state error exceeds a specified threshold. The "online" nature of event-triggering makes it possible to reduce the frequency with which agents communicate

Both authors are with the department of Electrical Engineering, Univ. of Notre Dame, Notre Dame, IN 46556; e-mail: xwang13, lemmon@nd.edu. The authors gratefully acknowledge the partial financial support of the National Science Foundation (NSF-CNS-07-20457)

and therefore reduces the bandwidth requirements on the network. Event-triggering was originally proposed in [13] and has appeared under a number of names that include interrupt-based feedback [14], Lebesgue sampling [15], state-triggered feedback [16], and self-triggered feedback [17]. All of this prior work, however, has focused on using event-triggered feedback in single processor real-time systems.

A Lyapunov analysis similar to that suggested by Tabuada et al. [16] is used in this paper to design decentralized event triggering rules that allow agents to adapt their broadcast periods to the current activity level in the system. We show that such a design can be done in a distributed manner. The resulting decentralized event-triggering rule based only on the subsystem's local state error guarantees the asymptotic stability for the entire group. The analysis is valid for linear time-invariant subsystems that have full access to their local state. We establish bounds on the "time to next broadcast". We use simulation results to examine the scheme's scalability and to study how the approach adapts the broadcast rate to variations in a subsystem's external disturbance environment.

This paper is organized as follows. In section II, the problem is formulated. Section III presents the decentralized event-triggering scheme. An estimate of broadcast period for subsystems is provided in section IV. Simulation results are presented in section V. Finally, the results are summarized in section VI.

II. Problem Statement

This section formally presents the assumed system model and establishes some of the necessary mathematical notation.

Notational Conventions: If $V : \Re^n \to \Re$ is a function then its directional derivative with respect to the differential equation $\dot{x} = f(x)$ is $L_f V = \frac{\partial V}{\partial x} f(x)$. If $x \in \Re^n$, then we let $\|x\|_2$ denote the Euclidean 2-norm of this vector. If $A \in \Re^{n\times m}$ is a real matrix we let $\|A\|$ denote the matrix gain induced with respect to the Euclidean 2-norm. We let $\mathcal{N}$ denote the set $\{1, 2, \cdots, N\}$ of N integers and we let $|\mathcal{N}|$ denote the number of elements in that set.

The system under study is a group of N linear time-invariant systems. The local state of the ith subsystem (also called an *agent*) is a function $x_i : \Re \to \Re^{n_i}$ where n_i is the local state space dimension and $i \in \mathcal{N}$. This function satisfies the linear differential equation

$$\begin{aligned} \dot{x}_i(t) &= A_i x_i(t) + B_i u_i(t) + \textstyle\sum_{j\in\mathcal{N}_i} H_{ij} x_j(t) \\ x_i(0) &= x_{i0} \end{aligned} \quad (1)$$

where $x_{i0} \in \Re^{n_i}$ is the initial state and $\mathcal{N}_i \subseteq \mathcal{N}$ is the set of *neighbors* for agent i. We assume that the neighborhood sets $\mathcal{N}_i$ are such that $i \notin \mathcal{N}_i$. We further assume that being in a neighborhood is a symmetric relation in the sense that $j \in \mathcal{N}_i$ if and only if $i \in \mathcal{N}_j$. The signal $u_i : \Re \to \Re^{m_i}$ is the local control signal generated by agent i's controller where m_i is the dimension of the control set. In this paper, we assume agent i can receive information from agents in $\mathcal{N}_i$. $A_i \in \Re^{n_i\times n_i}$, $B_i \in \Re^{n_i\times m_i}$, and $H_{ij} \in \Re^{n_i\times n_j}$ are matrices of appropriate dimension.

For each $i \in \mathcal{N}$ we assume there exist $K_i \in \Re^{m_i\times n_i}$, $P_i \in \Re^{n_i\times n_i}$, and $Q_i \in \Re^{n_i\times n_i}$ such that

$$A_{K_i}^T P_i + P_i A_{K_i} \le -Q_i \quad (2)$$

where $A_{K_i} = A + B_i K_i$.

Note that this inequality is equivalent to requiring that the function $V_i : \Re^{n_i} \to \Re$, define as $V_i(x_i) = x_i^T P_i x_i$, is a control Lyapunov function for the *decoupled* system,

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t).$$

Consider the use of a digital communication network so that every agent can only broadcast its state information to its neighbors in a discrete-time manner. The control for agent i is therefore generated by the following equation

$$u_i(t) = K_i \hat{x}_i(t) + \textstyle\sum_{j\in\mathcal{N}_i} L_{ij} \hat{x}_j(t) \quad (3)$$

where K_i is the state feedback gain satisfying (2), $L_{ij} \in \Re^{m_i\times n_j}$ is a set of *decoupling* gains, and $\hat{x}_j(t)$ is the latest state that was broadcast by agent j at time t (also called the "measured state"). The state equation in (1) comes from earlier models for decentralized control systems used by Siljak [18]. The structure of the controller differs from Siljak's earlier work because our model specifically considers discrete broadcasts between agents.

Note that we distinguish between the measured feedback state and the actual state of the agent. This is because a subsystem can only broadcast its state information at discrete times. We model this discrete transmission by associating a monotone increasing sequence of *broadcast times*, $\{b_j[k]\}_{k=0}^{\infty}$, with the jth agent. The broadcast times are increasing in the sense that $b_j[k] < b_j[k+1]$ for all k. The time $b_j[k]$ denotes the kth consecutive time instant when the jth agent broadcasts its local state $x_j(b_j[k])$ to all of its neighbors in $\mathcal{N}_j$.

The "measured" states used by agent i in (3) are the functions $\hat{x}_j : \Re \to \Re^{n_j}$ where $j \in \mathcal{N}_i$ and

$$\hat{x}_j(t) = x_j(b_j[k]) \quad (4)$$

for $t \in [b_j[k], b_j[k+1])$ and all $k = 0, \cdots, \infty$. The measured state, therefore, is a sampled version of the neighbor's state trajectory where the sampling instants are the broadcast times. For simplicity we assume that all neighbors receive the broadcasted state without any delay. This paper's purpose is to provide a distributed scheme that enables every agent to locally identify K_i, L_{ij}, and its broadcast time sequence ($\{b_i[k]\}_{k=0}^{\infty}$) such that the overall system is guaranteed to be asymptotically stable.

III. Event Triggering for Asymptotic Stability

This section derives the event-triggering rule that assures the entire system is asymptotically stable. The first lemma characterizes the directional derivative of the function $V_i(x_i) = x_i^T P_i x_i$, where P_i satisfies (2). The results in lemma 3.1 are used to characterize the directional derivative of the function $V : \Re^{\sum_i n_i} \to \Re$ defined as $V(x_1, x_2, \cdots, x_N) = \sum_i x_i^T P_i x_i$ that is used in theorem 3.2 to establish a condition for event triggering.

Lemma 3.1: Consider the system in (1) where

1) the control u_i is the distributed control in (3) using measured states defined by (4),
2) P_i, K_i, and Q_i satisfy (2),
3) and $e_i(t) = \hat{x}_i(t) - x_i(t)$ is the error between the measured state and the actual state.

The directional derivative of $V_i(x_i) = x_i^T P_i x_i$ satisfies

$$\begin{aligned}\frac{\partial V_i}{\partial x_i}\dot{x}_i \quad \leq \quad & -\left(\underline{\lambda}(Q_i) - (|\mathcal{N}_i|+1)\delta\right)\|x_i\|_2^2 \\ & + \sum_{j\in\mathcal{N}_i} \frac{2\|P_iB_iL_{ij}\|^2}{\delta}\|e_j\|_2^2 + \frac{\|P_iB_iK_i\|^2}{\delta}\|e_i\|_2^2 \\ & + \sum_{j\in\mathcal{N}_i} \frac{2\|P_i(B_iL_{ij}+H_{ij})\|^2}{\delta}\|x_j\|_2^2 \end{aligned} \tag{5}$$

for all $i \in \mathcal{N}$ where δ is any positive real constant and where $\underline{\lambda}(Q_i)$ is the minimum eigenvalue of Q_i.

Proof: A direct computation shows that

$$\begin{aligned}\frac{\partial V_i}{\partial x_i}\dot{x}_i \quad = \quad & x_i^T P_i\left(A_ix_i + B_iK_i\hat{x}_i\right) \\ & + x_i^T P_i \sum_{j\in\mathcal{N}_i}\left(B_iL_{ij}\hat{x}_j + H_{ij}x_j\right) \\ & + \text{transposed terms}\end{aligned} \tag{6}$$

Note that $\hat{x}_i = x_i + e_i$ so we can rewrite (6) in terms of x_i and e_i to obtain

$$\begin{aligned}\frac{\partial V_i}{\partial x_i}\dot{x}_i \quad = \quad & x_i^T P_i(A_{K_i}x_i + B_iK_ie_i) \\ & + x_i^T P_i \sum_{j\in\mathcal{N}_i}\left(\Delta_{ij}x_j + B_iL_{ij}e_j\right) \\ & + \text{transposed terms}\end{aligned} \tag{7}$$

where $\Delta_{ij} = B_iL_{ij} + H_{ij}$ and $A_{K_i} = A_i + B_iK_i$.

Using the fact that

$$\delta\|z\|_2^2 + \frac{\|Ry\|_2^2}{\delta} - 2z^TRy \geq \|\delta z - Ry\|_2^2 \geq 0, \text{ for } \delta > 0,$$

(7) can be rewritten as

$$\begin{aligned}\frac{\partial V_i}{\partial x_i}\dot{x}_i \quad \leq \quad & -x_i^TQ_ix_i + \delta\|x_i\|_2^2 + \frac{\|P_iB_iK_i\|^2}{\delta}\|e_i\|_2^2 \\ & + \sum_{j\in\mathcal{N}_i}\left(\frac{\delta}{2}\|x_i\|_2^2 + \frac{2\|P_i\Delta_i\|^2}{\delta}\|x_j\|_2^2\right) \\ & + \sum_{j\in\mathcal{N}_i}\left(\frac{\delta}{2}\|x_i\|_2^2 + \frac{2\|P_iB_iL_{ij}\|^2}{\delta}\|e_j\|_2^2\right)\end{aligned}$$

where δ is any positive real constant. Collecting the terms in $\|x_i\|_2^2$ and recognizing that

$$-x_i^TQ_ix_i \leq -\underline{\lambda}(Q_i)\|x_i\|_2^2 \tag{8}$$

yields (5). ■

Given the characterization of the V_i's directional derivative in (5), we can now state and prove the following theorem regarding the asymptotic stability of the entire system. This theorem presumes the decoupling gains, L_{ij}, were chosen to satisfy the *matching condition*, $B_iL_{ij} = -H_{ij}$, which essentially assures perfect decoupling of the subsystems.

Theorem 3.2: Assume the matching condition $B_iL_{ij} = -H_{ij}$ holds for all i and j. Under the assumptions of lemma 3.1, the networked system in (1) under the control in (3) is asymptotically stable, if

$$\beta_i\|e_i(t)\|_2^2 \leq \rho_i\|x_i(t)\|_2^2 \tag{9}$$

for all $i \in \mathcal{N}$ and all t, where

$$\beta_i = \frac{\|P_iB_iK_i\|^2}{\delta} + \sum_{j\in\mathcal{N}_i}\frac{2\|P_jB_jL_{ji}\|^2}{\delta}, \tag{10}$$

$$0 < \delta < \min_{i\in\mathcal{N}}\left\{\frac{\underline{\lambda}(Q_i)}{|\mathcal{N}_i|+1}\right\}, \tag{11}$$

$$0 < \rho_i < \underline{\lambda}(Q_i) - (|\mathcal{N}_i|+1)\delta. \tag{12}$$

Proof: Consider the candidate Lyapunov function $V(x_1,\cdots,x_N) = \sum_{i\in\mathcal{N}} V_i(x_i)$. Using lemma 3.1, its directional derivative may be written as

$$\begin{aligned}\frac{\partial V}{\partial x}\dot{x} \quad \leq \quad & -\sum_{i\in\mathcal{N}}\left(\underline{\lambda}(Q_i) - (|\mathcal{N}_i|+1)\delta\right)\|x_i\|_2^2 \\ & + \sum_{i\in\mathcal{N}}\frac{\|P_iB_iK_i\|^2}{\delta}\|e_i\|_2^2 \\ & + \sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}_i}\frac{2\|P_iB_iL_{ij}\|^2}{\delta}\|e_j\|_2^2\end{aligned} \tag{13}$$

Recall that neighborhood membership is a symmetric relation, so that $j \in \mathcal{N}_i$ whenever $i \in \mathcal{N}_j$. Therefore, we can redistribute the terms in the third line of (13) to group together terms indexed by $\|e_i\|_2^2$ and obtain

$$\begin{aligned}\frac{\partial V}{\partial x}\dot{x} \quad \leq \quad & -\sum_{i\in\mathcal{N}}\left(\underline{\lambda}(Q_i) - (|\mathcal{N}_i|+1)\delta\right)\|x_i\|_2^2 \\ & + \sum_{i\in\mathcal{N}}\frac{\|P_iB_iK_i\|^2}{\delta}\|e_i\|_2^2 \\ & + \sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}_i}\frac{2\|P_jB_jL_{ji}\|^2}{\delta}\|e_i\|_2^2 \\ = \quad & -\sum_{i\in\mathcal{N}}\left(\underline{\lambda}(Q_i) - (|\mathcal{N}_i|+1)\delta\right)\|x_i\|_2^2 \\ & + \sum_{i\in\mathcal{N}}\beta_i\|e_i\|_2^2\end{aligned} \tag{14}$$

where β is defined in (10). By the assumption in (9), (14) can be further reduced as

$$\frac{\partial V}{\partial x}\dot{x} \leq -\sum_{i\in\mathcal{N}}\left(\underline{\lambda}(Q_i) - (|\mathcal{N}_i|+1)\delta - \rho_i\right)\|x_i\|_2^2. \tag{15}$$

By (12), we have $\underline{\lambda}(Q_i) - (|\mathcal{N}_i|+1)\delta - \rho_i > 0$ for all $i \in \mathcal{N}$. Therefore, the term on the right-hand side of (15) is negative definite, which implies the asymptotic stability of the equilibrium point. ■

Theorem 3.2 is interesting because the error condition in (9) is only dependent on what the ith subsystem can directly measure. In other words, if all agents cooperate in the sense of broadcasting their states so that the threshold condition in (9) is always satisfied, we can assure the entire system's asymptotic stability.

The inequality in (9) can be used as the basis for event-triggering the broadcast of an agent's state. Note that the inequality is trivially satisfied by the ith agent at broadcast time $t = b_i[k]$. So if we trigger the next broadcast, $b_i[k+1]$, any time before (9) is violated and if we can guarantee this behavior across all agents in the system, the entire networked system is asymptotically stable.

The matching condition assumed in theorem 3.2 is exceptionally restrictive. The following theorem relaxes this assumption.

Theorem 3.3: Assume that the hypotheses in lemma 3.1 are true and assume that for all j

$$W_i \equiv \sum_{j\in\mathcal{N}_i}\|P_j(B_jL_{ji} + H_{ji})\|^2 \leq \frac{\underline{\lambda}(Q_i)}{8(1+|\mathcal{N}_i|)} \tag{16}$$

for all $i \in \mathcal{N}$ The networked system in (1) under the control in (3) is asymptotically stable, if

$$\beta_i\|e_i(t)\|_2^2 \leq \alpha_i\|x_i(t)\|_2^2 \tag{17}$$

for all $i \in \mathcal{N}$ and all t, where β_i is defined in (10),

$$\delta < \min_i\left\{\frac{\underline{\lambda}(Q_i)}{2(|\mathcal{N}_i|+1)}\left(1 + \sqrt{1 - \frac{8(1+|\mathcal{N}_i|)W_i}{\underline{\lambda}^2(Q_i)}}\right)\right\} \tag{18}$$

$$0 < \alpha_i < \underline{\lambda}(Q_i) - (1+|\mathcal{N}_i|)\delta - \frac{2W_i}{\delta}. \tag{19}$$

Proof: Notice that the definition of W_i in (16) ensures the term on the right-hand side of (18) is positive. Consequently, (18) implies the term in the most right side of (19) is positive. We now consider the candidate Lyapunov function $V(x_1, \cdots, x_N) = \sum_{i\in\mathcal{N}} V_i(x_i)$.

From lemma 3.1, the directional derivative of V becomes

$$\begin{array}{rcl}\frac{\partial V}{\partial x}\dot{x} & \leq & -\sum_{i\in\mathcal{N}}\left(\underline{\lambda}(Q_i)-(|\mathcal{N}_i|+1)\delta\right)\|x_i\|_2^2 \\ & & +\sum_{i\in\mathcal{N}}\frac{\|P_iB_iK_i\|^2}{\delta}\|e_i\|_2^2 \\ & & +\sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}_i}\frac{2\|P_iB_iL_{ij}\|^2}{\delta}\|e_j\|_2^2 \\ & & +\sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}_i}\frac{2\|P_i(B_iL_{ij}+H_{ij})\|^2}{\delta}\|x_j\|_2^2.\end{array} \tag{20}$$

Since the neighborhood relation is symmetric, we can redistribute the terms in the third and fourth lines of (20) to obtain

$$\begin{array}{rcl}\frac{\partial V}{\partial x}\dot{x} & \leq & -\sum_{i\in\mathcal{N}}\left[\underline{\lambda}(Q_i)-(|\mathcal{N}_i|+1)\delta-2\frac{W_i}{\delta}\right]\|x_i\|_2^2 \\ & & +\sum_{i\in\mathcal{N}}\beta_i\|e_i\|_2^2\end{array} \tag{21}$$

where W_i and β_i are defined in (16) and (10), respectively.

Applying (17) into (21), we have

$$\frac{\partial V}{\partial x}\dot{x} \leq -\sum_{i\in\mathcal{N}}\left[\underline{\lambda}(Q_i)-(|\mathcal{N}_i|+1)\delta-2\frac{W_i}{\delta}-\alpha_i\right]\|x_i\|_2^2. \tag{22}$$

By (19), we have $\underline{\lambda}(Q_i)-(|\mathcal{N}_i|+1)\delta-2\frac{W_i}{\delta}-\alpha_i > 0$ for all $i \in \mathcal{N}$. Therefore, the term on the right-hand side of (22) is negative definite, which implies the asymptotic stability of the equilibrium point. ∎

Theorem 3.3 relaxes the matching condition of theorem 3.2. In this case, then we require that there exists symmetric matrices P_i and Q_i as well as K_i and L_{ij} such that

$$A_{K_i}^T P_i + P_i A_{K_i} \leq -Q_i \tag{23}$$

$$\sum_{j\in\mathcal{N}_i}\|P_j(B_jL_{ji}+H_{ji})\|^2 \leq \frac{\underline{\lambda}(Q_i)}{8(|\mathcal{N}_i|+1)} \tag{24}$$

One traditional way of interpreting these equations is to assume that P_i and Q_i are fixed. We would then use (23) and (24) to determine the control gains K_i and decoupling gains, L_{ij}. An alternative approach assumes we select K_i to stabilize the decoupled systems with a given level of robust stability. We would then use (23) and (24) to determine the matrices P_i and gains L_{ji}. In this particular case we can view V_i as robust control Lyapunov functions [19] for the networked system.

IV. Broadcast Period

This section presents preliminary results bounding the time between broadcasts when the matching condition holds. We define the *broadcast period* of agent i as

$$B_i[k] = b_i[k+1] - b_i[k]. \tag{25}$$

The main result of this section shows that agent i can compute its expected "time" to its next broadcast in a rather simple manner that is a function of the states in the agent's neighborhood. This means that broadcast frequency is really a function of the activity level in an agent's neighborhood. Moreover, these results show that the time between consecutive broadcasts by the ith agent should be bounded away from zero.

To bound the time between broadcasts, however, we first need the following weaker version of theorem 3.2. A similar corollary can be established under the relaxed form of the matching condition in (16).

Corollary 4.1: Consider the networked control system in (1) using the control in (3). If the matching condition holds under the assumptions of lemma 3.1 and the sequence of agent broadcasts can ensure that

$$(\beta_i + \rho_i)\|e_i(t)\|_2^2 \leq \rho_i\|\hat{x}_i(t)\|_2^2 \tag{26}$$

for all i and all $t \in [b_i[k], b_i[k+1])$, where β_i and ρ_i are defined in (10) and (12), respectively, then the networked system is asymptotically stable.

Proof: For notational simplicity let x_b denote $x_i(b_i[k])$, then the condition in corollary 4.1 can be rewritten as

$$\begin{array}{rcl}\beta_i\|e_i(t)\|_2^2 & \leq & \rho_i(\|x_b\|_2^2 - \|e_i(t)\|_2^2) \\ & \leq & \rho_i\|x_b-(x_b-x_i(t))\|_2^2 = \rho_i\|x_i(t)\|_2^2.\end{array}$$

So by theorem 3.2, we can immediately conclude the entire system is asymptotically stable. ∎

Corollary 4.1 is clearly a weaker condition than that used in theorem 3.2. But we can use it to bound the broadcast period of a given agent. In particular, let's assume that the hypotheses of theorem 3.2 hold and let's further require that an agent broadcasts its state whenever the condition in corollary 4.1 is about to be violated.

Let's assume that agent i broadcasts its state at time $b_i[k] = r_0$. Between this broadcast and the next broadcast by agent i, it is quite possible that agent i will receive broadcasts from any of its neighbors. Let M be the number of messages agent i received from its neighbors during the time interval $[b_i[k], b_i[k+1])$ and r_m denote the mth message agent i received during $[b_i[k], b_i[k+1])$. We may therefore order these times as $b_i[k] = r_0 < r_1 < \cdots < r_{M+1} = b_i[k+1]$.

We now study the behavior of the state error e_i between any two consecutive times r_m and r_{m+1}. To simplify notation we let $z_i(t) = \|e_i(t)\|_2$. We can show that

$$\begin{array}{rcl}\dot{z}_i & \leq & \|\dot{e}_i\|_2 = \|\dot{x}_i\|_2 \\ & = & \left\|A_ix_i + B_iK_i\hat{x}_i + \sum_{j\in\mathcal{N}_i}(B_iL_{ij}\hat{x}_j + H_{ij}x_j)\right\|_2 \\ & = & \left\|A_{K_i}\hat{x}_i - B_iK_ie_i - \sum_{j\in\mathcal{N}_i}H_{ij}e_j\right\|_2 \\ & \leq & \|A_{K_i}\hat{x}_i\|_2 + \|B_iK_i\|\|e_i\|_2 + \sum_{j\in\mathcal{N}_i}\|H_{ij}\|\|e_j\|_2\end{array} \tag{27}$$

where we used $B_iL_{ij}+H_{ij} = 0$ (i.e. the matching condition).

By the event-triggering rule in corollary 4.1, agent j only broadcasts if it is about to violate the inequality

$$\|e_j(t)\|_2 \leq \gamma_j\|\hat{x}_j(t)\|_2 \tag{28}$$

for any j where $\gamma_j = \sqrt{\frac{\rho_j}{\beta_j+\rho_j}}$.

Between any two times (say r_m and r_{m+1}) when a message is received (or broadcast) by agent i, we know the measured state $\hat{x}_j$ is constant for any $j \in \mathcal{N}_i$. Therefore (27) can be reduced to

$$\dot{z}_i(t) \quad \leq \quad \sigma z_i(t) + \mu \tag{29}$$

for any $t \in [r_m, r_{m+1})$ where $\sigma = \|B_i K_i\|$ and $\mu = \|A_{K_i}\hat{x}_i\|_2 + \sum_{j\in\mathcal{N}_i} \gamma_j \|H_{ij}\| \|\hat{x}_j\|_2$.

Note that μ is constant between any two consecutive receptions. We can therefore solve the differential inequality in (29) to show that

$$z_i(t) \le e^{\sigma(t-r_m)} z_i(r_m) + \frac{\mu}{\sigma}(e^{\sigma(t-r_m)} - 1) \quad (30)$$

for $t \in [r_m, r_{m+1})$.

When $m = M$, we have

$$z_i(b_i[k+1]) = z_i(r_{M+1}) = \gamma_i \|x_i(r_0)\|_2. \quad (31)$$

We can use our expression for $z_i(t)$ in (30) to solve for $b_i[k+1]$ in (31). This yields

$$b_i[k+1] - r_M \ge \frac{1}{\sigma} \ln\left(1 + \frac{\gamma_i\|x_i(r_0)\|_2 - z_i(r_M)}{z_i(r_M) + \mu/\sigma}\right).$$

Clearly the broadcast period can be bounded as

$$\begin{aligned} B_i &= b_i[k+1] - r_M + r_M - r_0 \\ &\ge \frac{1}{\sigma}\ln\left(1 + \frac{\gamma_i\|x_i(r_0)\|_2 - z_i(r_M)}{z_i(r_M)+\mu/\sigma}\right) + r_M - r_0. \end{aligned} \quad (32)$$

First notice that $z_i(r_M) \le \gamma_i\|x_i(r_0)\|_2$. If $M \ge 1$, then we have $B_i \ge r_M - r_0 > 0$. If $M = 0$, then $z_i(r_M) = z_i(r_0) = 0$. By (32), we have

$$B_i \ge \frac{1}{\sigma}\ln\left(1 + \frac{\gamma_i\|x_i(r_0)\|_2}{\mu/\sigma}\right) > 0 \text{ if } x_i(r_0) \neq 0.$$

Therefore, we can conclude that the time between consecutive broadcasts of the same agent must be greater than zero.

V. Simulation Results

This section presents simulation results demonstrating event triggering in a networked control system. The system under study is a collection of N inverted pendulums (Fig. 1) whose pendulum arms are coupled together by springs. The basic system matrices for the ith pendulum are

$$A_i = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} - \frac{a_i k}{m\ell^2} & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ \frac{1}{m\ell^2} \end{bmatrix}$$

where $g = 10$ is gravitational acceleration, $\ell = 2$ is the length of the pendulum, $m = 1$ is the mass of the pendulum bob, $k = 5$ is the spring constant, and $a_1 = a_N = 1$, $a_i = 2, \forall i \notin \{1, N\}$ are the number of pendulums connected to the ith pendulum. The coupling matrices, H_{ij}, have the form

$$H_{ij} = \begin{bmatrix} 0 & 0 \\ \frac{c_{ij}k}{m\ell^2} & 0 \end{bmatrix}$$

where $c_{ij} = 1$ for $i \notin \{1, N\}$, $j \in \{(i-1, i+1\}$. Also $c_{12} = c_{N,N-1} = 1$. Otherwise $c_{ij} = 0$.

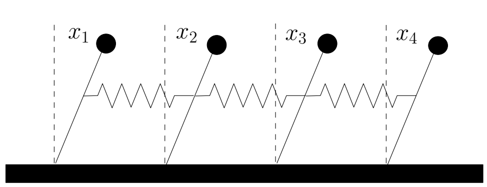

Fig. 1. Network of inverted pendulums

A local set of control gains, K_i, were obtained to place the decoupled system's poles at -1 and -2. This resulted in $K_i = [-23, -12]$ for $i = 1$ and N. The gain for other agents was $K_i = [-18, -12]$. In this problem the matching condition, $B_i L_{ij} = -H_{ij}$, can be used if we select $L_{ij} = [-5, 0]$ for nonzero H_{ij} and $L_{ij} = [0, 0]$ otherwise.

The candidate control Lyapunov function V_i for agent i was chosen to be $x_i^T P_i x_i$ where $P_i = \begin{bmatrix} 1.25 & .25 \\ .25 & .25 \end{bmatrix}$ for all i. The matrices P_i were obtained by solving the following Lyapunov equation $(A_i + B_i K_i)^T P_i + P_i(A_i + B_i K_i) = -I$, where I is a 2×2 identity matrix.

With this setup we computed the coefficients β_i and ρ_i in the event-triggering inequality (9). Our simulation then triggered agent i to broadcast its state whenever

$$-0.5\|x_i\|_2^2 + \beta_i\|e_i\|_2^2 > 0$$

where $\beta_1 = \beta_N = 32.7177$ and $\beta_i = 24.2812$ for $i \neq 1, N$. These values were obtained for a δ that was one half of its maximum possible value in (12).

The simulation results are shown in Fig. 2 where $N = 100$ and the initial states were randomly generated. The simulation ran for 16 seconds, with a large disturbance being applied to the third system halfway through the simulation. The top plot in Fig 2 is the state time history for all 100 inverted pendulums. Note that the system is stable. The bottom plot in Fig. 2 is the history of broadcast periods of the first three pendulums generated by the event-triggering scheme. Note that the broadcast periods vary considerably over those intervals when the state has been perturbed away from its equilibrium point. This shows that our event triggering scheme indeed adjusts broadcast periods in response to what is happening in the plant. We computed the average broadcast period, $\overline{B}$, for the 100 inverted pendulums simulated in Fig. 2 to be 0.1157.

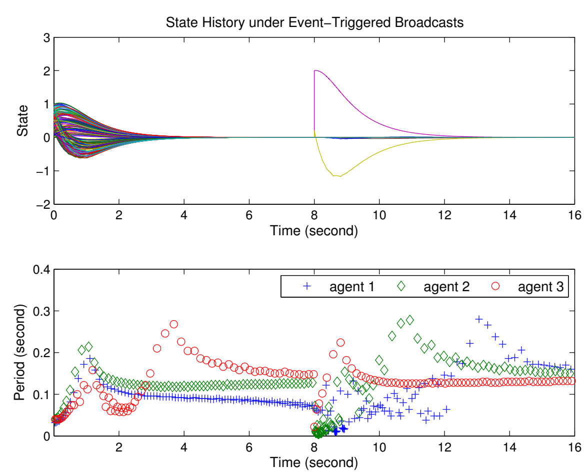

Fig. 2. Event-triggered broadcast simulation results

Let's now compare the average periods of event-triggered systems against MATI in [12] for different N. Recall that

MATI in [12] is defined by

$$B_{MATI} = \begin{cases} \frac{1}{Lr}\arctan\frac{r(1-\bar\rho)}{2\frac{\bar\rho}{1+\bar\rho}\left(\frac{\bar\gamma}{\bar L}-1\right)+1+\bar\rho} & \bar\gamma > \bar L \\ \frac{1-\bar\rho}{L(1+\bar\rho)} & \bar\gamma = \bar L \\ \frac{1}{Lr}\text{arctanh}\frac{r(1-\bar\rho)}{2\frac{\bar\rho}{1+\bar\rho}\left(\frac{\bar\gamma}{\bar L}-1\right)+1+\bar\rho} & \bar\gamma < \bar L \end{cases}, \quad (33)$$

where, assuming the state equation of the entire system is $\dot x = Ax + BK\hat x$, $\bar L = \max\{0.5\overline{\lambda}(-BK - K^TB^T), 0\}$, $\bar\gamma$ is the $\mathcal{L}_2$ gain for the system $\dot x = (A+BK)x + BKe$ from e to $-(A+BK)x$, $\bar\rho = \sqrt{\frac{N-1}{N}}$, and $r = \sqrt{\left|\frac{\bar\gamma^2}{\bar L^2} - 1\right|}$.

Table I shows the comparison results. It is obvious that event-triggering scheme can provide a much longer average broadcast period than MATI. Also notice that by event-triggering the average periods change little as N increases. This shows that the average broadcast period by event-triggering scales well with respect to the size of the system.

TABLE I

COMPARISON RESULTS

Number of Pendulums	Average Period by Event-triggering	MATI in [12]
$N = 10$	0.1149	3.70×10^{-3}
$N = 50$	0.1175	7.05×10^{-4}
$N = 100$	0.1152	3.51×10^{-4}
$N = 150$	0.1180	2.34×10^{-4}
$N = 200$	0.1177	1.75×10^{-4}

VI. SUMMARY

This paper presented an event-triggering approach to broadcasting state data in distributed control systems implemented over ad hoc wireless networks. Broadcasts are triggered in a decentralized manner, so that all agents make their broadcast decisions solely on the basis of their own measured states. Information from neighboring subsystems is used to adjust the event-triggering level. This approach therefore allows a subsystem to adjust its broadcast rate to the amount of activity in its immediate neighborhood. We were able to bound the time between broadcast events and simulation results supported our contention that event-triggering provides an effective means of adapting broadcast rates in sensor-actuator networks.

The work presented in this paper is preliminary in nature. There are a number of important issues that will need to be addressed in our future work. Some of these issues are itemized below.

- It would be valuable to see how we can take advantage of the relaxed matching condition in controller synthesis. As noted above, we can use the conditions in theorem 3.3 to design both the decoupling gains, L_{ij}, and robust control Lyapunov functions for the networked systems. Precisely how such distributed controllers can be synthesized is a topic for future study.
- The current work restricts its attention to linear time-invariant systems. It would be valuable to extend this to networks of nonlinear systems. We believe this may be possible for nonlinear systems that are affine in the controls. Once again the matching condition becomes a major concern in such analyses.
- This paper did not address the issue of message collisions. In practice, such collisions will delay the delivery of messages in a way that can adversely affect system stability. Bounding delays as was done in [20] may help in analyzing the impact message collisions have on overall system stability.

REFERENCES

[1] E. Lee, "Computing Foundations and Practice for Cyber-Physical Systems: A Preliminary Report," Technical Report No. UCB/EECS-2007-72, Electrical Engineering and Computer Sciences, University of California at Berkeley, May 21, 2007.

[2] B. Bamieh, F. Paganini, and M. Dahleh, "Distributed control of spatially invariant systems," *IEEE Transactions on Automatic Control*, vol. 47, no. 7, pp. 1091–1107, 2002.

[3] E. Camponogara, D. Jia, B. Krogh, and S. Talukdar, "Distributed model predictive control," *IEEE Control Systems*, vol. 22, no. 1, pp. 44–52, 2002.

[4] W. Dunbar, "A distributed receding horizon control algorithm for dynamically coupled nonlinear systems," in *IEEE Conference on Decision and Control*, 2005.

[5] R. D'Andrea and G. Dullerud, "Distributed control design for spatially interconnected systems," *IEEE Transactions on Automatic Control*, vol. 48, no. 9, pp. 1478–1495, 2003.

[6] C. Langbort, R. Chandra, and R. D'Andrea, "Distributed control design for systems interconnected over an arbitrary graph," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1502–1519, 2004.

[7] M. Jovanovic and B. Bamieh, "Lyapunov-based distributed control of systems on lattices," *IEEE Transactions on Automatic Control*, vol. 50, no. 4, pp. 422–433, 2005.

[8] F.-L. Lian, J. Moyne, and D. Tilbury, "Network design consdieration for distributed control systems," *IEEE Transactions on Control Systems Technology*, vol. 10, no. 2, pp. 297–307, 2002.

[9] G. Walsh, H. Ye, and L. Bushnell, "Stability analysis of networked control systems," *IEEE Transactions on Control Systems Technology*, vol. 10, no. 3, pp. 438–446, 2002.

[10] W. Zhang, M. Branicky, and S. Phillips, "Stability of networked control systems," *IEEE Control Systems Magazine*, vol. 21, pp. 84–99, 2001.

[11] D. Nesic and A. Teel, "Input-output stability properties of networked control systems," *IEEE Transactions on Automatic Control*, vol. 49, pp. 1650–1667, 2004.

[12] D. Carnevale, A. R. Teel, and D. Nesic, "Further results on stability of networked control systems: a lyapunov approach," *IEEE Transactions on Automatic Control*, vol. 52, pp. 892–897, 2007.

[13] K. Arzen, "A simple event-based pid controller," in *Proceedings of the 14th IFAC World Congress*, 1999.

[14] D. Hristu-Varsakelis and P. Kumar, "Interrupt-based feedback control over a shared communication medium," in *IEEE Conference on Decision and Control*, 2002.

[15] K. Astrom and B. Bernhardsson, "Comparison of Riemann and Lebesgue sampling for first order stochastic systems," in *IEEE Conference on Decision and Control*, 1999.

[16] P. Tabuada and X. Wang, "Preliminary results on state-triggered scheduling of stabilizing control tasks," in *IEEE Conference on Decision and Control*, 2006.

[17] M. Lemmon, T. Chantem, X. Hu, and M. Zyskowski, "On self-triggered full information H-infinity controllers," in *Hybrid Systems: computation and control*, 2007.

[18] D. Siljak, *Decentralized Control of Complex Systems*. Cambridge,MA:Academic Press, 1991.

[19] R. Freeman and P. Kokotovic, *Robust nonlinear control design: state-space and Lyapunov techiques*. Birkhauser, 1996.

[20] X. Wang and M. Lemmon, "Self-triggered feedback control systems with finite-gain L_2 stability," submitted to IEEE Transactions on Automatic Control, 2007.