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Optimization based Option Pricing Bounds via Piecewise Polynomial Super- and Sub-Martingales

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Abstract— **In this paper we first prove sufficient conditions for a continuous function of a diffusion process to be a super- or sub-martingale. This result is then used to create piecewise polynomial super- and sub-martingale bounds on option prices via a polynomial optimization problem. The polynomial optimization problem is solved under a sum-of-squares paradigm and thus uses semi-definite programming. The results are tested on a Black-Scholes example where a piecewise polynomial function of degree four in both the stock value and time is used to compute upper and lower bounds.**

I. INTRODUCTION

The connections between martingales and option pricing theory are well established. In particular, the fundamental theorem of asset pricing asserts that for no arbitrage to exist, the discounted price of an option must be a martingale under the so-called risk neutral measure [6]. This result allows options to be priced by computing expectations, and has led to the successful use of Monte-Carlo methods in option pricing theory.

In recent years researchers have developed alternative computational methods that compute *hard* bounds on option prices via optimization methods (as opposed to soft bounds that accompany Monte-Carlo methods). Work in this area includes [13], [1], [2], [7], [11]. These results share a common root in generalized Chebyshev bounds, and most can be interpreted as static arbitrage bounds using special securities with monomial payoffs. An exception is [11] which allowed certain discrete time dynamic strategies.

This paper presents a new optimization based hard bound approach that uses martingale theory to construct bounds. We first derive conditions under which a continuous function (not necessarily C^2) is a super- or sub-martingale. This result allows us to construct super- and sub-martingale bounds on an option price via piecewise polynomial functions. The use of piecewise polynomial functions allows us to optimize the bound via a polynomial programming problem that is replaced by a sum-of-squares formulation using the software package SOStools [10]. Thus, semi-definite programming is ultimately used to compute the bound.

The paper is organized as follows. In Section II we set up the problem and review basic facts from martingale pricing. In Section III we develop some theoretical preliminaries. Section IV presents the main theorem that gives sufficient conditions for a continuous function to be a super- or sub-martingale. Section V uses the main theorem to justify the use of piecewise polynomial functions in an optimization problem that computes hard upper or lower bounds. Numerical results for the Black-Scholes setup are given in Section VI. Conclusions are discussed in Section VII.

II. PROBLEM SETUP AND MARTINGALE PRICING

For simplicity of exposition, we present the results for an option on a single underlying asset. However, it may be easily verified that all the results in this paper are directly applicable to options on multiple underlying assets.

We consider the risk neutral pricing problem where under the risk neutral measure Q, the underlying asset $S(t)$, $t \in [0,T]$ satisfies the stochastic differential equation

$$dS = a(S,t)dt + b(S,t)dz(t) \tag{1}$$

where $z(t)$ is a standard Brownian motion. We assume that

(i) $a(S,t)$ and $b(S,t)$ are continuous on $\mathbf{R} \times [0,T]$.
(ii) There exists an M such that $\|a(S,t)\| \le M(1+\|S\|)$ and $\|b(S,t)\| \le M(1+\|S\|)$ for all $(S,t) \in \mathbf{R} \times [0,T]$.
(iii) For each c, there exists K_c such that $\|a(S_1,t) - a(S_2,t)\| \le K_c\|S_1 - S_2\|$ and $\|b(S_1,t) - b(S_2,t)\| \le K_c\|S_1 - S_2\|$ whenever $\|S_1\| \le c$, $\|S_2\| \le c$.

Under these conditions, (1) has a pathwise unique solution which is a Markov diffusion process (See Chapter 5, [4]).

A. Martingale Pricing

Let $c(S,t)$ be the value of a European call option with expiration T and strike price K. By the fundamental theorem of asset pricing [3], the value of this option at time 0 with $S(0) = S_0$ is given by

$$c(S_0,0) = e^{-rT} E_{0,S_0}\left[(S(T)-K)^+\right]. \tag{2}$$

where $E_{t,S_t}[\cdot]$ denotes the expectation under Q conditional upon time t and price $S(t) = S_t$. In particular, the quantity

$$e^{-r(T-t)}c(S(t),t) \tag{3}$$

is a Q-martingale.

Furthermore, assuming that $c(S,t) \in C^{2,1}$, then by the Feynman-Kac theorem [8], $c(S,t)$ satisfies the Black-Scholes partial differential equation

$$c_t + a(S,t)c_S + \frac{1}{2}b^2(S,t)c_{SS} - rc = 0, \quad c(S,T) = (S-K)^+ \tag{4}$$

where $c_t = \frac{\partial c}{\partial t}$, $c_S = \frac{\partial c}{\partial S}$, and $c_{SS} = \frac{\partial^2 c}{\partial S^2}$.

For simplicity (and without loss of generality) in what follows we assume that $r = 0$. Thus, $c(S,t)$ is a Q-martingale.

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B. Approach via Super- and Sub-Martingales

To explain our basic approach, consider the problem of computing an upper bound on $c(S_0,0)$. Now, any super-martingale satisfying $V(S,T) \geq (S-K)^+$ will provide an upper bound via

$$V(S_0,0) \geq E_{0,S_0}[V(S(T),T)] \geq E_{0,S_0}\left[(S(T)-K)^+\right] = c(S_0,0).$$

A similar argument shows that any sub-martingale satisfying $V(S,T) \leq (S-K)^+$ will give a lower bound.

Our computational approach will be to search for super- and sub-martingales that are piecewise polynomial. But first we need convenient characterizations of these super- and sub-martingales that allow for computation and optimization. In particular, our piecewise polynomial super- and sub-martingales will not be everywhere $C^{2,1}$. Thus, we prove a result that provides sufficient conditions for a continuous function to be a super- or sub-martingale. This condition is then used as a constraint in our optimizations, which are conveniently solved using a sum-of-squares polynomial programming approach.

The next two sections develop the theory behind our computational approach. We begin with preliminary facts and results that are used in the proof of the main theoretical result in Section IV.

III. Theoretical Preliminaries

As preliminaries for the main theoretical result, we review conditions for Dynkin's formula to hold, provide a generalization of the standard Newton-Leibnitz calculus formula, and prove a simple result on the extension of functions.

A. Diffusions and Dynkin's Formula

Following the development on page 128-129 of [5], from the assumptions of (i), (ii), and (iii) in Section II, we can further assert that for each $m = 1, 2, \ldots$, there exists a constant C_m (depending on m and T) such that

$$E_{0,S_0}\|S(t)\|^m \leq C_m(1+\|S_0\|^m),\ 0 \leq t \leq T \tag{5}$$

Let $C_p^{2,1}$ denote the space of $f \in C^{2,1}$ such that f, f_t, f_x, f_{xx} satisfy the polynomial growth condition

$$\|f(x,t)\| \leq k(1+\|x\|^m),\ \forall (x,t) \in \mathbf{R} \times [0,T] \tag{6}$$

for some constants k and m. Then it follows that $f(S(t),t) \in C_p^{2,1}$ satisfies Dynkin's formula,

$$E_{t_1,S_1}[f(S(t_2),t_2)] - f(S_1,t_1) = E_{t_1,S_1}\left[\int_{t_1}^{t_2} \mathscr{A} f(S(\tau),\tau)d\tau\right], \tag{7}$$

where $\mathscr{A}$ is the differential operator

$$\mathscr{A} f = f_t + a(S,t)f_x + \frac{1}{2}b(S,t)^2 f_{xx}. \tag{8}$$

See [5] page 129 for more details.

Now, by (6), for $f \in C_p^{2,1}$, $\|\mathscr{A} f(x,t)\| \leq k(1+\|x\|^m)$ for some k and m. Coupling this with (5) guarantees the absolute integrability of the right hand side of (7). Therefore Fubini's theorem [12] justifies an exchange of the expectation and integral. Hence, one may assert from (7) that the generator is given by

$$\lim_{h \downarrow 0} \frac{E_{t,S(t)}[f(S(t+h),t+h)] - f(S(t),t)}{h} = \mathscr{A} f \tag{9}$$

for all $f \in C_p^{2,1}$.

The characterization of the generator in (9) is important since we will use a condition based on the generator $\mathscr{A}$ to characterize super- and sub-martingales in the main theorem.

B. Generalization of Newton-Leibnitz Formula

Since we will be dealing with functions that are only continuous, we will need the following result from [15].

Lemma 3.1: Let $g \in C[0,T]$. Extend g to $(-\infty,+\infty)$ with $g(t) = g(T)$ for $t > T$, and $g(t) = g(0)$ for $t < 0$. Suppose there is a $\rho \in L^1(0,T)$ such that

$$\limsup_{h \to 0} \frac{g(t+h)-g(t)}{h} \leq \rho(t), \quad a.e.\ t \in [0,T]. \tag{10}$$

Then

$$g(\beta) - g(\alpha) \leq \int_\alpha^\beta \limsup_{h \to 0+} \frac{g(\tau+h)-g(\tau)}{h} d\tau, \tag{11}$$

for $0 \leq \alpha \leq \beta \leq T$.

For a proof of this result, see [15], page 270.

C. Result on extension of functions

The final preliminary result is on the extension of functions. We will need to lower and upper bound our continuous function by a $C_p^{2,1}$ function that lies in the domain of the generator. The following lemma allows us to do this in a global manner. We give only the upper bound extension, with a lower bound extension following in exactly the same manner.

Lemma 3.2: Let $V(x,t) \in C_p$ for $(x,t) \in \mathbf{R} \times [0,T]$ (that is, $V(x,t)$ satisfies the bound in (6)). Extend $V(x,t)$ to $(x,t) \in \mathbf{R} \times \mathbf{R}$ by $V(x,\tau) = V(x,T)$ for $\tau > T$ and $V(x,\tau) = V(x,0)$ for $\tau < 0$. Let $f \in C^{2,1}$ such that $f \geq V$ in an open set $\mathscr{O}$ containing (x_0,t_0). Then there exists an open set $\mathscr{O}'$ containing (x_0,t_0) with $\mathscr{O}' \subset \mathscr{O}$ and a function $\tilde{f} \in C_p^{2,1}$ such that $\tilde{f} = f$ in $\mathscr{O}'$ and $\tilde{f}(y,\tau) \geq V(y,\tau)$ for all (y,τ).

Proof: Define the function g by

$$g(x,t) = M + k(1+\|x\|^m) \tag{12}$$

where k and $m \geq 2$ correspond to the bound in (6) for $V(x,t)$, $M = max_{(y,\tau)\in\overline{\mathscr{B}}_2(x_0,t_0)} V(y,\tau)$, and $\overline{\mathscr{B}}_2(x_0,t_0)$ is the closed ball of radius 2 centered around (x_0,t_0). Thus $g(x,t) \geq V(x,t)$ for all (x,t), and $g(x,t) \in C_p^{2,1}$.

Now, let $b(x,t)$ be a C^∞ cutoff function that is 1 on the compact set $\overline{\mathscr{B}}_\varepsilon(x_0,t_0)$ and zero outside $\mathscr{B}_{2\varepsilon}(x_0,t_0)$ where $\varepsilon > 0$ is chosen so that $\mathscr{B}_{2\varepsilon}(x_0,t_0) \subset \mathscr{O}$.

Then letting

$$\tilde{f}(x,t) = f(x,t)b(x,t) + g(x,t)(1-b(x,t)) \tag{13}$$

with $\mathscr{B}_\varepsilon(x_0,t_0)$ satisfies the theorem. $\square$

We are now ready to prove the main theoretical result.

IV. Main Theoretical Result

The following theorem gives sufficient conditions for a continuous function to be a super-martingale (the corresponding result for a sub-martingale follows easily by flipping inequalities). It justifies the computational approach taken later in the paper.

Theorem 4.1: Let $S(t)$ satisfy (1) and the conditions (i), (ii), and (iii), and assume that $V(x,t) \in C_p$. Let the support of $S(t)$, $t \in [0,T]$ be contained in the set $A \subseteq \mathbf{R}$. Assume that for any point $(x,t) \in A$, there exists a function $f \in C^{2,1}$ such that $f(x,t) = V(x,t)$ and for some open set $\mathscr{O}$ containing (x,t) we have

$$f(y,\tau) \geq V(y,\tau), \quad \forall (y,\tau) \in \mathscr{O} \tag{14}$$

with

$$\mathscr{A} f(x,t) \leq 0. \tag{15}$$

Then $V(S(t),t)$ is a super-martingale and satisfies

$$V(S_0,0) \geq E_{0,S_0}\left[V(S(T),T)\right]. \tag{16}$$

Proof: At any point $(S_t,t) \in A$ consider the $C^{2,1}$ function $f(S_t,t)$ from the theorem satisfying $V(S_t,t) = f(S_t,t)$ and $f \geq V$ in some open set $\mathscr{O}$. By Lemma 3.2 we can replace f by a $C^{2,1}$ function $\tilde{f}$ that globally satisfies $\tilde{f} \geq V$.

Note that at (S_t,t), $\tilde{f}$ satisfies

$$\lim_{h \downarrow 0} \frac{E_{t,S_t}\left[\tilde{f}(S(t+h),t+h)\right] - \tilde{f}(S_t,t)}{h} = \mathscr{A}\tilde{f} = \mathscr{A} f \leq 0. \tag{17}$$

Furthermore, since globally $\tilde{f} \geq V$, we have that

$$\begin{aligned} &\frac{E_{t,S_t}\left[V(S(t+h),t+h)\right] - V(S_t,t)}{h} \\ &\qquad \leq \frac{E_{t,S_t}\left[\tilde{f}(S(t+h),t+h)\right] - \tilde{f}(S_t,t)}{h}. \end{aligned} \tag{18}$$

Thus, combining (17) and (18), gives

$$\limsup_{h \downarrow 0} \frac{E_{t,S_t}\left[V(S(t+h),t+h)\right] - V(S_t,t)}{h} \leq 0. \tag{19}$$

Now, by Fatou's lemma [12] for $t_0 < t$,

$$\begin{aligned} &\limsup_{h \downarrow 0} \frac{E_{t_0,S_{t_0}}\left[V(S(t+h),t+h)\right] - E_{t_0,S_{t_0}}\left[V(S(t),t)\right]}{h} \\ &\quad \leq E_{t_0,S_{t_0}}\left[\limsup_{h \downarrow 0} \frac{E_{t,S(t)}\left[V(S(t+h),t+h)\right] - V(S(t),t)}{h}\right] \leq 0. \end{aligned}$$

Finally, letting $g(t) = E_{t_0,S_{t_0}}\left[V(S(t),t)\right]$, one may note that $g(t)$ satisfies the assumptions of Lemma 3.1. Thus

$$E_{t_0,S_{t_0}}\left[V(S(t),t)\right] - V(S_{t_0},t_0) \leq 0 \tag{20}$$

showing that $V(S(t),t)$ is a super-martingale. Setting $t_0 = 0$ gives (16). □

With this theoretical result in hand, we now proceed to developing an optimization based bounding approach that uses piecewise polynomial functions.

V. A Numerical Optimization Approach

We can use Theorem 4.1 to construct upper and lower bounds on option prices via piecewise polynomial super- and sub-martingales. As before, we present the case of $S(t)$ being one dimensional with the multidimensional case following along similar lines.

Our approach is to search over piecewise polynomial $V(S,t)$ that satisfy the conditions of Theorem 4.1. To construct such a piecewise polynomial $V(S,t)$, we first select "break" points for S, denoted $a_1 < a_2 < \ldots < a_{n+1}$ and let the support of $S(t)$ be a subset of $[a_1, a_{n+1}]$. $V(S,t)$ is then pieced together as

$$V(S,t) = f^{(i)}(S,t), \quad S \in [a_i, a_{i+1}], \quad t \in [0,T] \tag{21}$$

for $i = 1, \ldots, n$ where each $f^{(i)}$ is a polynomial in S and t.

Now, in order to make $V(S,t)$ continuous, we require that

$$f^{(i-1)}(a_i,t) = f^{(i)}(a_i,t), \quad t \in [0,T], \; i = 2, \ldots, n. \tag{22}$$

Additionally, following the conditions of Theorem 4.1 we need to guarantee that at every point a $C^{2,1}$ function exists that is locally an upper bound on V. At points where $V(S,t)$ is twice continuously differentiable, we can use the polynomial $f^{(i)}(S,t) = V(S,t)$ itself. At boundary points, we can require the derivative condition

$$f_x^{(i-1)}(a_i,t) > f_x^{(i)}(a_i,t), \quad t \in [0,T], \; i = 2, \ldots, n \tag{23}$$

which guarantees that both $f^{(i-1)}$ or $f^{(i)}$ are locally upper bounds.

Finally, as long as

$$\mathscr{A} f^{(i)}(S,t) \leq 0, \quad S \in [a_i, a_{i+1}], \; t \in [0,T], \; i = 1, \ldots, n \tag{24}$$

then one can easily verify that $V(S,t)$ defined in this manner satisfies the conditions of Theorem 4.1 and is a super-martingale. Hence, as long as $V(S,T) \geq (S-K)^+$, then $V(S_0,0)$ is an upper bound on the option price.

Our computational procedure for an upper bound is simply based on minimizing over piecewise polynomial functions satisfying the preceding conditions and upper bounding the payoff function of the option at expiration. That is, define the optimization problem $\mathscr{P}^u$ as

$$\mathscr{P}^u = \begin{cases} \min V(S_0,0) \\ V(S,T) \geq (S-K)^+ \\ V(S,t) = f^{(i)}(S,t) \quad S \in [a_i, a_{i+1}], \; i = 1 \ldots n \\ \mathscr{A} f^{(i)}(S,t) \leq 0, \quad S \in [a_i, a_{i+1}], \; i = 1 \ldots n \\ f^{(i-1)}(a_i,t) = f^{(i)}(a_i,t), \quad t \in [0,T], \; i = 2, \ldots, n. \\ f_x^{(i-1)}(a_i,t) > f_x^{(i)}(a_i,t), \quad t \in [0,T], \; i = 2, \ldots, n. \\ f^{(i)} \text{ a polynomial }, \; i = 1, \ldots, n \end{cases} \tag{25}$$

This is a *polynomial* program that computes an upper bound on the call option price $c(S_0,0)$. In a completely analogous manner, a lower bound optimization $\mathscr{P}^l$ can also be formulated as a polynomial optimization.

While this optimization problem is difficult, the inequalities can be replaced by a more restrictive sum-of-squares condition that then allows for semi-definite programming methods to be used [9]. Additionally, the freely available software package SOStools [10] automates this process. Details can be found in the SOStools manual [10].

In the following section, we solve the optimization problems $\mathscr{P}^u$ and $\mathscr{P}^l$ and test the effectiveness of this super- and sub-martingale based optimization approach on the familiar Black-Scholes example.

VI. Black-Scholes Numerical Example

In this section, we use a Black-Scholes example with the underlying asset following geometric Brownian motion. We solve the upper bound optimization $\mathscr{P}^u$ as well as the corresponding lower bound problem $\mathscr{P}^l$ and explore the tightness of the bounds compared to the actual Black-Scholes solution.

A. Model

Let $S(t)$ satisfy

$$dS = \sigma S dz \quad (26)$$

with $\sigma = 0.30$. We considered the pricing of a European call options with strike price $K = 1$ and expiration time $T = 0.4$.

The optimization $\mathscr{P}^u$ was used to compute upper and lower bounds. This optimization was solved using SOStools [10] which linked to the semi-definite programming solver SeDuMi [14].

Specifically, we constructed $V(S,t)$ by piecing together four polynomials: $f^{(1)}(S,t)$ on $S \in [0, 0.9]$, $f^{(2)}(S,t)$ on $S \in [0.9, 1]$, $f^{(3)}(S,t)$ on $S \in [1, 1.1]$, and $f^{(4)}(S,t)$ on $S \in [1.1, \infty)$. Each of the polynomials was fourth order in S and t. (That is, terms such as S^4t^4 were allowed, but not S^5t.)

B. Upper Bounds Results

We first computed the upper bound for an at the money option. That is, we selected $S(0) = 1$. This is the most difficult selection for $S(0)$ in terms of the tightness of the bounds. Thus, the results of this section show the bounds under the most challenging scenario.

Solving the optimization resulted in the polynomial functions $f^{(i)}$, $i = 1, 2, 3, 4$. At expiration, these functions (blue, magenta, black, and cyan) along with the payoff of the option (red) are shown in Figure 1. One can see that when pieced together at the break points 0.9, 1, 1.1, the functions $f^{(i)}$, $i = 1, 2, 3, 4$ form a fairly tight upper bound on the payoff function, especially around $S(0) = 1$. However, note that each polynomial $f^{(i)}$ individually need not be an upper bound on the payoff.

Figure 2 shows the upper bound piecewise polynomial function at time 0. Thus, the piecewise polynomial function is an upper bound on the Black-Scholes price of the option (which is given by the green line). For $S(0) = 1$, the upper bound and the Black-Scholes price are quite close, indicating that the bound is fairly effective. Again, we emphasize that pricing an at-the-money option leads to the *loosest* bound and other initial conditions only lead to tighter bounds. For reference, the upper bound value at-the-money is $UB = 0.07996$ while the Black-Scholes price is 0.0756.

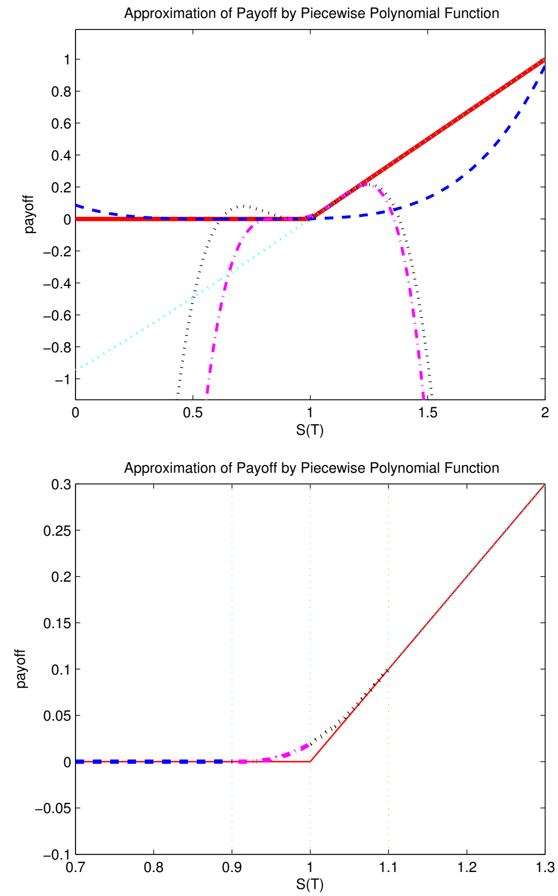

Fig. 1. Piecewise polynomial (fourth order in S and T) upper bound on the call option payoff (red). The upper plot shows the entire polynomial functions (blue, magenta, black, and cyan) that are pieced together to make up the upper bound super-martingale $V(S(T),T)$. The lower plot is a zoomed-in version of the piecewise polynomial function $V(S(T),T)$ where vertical lines show the breakpoints (0.9, 1, and 1.1) at which the polynomial functions from the upper plot are pieced together.

C. Lower Bound Results

Lower bound results are given in Figures 3 and 4. This time Figure 3 shows the piecewise polynomial function creating a lower bound on the payoff of the option. Figure 4 shows the bound on the price at time 0 given by the piecewise polynomial function in reference to the Black-Scholes price shown in green. Again, we see that the piecewise polynomial function provides a reasonably tight lower bound. For reference, the lower bound price at-the-money is $LB = 0.06721$ while the Black-Scholes price is 0.0756.

D. Discussion

To compute these examples, we only used up to fourth order polynomials in S and t. Higher order polynomials lead to tighter bounds but require increased computational effort. SOStools was able to solve the problem using fourth order polynomials in a matter of a few seconds. Thus, for one dimensional problems, computation time was not an issue.

Rather than using polynomials, one may also consider using other functional forms. We tested polynomials in S

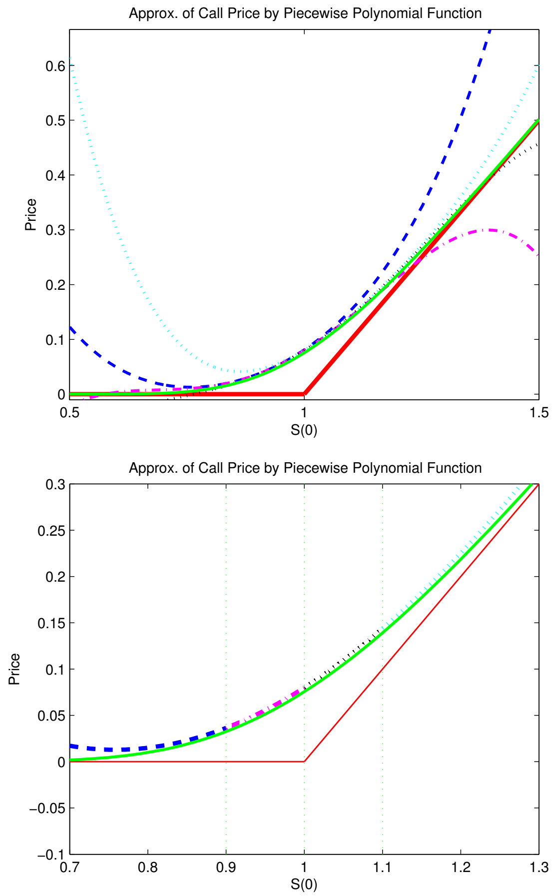

Fig. 2. Upper bound on the price given by the piecewise polynomial (fourth order in S and t) function $V(S(0),0)$. In both plots, the green line is the Black-Scholes price of the option. The upper plot shows the entire polynomial functions (blue, magenta, black, and cyan) that are pieced together to make up the upper bound super-martingale $V(S(0),0)$. The lower plot is a zoomed-in version of the piecewise polynomial function $V(S(0),0)$ where vertical lines show the breakpoints (0.9, 1, and 1.1) at which the polynomial functions from the upper plot are pieced together. The red line is the payoff function for the option and is provided for reference.

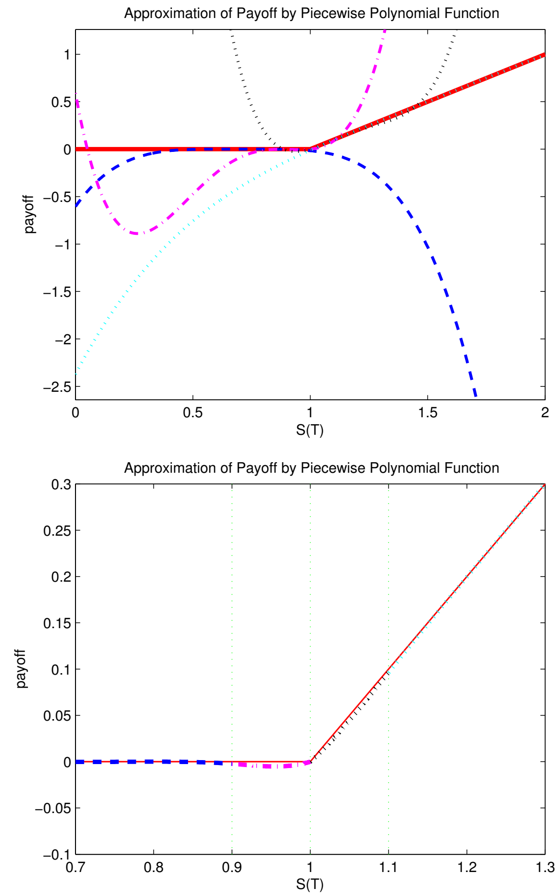

Fig. 3. Piecewise polynomial (fourth order in S and T) lower bound on the call option payoff (red). The upper plot shows the entire polynomial functions (blue, magenta, black, and cyan) that are pieced together to make up the lower bound sub-martingale $V(S(T),T)$. The lower plot is a zoomed-in version of the piecewise polynomial function $V(S(T),T)$ where vertical lines show the breakpoints (0.9, 1, and 1.1) at which the polynomial functions from the upper plot are pieced together.

and exponentials in t. The results were similar to simply using a polynomial in t, so we did not report those results in this paper.

As mentioned previously, this approach also applies to pricing options on more than a single underlying asset, although we do not report results here. Of course, using higher dimensions leads to increased computational times and questions regarding efficient methods for piecing polynomials together.

VII. CONCLUSIONS

In this paper we first derived sufficient conditions for a continuous function to be a super- or sub-martingale. This characterization allowed us to use piecewise polynomial super- and sub-martingales in an optimization to bound the price of an option. By using polynomials, the optimization was formulated as a polynomial programming problem. We used the software package SOStools which replaces the problem with a sum-of-squares program, allowing it to be solved as a semi-definite program.

The results were tested on a Black-Scholes example using four polynomials pieced together to create an upper and lower bound. The results showed that we achieved reasonably tight bounds with four fourth order polynomials, even for an at-the-money option.

VIII. ACKNOWLEDGMENTS

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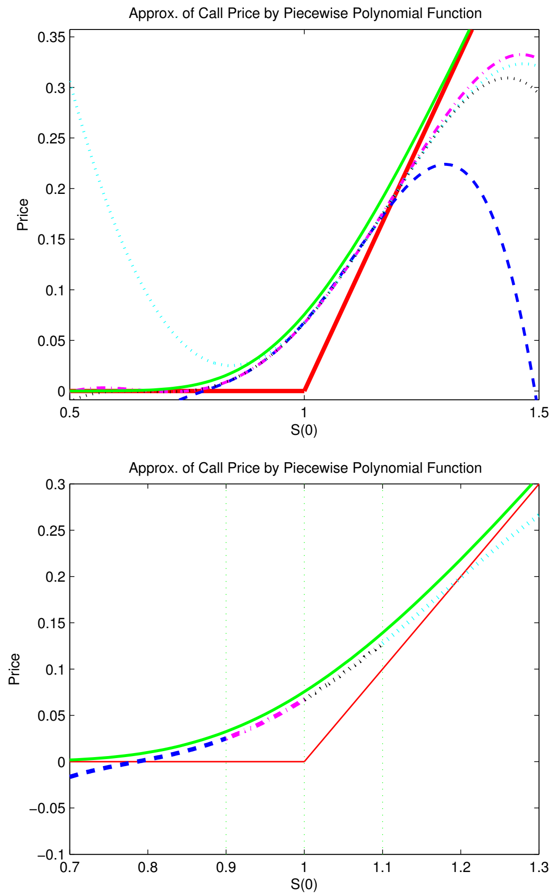

Fig. 4. Lower bound on the price given by the piecewise polynomial (fourth order in S and t) function $V(S(0),0)$. In both plots, the green line is the Black-Scholes price of the option. The upper plot shows the entire polynomial functions (blue, magenta, black, and cyan) that are pieced together to make up the lower bound sub-martingale $V(S(0),0)$. The lower plot is a zoomed-in version of the piecewise polynomial function $V(S(0),0)$ where vertical lines show the breakpoints (0.9, 1, and 1.1) at which the polynomial functions from the upper plot are pieced together. The red line is the payoff function for the option and is provided for reference.

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