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Adaptive Output-Feedback Control of MIMO Plants with Unknown, Time-Varying State Delay

Boris Mirkin, Per-Olof Gutman and Paolo Falcone

Abstract— **In this paper, we develop a simple model reference adaptive control (MRAC) scheme for a class of multi-input-multi-output (MIMO) linear dynamic systems with unknown time-varying state delay which is also robust with respect to an external disturbance with unknown bound. A suitable Lyapunov-Krasovskii type functional with "virtual" gain is introduced to design the adaptation algorithms and to prove stability.**

I. INTRODUCTION

Adaptive control is an effective method for controlling systems with uncertainties and delays. Recent research in this area can be found in [1] – [7] and the references therein.

Adaptive signal tracking based on state feedback can be found in, e.g. [1], and, in the papers [2], [3], [4], [9], based on output feedback. The problem of output reference model signal tracking by state feedback for linear systems with bounded multiple delayed nonlinear state perturbations, and a bounded disturbance was considered in [1], where the reference model was an autonomous dynamic system. The output adaptive tracking control of a class of linear, minimum-phase, single-input-single-output (SISO) and MIMO systems of relative degree one described by functional differential equations was considered in the framework of functional differential inclusions in [2], [3]. Tools from differential geometry were used in [4] for a special type of nonlinear tracking problem. In [9] the backstepping technique was used to form an adaptive control scheme for a class of SISO parametric-strict-feedback nonlinear systems with unknown state time delays whereby the sizes of the unknown time delays are bounded by known constants.

Most of the developed results with output feedback tracking are applicable to SISO systems with known delay. Only small progress was made towards the extension of these ideas to the MIMO case. Recently a new approach, [7], was developed for the output model reference adaptive control of linear continuous-time MIMO plants with *known* state delay. The main idea is to treat the state delay element not as a part of the plant but rather as the input to the system without delay and then decompose the control law into two components.

The present paper addresses further the MIMO MRAC problem considered in our paper [7]. Here we make an initial step in the direction of a more realistic situation, where the plant state delay is *unknown*. To the best of the authors' knowledge, the MIMO output tracking problem within the framework of MRAC was not previously solved for plants with unknown delays. We focus on the case with a single time-varying delay, and propose a simple MRAC scheme which is also robustly stable with respect to an additive bounded disturbance vector where an *a priori* bound on its magnitude is *not* known.

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B. Mirkin and Per-Olof Gutman are with Faculty of Civil and Environmental Engineering, Technion – Israel Institute of Technology, Haifa 32000, Israel (bmirkin),(peo)@technion.ac.il

P. Falcone is with Chalmers University of Technology, 41296 Goteborg, Sweden, paolo.falcone@unisannio.it

II. PROBLEM STATEMENT

In this section we formulate the control problem, including the state delay plant model and the reference model, assumptions and control objective. The uncertain multi-input ($u(t)$) multi-output ($y(t)$) linear continuous-time plant with state delay is of the form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + A_\tau x(t-\tau(t)) + Bu(t) + B\mu(t)\\ y(t) &= Cx(t)\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^m$, $u(t) \in \mathbb{R}^m$ $\mu(t) \in \mathbb{R}^m$ are, respectively, the state, output, control input and disturbance. The constant matrices A, A_τ and B of appropriate dimensions have unknown elements. The time-varying delay $\tau(t)$ is assumed to be *unknown*. $\tau(t)$ is a nonnegative differentiable function, satisfying

$$0 \le \tau(t) \le \tau_{max},\ \dot{\tau}(t) \le \tau^* < 1 \tag{2}$$

where τ_{max} and τ^* are some unknown positive constants. It is also assumed that the states are not accessible and only input-output measurements are available.

The specification is that all signals of the closed loop system remain bounded, and that the plant output $y(t)$ asymptotically exact follows the output $y_r(t)$ of a stable reference model with the transfer function

$$y_r(t) = W_r(s)r(t) \tag{3}$$

where $W_r(s) \in \mathbb{R}^{m\times m}$ is a stable rational transfer matrix, and $r(t) \in \mathbb{R}^m$ is a bounded reference input signal. Asymptotic tracking is demanded, i.e. $\lim_{t\to\infty} e(t) = 0$, with $e(t) = y(t) - y_r(t)$.

The following assumptions are made on the plant (1) and the reference model (3): (**A1**) When there is no term with state delay, the plant (1) can be described by

$$y = W_0(s)u \quad W_0(s) = C(Is-A)^{-1}B \in \mathbb{R}^{m\times m} \tag{4}$$

where $W_0(s)$ is the transfer matrix associated with the undelayed plant. (**A2**) The observability index ν of $W_0(s)$ is known. (**A3**) The transmission zeros of $W_0(s)$ have negative

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real parts (minimum phase plants). (**A4**) $W_0(s)$ is strictly proper, full rank, and has vector relative degree 1, i.e., $rank(CB) = m$. (**A5**) $A_\tau = BA_\tau^{*T}$, where the constant matrix A_τ^* is unknown. (**A6**) Because of the assumption (A4), and without loss of generality, we select a diagonal SPR reference model as

$$W_r(s) = diag\left[\frac{1}{s+a_{ri}}\right] \qquad a_{ri} > 0,\ i = 1,\dots,m. \quad (5)$$

(**A7**) The signs of the leading principal minors of the high frequency gain matrix $K_p = \lim_{s\to\infty} sW_0(s)$ are known. (**A8**) $\|\mu(t)\| \le \mu^*$, where μ^* is unknown.

The minimum phase assumption (A3) is fundamental in MRAC schemes [**?**], [8]. Assumption (A4) focusses on the simplest case amenable to Lyapunov designs. We hope, however, that the our idea of adaptive control parametrization in this note can be extended to higher relative degree. For the case of relative degree greater than two, it is required to use "error augmentation" and/or "tuning error normalization" [8]. Assumption (A5) is a matching condition usual in many MRAC schemes without delay, see again [**?**], [8].

III. Proposed Error Equation Parametrization

Let us assume that all the parameters of (1) are known, and let us define u_1^* as the standard matching control [10], [8] for the plant without delay (4)

$$u_1^*(t) = \theta_e^* y(t) + \theta_1^{*T} x_1(t) + \theta_2^{*T} x_2(t) + \theta_r^* r(t) \quad (6)$$

where

$$x_1 = H_m(s)[u_1^*] \quad x_1 \in \mathbb{R}^{m(\nu-1)} \quad (7)$$

$$x_2 = H_m(s)[y] \quad x_2 \in \mathbb{R}^{m(\nu-1)} \quad (8)$$

$$H_m(s) = \frac{[I_{m\times m}s^{\nu-2}\ \dots\ I_{m\times m}s\ I_{m\times m}]^T}{\Lambda(s)} \quad H_m(s) \in \mathbb{R}^{m(\nu-1)\times m} \quad (9)$$

$\theta_1^* = [\theta_{11}^{*T}, \theta_{12}^{*T}, \dots, \theta_{1\nu-1}^{*T}]^T$, $\theta_2^* = [\theta_{21}^{*T}, \theta_{22}^{*T}, \dots, \theta_{2\nu-1}^{*T}]^T$ with $\theta_{ij}^* \in \mathbb{R}^{m\times m}, i = 1,2; j = 1,\dots,\nu-1$, $\theta_e^* \in \mathbb{R}^{m\times m}$ $\theta_r^* \in \mathbb{R}^{m\times m}$, $\Lambda(s) = s^{\nu-1} + \dots + \lambda_m s + \lambda_0$ is a monic Hurwitz polynomial, and $I_{m\times m} \in \mathbb{R}^{m\times m}$ is the identity matrix.

With the definition of $\Lambda(s)$, $H_m(s)$ and $W_0(s)$ in (4), there exist $\theta_r^* = K_p^{-1}$, θ_e^*, θ_1^* and θ_2^* [10], [8] such that

$$\begin{aligned}&\theta_r^* W_r^{-1}(s) W_0(s) = \\ &I_{m\times m} - \theta_e^* W_0(s) - \theta_1^{*T} H_m(s) - \theta_2^{*T} H_m(s) W_0(s) \quad (10)\end{aligned}$$

When applying (6) to the actual plant (1), then from (1) and (10) and for any u, the tracking error $e = y - y_r$ is given by

$$\begin{aligned} e =& W_r(s)K_p\Big[u - \theta_e^* y - \theta_1^{*T} x_1 - \theta_2^{*T} x_2 - \theta_r^* r + A_\tau^{*T} x(t-\tau(t)) \\ &- \theta_1^{*T} H_m(s) A_\tau^{*T} x(t-\tau(t)) + \hat{\mu}(t)\Big]. \quad (11)\end{aligned}$$

where $\hat{\mu}(t) = \hat{H}_m(s)\mu(t)$ and $\hat{H}_m(s) = I - \theta_1^{*T} H_m(s)$ is a stable matrix, see e.g., [8].

To find a suitable error equation parametrization, we manipulate the term $-\theta_1^{*T} H_m(s) A_\tau^{*T} x(t-\tau(t))$ in (11). First, we introduce a new dynamical system

$$z(t) = \theta_1^{*T} H_m(s)[A_\tau^{*T} x(t-\tau(t))] = \theta_z^{*T} z_x(t) \quad (12)$$

where $\theta_z^{*T} = [\theta_{11}^{*T} A_\tau^{*T}, \theta_{12}^{*T} A_\tau^{*T}, \dots, \theta_{1\nu-1}^{*T} A_\tau^{*T}]$ and

$$z_x(t) = H_n(s)[x(t-\tau(t))] \quad (13)$$

$$H_n(s) = \frac{[I_{n\times n}s^{\nu-2},\ \dots,\ I_{n\times n}s,\ I_{n\times n}]^T}{\Lambda(s)} \quad (14)$$

Here $z_x \in \mathbb{R}^{n(\nu-1)}$, $H_n(s) \in \mathbb{R}^{n(\nu-1)\times n}$ and $I_{n\times n}$ is the $n\times n$ identity matrix.

Remark 1: The transfer function matrix $H_n(s)$ in (14) has the same structure as the transfer matrix $H_m(s)$ in (9), only instead of the identity matrix $I_{m\times m}$ in the numerator of (9) we have the identity matrix $I_{n\times n}$.

Secondly, we decompose the signals z_x in (13) into two components $z_x(t) = z_e(t) + z_r(t)$ where

$$\begin{aligned} z_e(t) =& H_n(s)\big[e_x(t-\tau(t))\big],\ z_r(t) = H_n(s)\big[x_r(t-\tau(t))\big] \\ e_x(t-\tau(t)) =& x(t-\tau(t)) - x_r(t-\tau(t)) \quad (15)\end{aligned}$$

where $x_r(t) \in \mathbb{R}^n$ is the state of the reference model (5) with the state space triple (A_r, B_r, C_r).

Then, using (12) and (15) from (11) we obtain the error equation

$$\begin{aligned} e(t) =& W_r(s)K_p\Big[u(t) - \theta_e^* e(t) - \theta_1^{*T} x_1(t) - \theta_2^{*T} x_2(t) - \theta_r^* r(t) \\ &- \theta_{x_r}^{*T} x_r(t) - \theta_\tau^{*T} x_r(t-\tau(t)) - \theta_z^{*T} z_r(t) \\ &- \theta_\tau^{*T} e_x(t-\tau(t)) + \theta_z^{*T} z_e(t) + \hat{\mu}(t)\Big] \quad (16)\end{aligned}$$

where $\theta_\tau^* = -A_\tau^*$ and $\theta_{x_r}^* = C_r^T \theta_e^{*T}$.

Remark 2: Note that $e_x(t)$ and $z_e(t)$ are not available for measurement and we shall use them only for analysis.

The following two lemmas are useful in the sequel:

Lemma 1 [11]: Every $m\times m$ matrix K_p with nonzero leading principal minors $\Delta_1, \dots \Delta_m$ can be factored as $K_p = SDU$ where S is symmetric positive definite, D is diagonal, and U is unity upper triangular.

Lemma 2 [11]: For any $W_r(s)$ from (5) *a positive definite $S = S^T$ exists such that $W_r(s)S$ is SPR.*

By substituting the high-frequency gain matrix decomposition $K_p = SDU$ in (16), we obtain

$$\begin{aligned} e(t) =& W_r(s)SD\Big[u(t) - (I-U)u(t) - U\theta_e^* e(t) - U\theta_1^{*T} x_1(t) \\ &- U\theta_2^{*T} x_2(t) - U\theta_{x_r}^{*T} x_r(t) - U\theta_\tau^{*T} x_r(t-\tau(t)) \\ &- U\theta_r^* r(t) - U\theta_z^{*T} z_r(t) + U\hat{\mu}(t) \\ &- U\theta_\tau^{*T} e_x(t-\tau(t)) + U\theta_z^{*T} z_e(t)\Big] \quad (17)\end{aligned}$$

By defining $\hat{\theta}_e^* = U\theta_e^*$, $\hat{\theta}_1^{*T} = U\theta_1^{*T}$, $\hat{\theta}_2^{*T} = U\theta_2^{*T}$, $\hat{\theta}_r^* = U\theta_r^*$, $\hat{\theta}_u^* = (I_{m\times m} - U)$, $\hat{\theta}_{x_r}^{*T} = U\theta_{x_r}^{*T}$, $\hat{\theta}_\tau^{*T} = U\theta_\tau^{*T}$, $\hat{\theta}_z^{*T} = U\theta_z^{*T}$, and $\hat{\theta}_z^{*T} = U\theta_z^{*T}$, we obtain from (17) the *basic error equation*

$$\begin{aligned} e(t) =& W_r(s)SD\Big[u(t) - \hat{\theta}_e^* e(t) - \hat{\theta}_1^{*T} x_1(t) - \hat{\theta}_2^{*T} x_2(t) - \hat{\theta}_u^* u(t) \\ &- \hat{\theta}_r^* r(t) - \hat{\theta}_{x_r}^{*T} x_r(t) - \hat{\theta}_\tau^{*T} x_r(t-\tau(t)) - \hat{\theta}_z^{*T} z_r(t) \\ &- \hat{\theta}_\tau^{*T} e_x(t-\tau(t)) + \hat{\theta}_z^{*T} z_e(t) + U\hat{\mu}(t)\Big]. \quad (18)\end{aligned}$$

where the matrix $\hat{\theta}_u^*$ has the specific upper triangular form with zero diagonal elements

$$\hat{\theta}_u^* = \begin{bmatrix} 0 & \theta_u^{*12} & \theta_u^{*13} & \dots & \theta_u^{*1m} \\ 0 & 0 & \theta_u^{*23} & \dots & \theta_u^{*2m} \\ & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \theta_u^{*(m-1)\,m} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (19)$$

IV. Proposed Controller Structure

For the output feedback control problem we look for the adaptive controller in the class defined by

$$\begin{aligned} u(t) =& \theta_e(t)e(t) + \theta_{x1}^T(t)x_1(t) + \theta_{x2}^T(t)x_2(t) + \theta_u(t)u(t) \\ & - \Gamma_I \mathrm{Sgn}(e(t)) \int_0^t |e(t)|\,dt \qquad (20) \\ =& \ \theta^T(t)\omega(t) + \theta_u(t)u(t) - \Gamma_I \mathrm{Sgn}(e(t)) \int_0^t |e(t)|\,dt \end{aligned}$$

where $\theta_{x1}(t),\ \theta_{x2}(t) \in \mathbb{R}^{m(\nu-1)\times m}$, $\theta_e(t) \in \mathbb{R}^{m\times m}$ and $\theta_u(t)$ are the adaptation gain matrices, $x_1 = H_m(s)[u] \in \mathbb{R}^{m(\nu-1)}$, and $x_2 = H_m(s)[y] \in \mathbb{R}^{m(\nu-1)}$, taken from (8)-(9). The diagonal matrix

$$\mathrm{Sgn}(e(t)) = \mathrm{diag}[\mathrm{sgn}(e_1(t))\dots\mathrm{sgn}(e_k(t))\dots\mathrm{sgn}(e_m(t))]$$

with $\mathrm{sgn}(e_k(t)) = 1$, if $e_k(t) > 0$; $\mathrm{sgn}(e_k(t)) = 0$, if $e_k(t) = 0$; and $\mathrm{sgn}(e_k(t)) = -1$, if $e_k(t) < 0$, $k = 1,\dots,m$. *The constant design matrix* Γ_I is a some diagonal matrix with constant entries γ_{Ik}, $k = 1,\dots,m$.

The coefficient matrix $\theta_u(t)$ has a specific upper triangular structure with zero diagonal elements like in [11] for the case without delay, i.e.

$$\theta_u(t) = \begin{bmatrix} 0 & \theta_u^{12}(t) & \theta_u^{13}(t) & \dots & \theta_u^{1m}(t) \\ 0 & 0 & \theta_u^{23}(t) & \dots & \theta_u^{2m}(t) \\ & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \theta_u^{m-1\,m}(t) \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (21)$$

This upper triangular matrix structure guarantees that the control law is implementable without singularity, i.e.,

$$\begin{aligned} u_1(t) =& \theta_1^T(t)\omega(t) + \sum_{k=2}^{m} \theta_u^{1k}(t)u_k - \gamma_{I1}\mathrm{sgn}(e_1) \int_0^t |e_1(t)|\,dt \\ u_2(t) =& \theta_2^T(t)\omega(t) + \sum_{k=3}^{m} \theta_u^{2k}(t)u_k - \gamma_{I2}\mathrm{sgn}(e_2) \int_0^t |e_2(t)|\,dt \\ & \vdots \\ u_m(t) =& \theta_m^T(t)\omega(t) - \gamma_{Im}\mathrm{sgn}(e_m) \int_0^t |e_m(t)|\,dt \qquad (22) \end{aligned}$$

where the vector $\theta_i(t)$ $(i = 1,\dots,m)$ is the ith row of the matrix $\theta(t) = [\theta_e(t)\ \theta_{x1}^T(t)\ \theta_{x2}^T(t)]^T$ and $\omega(t) = [e(t)\ x_1(t)\ x_2(t)]^T$.

If we denote

$$\begin{aligned} \Theta_1(t) =& [\theta_1^T(t)\ \theta_u^{12}(t)\ \theta_u^{13}(t)\ \dots\theta_u^{1m}(t)]^T \\ \Theta_2(t) =& [\theta_2^T(t)\ \theta_u^{23}(t)\ \theta_u^{24}(t)\ \dots\theta_u^{2m}(t)]^T \\ & \vdots \\ \Theta_{m-1}(t) =& [\theta_{m-1}^T(t)\ \theta_u^{(m-1)m}(t)]^T \\ \Theta_m(t) =& [\theta_m^T(t)]^T \qquad (23) \end{aligned}$$

and

$$\begin{aligned} \Omega_1(t) &= [\omega^T(t)\ u_2\ u_3\dots u_{m-1}\ u_m]^T \\ \Omega_2(t) &= [\omega^T(t)\ u_3\dots u_{m-1}\ u_m]^T \\ & \vdots \\ \Omega_m(t) &= [\omega^T(t)]^T \qquad (24) \end{aligned}$$

we can rewrite the control (20) as

$$u(t) = \begin{bmatrix} \Theta_1^T(t)\Omega_1(t) - \gamma_{I1}\mathrm{sgn}(e_1(t)) \int_0^t |e_1(t)|\,dt \\ \vdots \\ \Theta_m^T(t)\Omega_m(t) - \gamma_{Im}\mathrm{sgn}(e_m(t)) \int_0^t |e_m(t)|\,dt \end{bmatrix} \quad (25)$$

Remark 3: The main contribution of the suggested parametrization of the adaptive control law is the introduction the new integral terms $\gamma_{Ik}\mathrm{sgn}(e_k(t)) \int_0^t |e_k(t)|\,dt$, $k = 1,\dots,m$ with *constant* gains γ_{Ik} which is present instead of an adjustable feedforward term driven by the reference signal. When comparing the new control $u(t)$ with our previous work [7], we notice the following main difference: in the adaptive control law all adjusted feedforward terms are deleted. Hence, the number of adaptive parameters is greatly reduced. Moreover, a nice feature is the fact that in the new control parametrization the time delay not only does not increase, but reduces the dimension of the adjusted parameter vector, even in comparison with the corresponding case of plants without delay, see e.g. the textbooks [8], [10]. Missing are the adjustable feedforward term driven by the reference signal $r(t)$, and the full matrix adaptation gain $\theta_r(t)$ found in the traditional adaptive control component $\theta_r(t)r(t)$.

V. Adaption Algorithms and Stability Analysis

A. Adaptation Algorithms

Let the adaptation algorithms be

$$\begin{aligned} \Theta_k(t) =& -\eta_k(0) - \eta_k(t) - \eta_k(t-h) - \int_0^t \eta_k(s)ds, \qquad (26) \\ & \eta_k(t) = \gamma_k \mathrm{sign}(d_k)\Omega_k(t)e_k(t), (k = 1,\dots,m). \end{aligned}$$

In differential form we have

$$\begin{aligned} \dot{\Theta}_k(t) =& -\eta_k(t) - \dot{\eta}_k(t) - \dot{\eta}_k(t-h), \qquad (27) \\ & \eta_k(t) = \gamma_k \mathrm{sign}(d_k)\Omega_k(t)e_k(t), (k = 1,\dots,m) \end{aligned}$$

where $\gamma_k > 0$, $k = 1,\dots,m$ and $h > 0$ are some design parameters and d_k are the entries of D.

Remark 4: For stability and exact asymptotic tracking, only the integral component $\eta_k(t)$ is needed in (27). The use of the proportional $\dot{\eta}_k(t)$ and the proportional delayed

$\dot{\eta}_k(t-h)$ terms in the adaptation algorithm (27) makes it possible, however, to achieve better adaptation performance than the traditional I and PI schemes, see e.g. [12]. This adaptation algorithm approximately realizes a proportional integral derivative (PID) law of parameter adjustment, since the component $\eta(t-h)$ is proportional, on the average over time h, to the derivative $\dot{\eta}(t)$. This follows from the fact that $-\eta(t-h) = \int_{t-h}^{t}\dot{\eta}(s)ds - \eta(t)$. The design parameter h is chosen in the same way as the traditional gains γ_k in (26), (27). The PITD (Proportional-Integral-Time Delay) algorithm (26) includes the traditional I and PI adaptation schemes as special cases.

B. Stability analysis

Now by introducing the parameter error vector $\widetilde{\theta}(t) = \theta(t) - \theta^*$ with $\theta^* = [\hat{\theta}_e^* \ \hat{\theta}_1^{*T} \ \hat{\theta}_2^{*T} \ \hat{\theta}_u^*]^T$ and using the control (20), (25) the equation for the tracking error follows from (18):

$$\begin{aligned} e(t) =& W_r(s)SD\Big[\widetilde{\theta}^T(t)\omega(t) - \Gamma_I \mathrm{Sgn}(e(t))\int_0^t |e(t)|dt \\ & - \hat{\theta}_r^* r(t) - \hat{\theta}_{x_r}^{*T}x_r(t) - \hat{\theta}_z^{*T}z_r(t) - \hat{\theta}_\tau^{*T}x_r(t-\tau(t)) \\ & + U\hat{\mu}(t) - \hat{\theta}_\tau^{*T}e_x(t-\tau(t)) + \hat{\theta}_z^{*T}z_e(t)\Big]. \end{aligned} \tag{28}$$

To design the mechanism of updating the controller matrices, the usual way of MRAC for delay free systems is used, see, e.g. [11]. The augmented vector $\bar{x}(t) = [x \ x_1 \ x_2]^T$ is introduced, and the state of the corresponding non-minimal realization $\bar{C}(sI-\bar{A})^{-1}\bar{B}$ of $W_r(s)S$ is denoted by $\bar{x}_r(t)$. Then in view of (25) we can write the following state space representation for (28)

$$\begin{aligned} \dot{\bar{e}}(t) =& \bar{A}\bar{e}(t) + \bar{B}D\Big(-\hat{\theta}_r^* r(t) - \hat{\theta}_{x_r}^{*T}x_r(t) - \hat{\theta}_\tau^{*T}x_r(t-\tau(t)) \\ & - \hat{\theta}_z^{*T}z_r(t) + U\hat{\mu}(t) - \hat{\theta}_\tau^{*T}e_x(t-\tau(t)) + \hat{\theta}_z^{*T}z_e(t)\Big) \\ & + \bar{B}D\begin{bmatrix} \widetilde{\Theta}_1^T(t)\Omega_1(t) - \gamma_{I1}\mathrm{sgn}(e_1(t))\int_0^t|e_1(t)|dt \\ \vdots \\ \widetilde{\Theta}_m^T(t)\Omega_m(t) - \gamma_{Im}\mathrm{sgn}(e_m(t))\int_0^t|e_m(t)|dt \end{bmatrix} \\ \dot{\bar{z}}_e(t) =& A_e\bar{z}_e(t) + B_e\hat{l}^T\bar{e}(t-\tau(t)) \\ z_e(t) =& C_e\bar{z}_e(t) \\ e(t) =& y(t) - y_r(t) = \bar{C}\bar{e}(t) \end{aligned} \tag{29}$$

where $\bar{e}(t) = \bar{x}e - \bar{x}_re$, the triple (A_e, B_e, C_e) is a minimal state space realization for the stable transfer matrix $H_n(s)$, and $\widetilde{\Theta}_k(t) = \Theta_k(t) - \Theta_k^*$, $k = 1, \dots, m$

Because $\bar{C}(sI-\bar{A})^{-1}\bar{B} = W_r(s)S$ is SPR [11], the triple $(\bar{A}, \bar{B}, \bar{C})$ satisfies the following equations [8]

$$\bar{A}^T\bar{P} + \bar{P}\bar{A} + \hat{Q} = 0 \qquad \bar{P}\bar{B} = \bar{C}^T \tag{30}$$

where $\bar{P} = \bar{P}^T > 0$ and $\hat{Q} = Q_e + Q$. Since A_e in (29) is stable, it also holds that

$$A_e^T P_z + P_z A_e + Q_z = 0 \tag{31}$$

where $P_z = P_z^T > 0$ and $Q_z = Q_z^T > 0$.

For the stability analysis we use the following Lyapunov-Krasovskii type functional

$$\begin{aligned} V =& V_1 + V_2 + V_3 \\ V_1 =& \bar{e}^T\bar{P}\bar{e} + \bar{z}_e^T P_z\bar{z}_e + \tfrac{1}{2}\int_{t-\tau(t)}^t \bar{e}^T(s)Q_e\bar{e}(s)ds \\ V_2 =& \sum_{k=1}^m \gamma_k^{-1}|d_k|\Big(\tilde{\eta}_k^T(t)\tilde{\eta}_k(t) + \int_{t-h}^t \eta_k^T(s)\eta_k(s)ds\Big) \\ V_3 =& \sum_{k=1}^m \sigma_k^{-1}(\beta_k(t) + \beta^*)^2 \end{aligned} \tag{32}$$

where $Q_e = Q_e^T > 0$, $\sigma_k > 0$, $k = 1, \dots, m$ and

$$\tilde{\eta}_k(t) = \tilde{\Theta}_k(t) + \eta_{0k} + \eta_k(t) + \eta_k(t-h) \tag{33}$$

The "artificial" vector

$$\eta_{0k} = r_0(2d_k)^{-1}g_k, \ k = 1, \dots, m \tag{34}$$

with

$$\begin{aligned} g_1 =& [\underbrace{1, 0, \dots, 0}, 0, \dots, 0] \\ g_2 =& [\underbrace{0, 1, \dots, 0}, 0, \dots, 0] \\ & \vdots \\ g_m =& [\underbrace{0, 0, \dots, m}, 0, \dots, 0] \end{aligned}$$

has the same dimension as Θ_k. The "virtual" scalar adaptation gains $\beta_k(t), k = 1, \dots, m$, the selective parameters r_0, σ_k and the positive constant $\beta^* > 0$ will be defined later.

Using (30) and (31), the time derivatives $\dot{V}_1(t)$ and $\dot{V}_3(t)$ of (32) along (29) can be written as

$$\begin{aligned} \dot{V}_1|_{(29)} =& -\bar{e}^T(t)Q\bar{e}(t) - \bar{z}_e^T(t)O_z\bar{z}_e(t) \\ & - (1-\dot{\tau}(t))\bar{e}^T(t-\tau(t))Q_e\bar{e}(t-\tau(t)) \\ & - 2\bar{e}^T(t)\bar{P}\bar{B}D\Big[\hat{\theta}_r^* r(t) + \hat{\theta}_{x_r}^{*T}x_r(t) + \hat{\theta}_\tau^{*T}x_r(t-\tau(t)) \\ & + \hat{\theta}_z^{*T}z_r(t) + U\hat{\mu}(t)\Big] - 2\bar{e}^T(t)\bar{P}\bar{B}D\hat{\theta}_\tau^{*T}\hat{l}^T\bar{e}(t-\tau(t)) \\ & - 2\bar{e}^T(t)\bar{P}\bar{B}D\hat{\theta}_z^{*T}C_e\bar{z}_e(t) + 2\bar{z}_e^T P_z B_e\hat{l}^T\bar{e}(t-\tau(t)) \\ & + 2\sum_{k=1}^m d_k e_k(t)\widetilde{\Theta}_k^T(t)\Omega(t) - 2\sum_{k=1}^m d_k\gamma_{Ik}|e_k|\int_0^t|e_k|dt \\ \dot{V}_3|_{(29)} =& 2\sum_{k=1}^m \sigma_k^{-1}|d_k|(\beta_k(t) + \beta^*)\dot{\beta}_k(t) \end{aligned} \tag{35}$$

with $Q = Q^T = \hat{Q} - Q_e > 0$.

To obtain the time derivative $\dot{V}_2(t)$, we at first note that, in view of (27), the time derivative of $\tilde{\eta}_k(t)$ from (33) is

$$\dot{\tilde{\eta}}_k(t) = \dot{\tilde{\Theta}}_k(t) + \dot{\eta}_k(t) + \dot{\eta}_k(t-h) = -\eta_k(t) \tag{36}$$

Then, using the last expression and (33), we have

$$\begin{aligned} \dot{V}_2(t)|_{(29)} =& -\sum_{k=1}^m \gamma_k^{-1}|d_k|\left\|\left(\eta_k(t) + \eta_k(t-h)\right)\right\|_2^2 \\ & - 2\sum_{k=1}^m \gamma_k^{-1}|d_k|\widetilde{\Theta}_k^T(t)\eta_k(t) \\ & - 2\sum_{k=1}^m \gamma_k^{-1}|d_k|\eta_{k0}^T\Gamma\eta_k(t) \end{aligned} \tag{37}$$

In view of (34), (30) and (26) we can write

$$-2\sum_{k=1}^{m}\gamma_k^{-1}|d_k|\eta_{0k}\eta_k(t) = -\sum_{k=1}^{m} r_0 e_k^2(t) = -r_0\bar{e}^T(t)\bar{P}^T\bar{B}\bar{B}^T\bar{P}\bar{e}(t) \quad (38)$$

Further using (2), (38), (35) and (37) and dropping negative terms we obtain

$$\begin{aligned}\dot{V}|_{(29)} \le & -\bar{e}^T(t)Q\bar{e}(t) - \bar{e}^T(t-\tau(t))\bar{\tau}Q_e\bar{e}(t-\tau(t)) \\ & -\bar{z}_e^T(t)Q_z\bar{z}_e(t) - 2\bar{e}^T(t)\bar{P}\bar{B}D\hat{\theta}_\tau^{*T}\hat{l}^T\bar{e}(t-\tau(t)) \\ & -2\bar{e}^T(t)\bar{P}\bar{B}D\Big[\hat{\theta}_r^* r(t) + \hat{\theta}_{x_r}^{*T}x_r(t) + \hat{\theta}_\tau^{*T}x_r(t-\tau(t)) \\ & + \hat{\theta}_z^{*T}z_r(t) + U\hat{\mu}(t)\Big] - r_0\bar{e}^T(t)\bar{P}^T\bar{B}\bar{B}^T\bar{P}\bar{e}(t) \quad (39) \\ & + 2\bar{z}_e^T P_z B_e \hat{l}^T\bar{e}(t-\tau(t)) - 2\bar{e}^T(t)\bar{P}\bar{B}D\hat{\theta}_z^{*T}C_e\bar{z}_e(t) \\ & -2\sum_{k=1}^{m}\gamma_k^{-1}|d_k|\widetilde{\Theta}_k^T(t)\eta_k(t) - 2\sum_{k=1}^{m}\gamma_k^{-1}|d_k|\eta_{k0}^T\Gamma\eta_k(t)\end{aligned}$$

where $\bar{\tau} = 1-\tau^*$.

Using the known fact that for any vectors x, y, and any positive-definite matrix $G = G^T > 0$ of appropriate dimensions, it holds that $2x^Ty \le x^TGx + y^TG^{-1}y$ by which we can estimate some of the terms in (39) as follows

$$\begin{aligned} 2\bar{e}^T(t)\hat{P}\hat{B}D\hat{\theta}_\tau^{*T}\hat{l}^T\bar{e}(t-\tau) \le & \bar{e}^T(t)\hat{P}\hat{B}\hat{\Psi}_1\hat{B}^T\hat{P}\bar{e}(t) \\ & + \bar{e}^T(t-\tau)G\bar{e}(t-\tau) \\ -2\bar{e}^T(t)\bar{P}\bar{B}D\hat{\theta}_z^{*T}C_e\bar{z}_e(t) \le & \bar{e}^T(t)\hat{P}\hat{B}\hat{\Psi}_2\hat{B}^T\hat{P}\bar{e}(t) \\ & + \bar{z}_e(t)G\bar{z}_e(t) \\ 2\bar{z}_e^T P_z B_e \hat{l}^T\bar{e}(t-\tau) \le & \bar{z}_e^T(t)G\bar{z}_e(t) \\ & + \bar{e}^T(t-\tau)\hat{\Psi}_3\bar{e}(t-\tau) \quad (40)\end{aligned}$$

where $\hat{\Psi}_1 = D\hat{\theta}_\tau^{*T}\hat{l}^TG^{-1}\hat{l}\hat{\theta}_\tau^*D^T$, $\hat{\Psi}_2 = D\hat{\theta}_z^{*T}C_eG^{-1}C_e^T\hat{\theta}_z^*D^T$ and $\hat{\Psi}_3 = \bar{l}B_e^T\hat{P}_zG^{-1}\hat{P}_zB_e\bar{l}^T$.

Let us denote

$$\Big[\star\Big] = \Big[\hat{\theta}_r^* r + \hat{\theta}_{x_r}^{*T}x_r + \hat{\theta}_\tau^{*T}x_r(t-\tau(t)) + \hat{\theta}_z^{*T}z_r + U\hat{\mu}\Big] \quad (41)$$

which can be viewed as "the total bounded input".

Then using boundedness of the reference and disturbance signals and the stability of the transfer functions $H_m(s)$, $H_n(s)$ and $\hat{H}_m(s)$ from (9), (14) and (11) we obtain

$$\begin{aligned}\Big\|\Big[\star\Big]\Big\|_1 \le & \big\|\hat{\theta}_r^* r(t)\big\|_1 + \big\|\hat{\theta}_{x_r}^{*T}x_r(t)\big\|_1 + \big\|\hat{\theta}_\tau^{*T}x_r(t-\tau(t))\big\|_1 \\ & + \big\|\hat{\theta}_z^{*T}z_r(t)\big\|_1 + \|U\hat{\mu}(t)\|_1 \le \beta^* \quad (42)\end{aligned}$$

where $\beta^* > 0$ is a some unknown constant and $\|.\|_1$ denotes a 1-norm.

Using (42) and (30) it can be easily verified that

$$-2\bar{e}^T(t)\bar{P}\bar{B}D\Big[\star\Big] \le 2\big\|e^TD\big\|_1\Big\|\Big[\star\Big]\Big\|_1 \le 2\sum_{k=1}^{m}|e_k|\,|d_k|\,\beta^* \quad (43)$$

Then applying (40) and (43) to (39) yields

$$\begin{aligned}\dot{V}|_{(28)} \le & -\bar{e}^T(t)Q\bar{e}(t) - \bar{z}_e^T(t)\Big[Q_z - 2G\Big]\bar{z}_e(t) \\ & -\bar{e}^T(t-\tau(t))\Big[\bar{\tau}Q_e - G - \hat{\Psi}_3\Big]\bar{e}(t-\tau(t)) \\ & -\bar{e}^T(t)\bar{P}^T\bar{B}\Big[r_0I - \hat{\Psi}_1 - \hat{\Psi}_2\Big]\bar{B}^T\bar{P}\bar{e}(t) \\ & -2\sum_{k=1}^{m}|e_k|\,d_k\gamma_{Ik}\int_0^t|e_k|\,dt + 2\sum_{k=1}^{m}|e_k|\,|d_k|\,\beta^* \\ & + 2\sum_{k=1}^{m}\sigma_k^{-1}\,|d_k|\,(\beta_k(t)+\beta^*)\,\dot{\beta}_k(t) \quad (44)\end{aligned}$$

Let us define $\gamma_{Ik} = \sigma_k \mathrm{sgn}(d_k)$, and the "virtual" adaptation gain $\beta(t)$ in (32) as

$$\dot{\beta}_k(t) = -\sigma_k\,|e_k(t)|\,, \quad \beta_k(0) = 0,\ k = 1,\ldots,m. \quad (45)$$

For convenience, let us define $\bar{\tau}Q_e = Q_{e1} + Q_{e2}$ and $Q_z = Q_{z1} + Q_{z2}$ with $Q_{e1} = Q_{e1}^T > 0$, $Q_{e2} = Q_{e2}^T > 0$, $Q_{z1} = Q_{z1}^T > 0$ and $Q_{z2} = Q_{z2}^T > 0$, and select values for r_0, Q_e and Q_z of (44) from the inequalities

$$\begin{aligned} r_0 > & \lambda_{max}\big(\hat{\Psi}_1 + \hat{\Psi}_2\big)\,,\ \lambda_{min}(Q_{z2}) > \lambda_{max}(2G)\,, \\ \lambda_{min}(Q_{e2}) > & 2\lambda_{max}(G + \Psi_3) \quad (46)\end{aligned}$$

where $\lambda_{min}(\bullet)$ and $\lambda_{max}(\bullet)$ are the minimum and maximum eigenvalues of $(\bullet)$ respectively.

Then we obtain from (44)

$$\begin{aligned}\dot{V} \le & -\bar{e}^T(t)Q\bar{e}(t) - \bar{e}^T(t-\tau(t))\bar{\tau}Q_{e1}\bar{e}(t-\tau(t)) \\ & -\bar{z}_e^T(t)Q_{z1}\bar{z}_e(t) \le 0 \quad (47)\end{aligned}$$

This implies [13] that V and, therefore, $\bar{e}(t)$, $e(t)$, $\bar{z}_e(t)$, $\widetilde{\Theta}_k \in \mathsf{L}_\infty$. The remainder of the stability analysis follows directly using the steps in [8]. Because $\bar{e}(t) = \bar{x}(t) - \bar{x}_r(t)$ and $\bar{x}_r(t) \in L_\infty$, it holds that $\bar{x}(t)$=$[x^T(t), x_1^T(t), x_2^T(t)]^T \in L_\infty$, which implies that $x(t), x_1(t), x_2(t)$ and $y(t) \in L_\infty$. Since $r(t)$ is uniformly bounded and the transfer matrix $H_n(s)$ from (15) is stable, $\Omega_k(t)$, $k = 1,\ldots,m$ are bounded and $u(t)$ is bounded. Therefore, all the signals in the closed-loop system are bounded. From (32) and (47) we establish that $\bar{e}(t)$ and therefore $e(t) \in L_2$. Furthermore, using $\bar{e}(t), \bar{z}_e(t), \Theta_k(t)$, $\Omega_k(t) \in L_\infty$ in (28) we have that $\dot{\bar{e}}(t), \dot{e}(t) \in L_\infty$. Hence, $e \in L_2 \bigcap L_\infty$, and $\dot{e}(t) \in L_\infty$, which by Barbălat's Lemma [8] implies that $e(t) \to 0$ as $t \to \infty$.

Remark 5: We note that the coefficient matrices Q, Q_z and Q_e are used only for analysis and do not influence the control law. Controller gains adjust automatically to counter the non-desirable effects of delayed state and parameter uncertainties.

VI. Example

To illustrate the application of the proposed adaptive scheme, let us consider an unstable system described by the model

$$\dot{x}(t) = \begin{bmatrix} 1.0 & 0 \\ 0 & 1.0 \end{bmatrix} x(t) + \begin{bmatrix} 2.0 & 3.0 \\ 4.0 & 5.0 \end{bmatrix} x(t-\tau)$$
$$+ \begin{bmatrix} -1.0 & 2.0 \\ 3.0 & 1.0 \end{bmatrix} u(t)$$
$$+ \begin{bmatrix} -1.0 & 2.0 \\ 3.0 & 1.0 \end{bmatrix} \begin{bmatrix} 2sin(3t) \\ 3sin(0.3t) \end{bmatrix}$$
$$y(t) = \begin{bmatrix} 1.0 & 0 \\ 0 & 1.0 \end{bmatrix} x(t), \quad x(0) = \begin{bmatrix} 1.0 \\ -1.0 \end{bmatrix} \tag{48}$$

To build the adaptive controller we choose the reference model

$$\dot{x}_r(t) = \begin{bmatrix} -1.0 & 0 \\ 0 & -1.0 \end{bmatrix} x_r(t) + \begin{bmatrix} 1.0 & 0 \\ 0 & 1.0 \end{bmatrix} r(t)$$
$$y_r(t) = \begin{bmatrix} 1.0 & 0 \\ 0 & 1.0 \end{bmatrix} x_r(t), \quad x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{49}$$

The simulation results was performed with the adaptive control

$$u_1(t) = \Theta^T(t)\Omega(t) + \theta_u(t)u_2(t) - \gamma_{I1}\mathrm{sgn}(e_1(t)) \int_0^t |e_1(t)|\, dt$$
$$u_2(t) = \Theta^T(t)\Omega(t) - \gamma_{I2}\mathrm{sgn}(e_2(t)) \int_0^t |e_2(t)|\, dt$$
$$\Omega(t) = [e_1(t)\ e_2(t)]^T \tag{50}$$

and update laws

$$\dot{\Theta}_1(t) = -\sigma_1 \mathrm{sign}(d_1)\Omega(t)e_1(t)$$
$$\dot{\theta}_u(t) = -\sigma_1 \mathrm{sign}(d_1)u_2(t)e_1(t)$$
$$\dot{\Theta}_2(t) = -\sigma_2 \mathrm{sign}(d_1)\Omega(t)e_2(t) \tag{51}$$

The some simulation results are shown in Figure1, where we show the robustness of the adaptive control law to delay uncertainty.

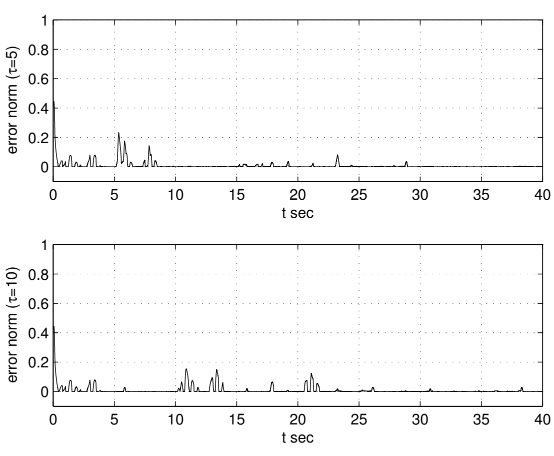

Fig. 1. Simulation of the adaptive control system for the plant with state delay (48) and the controller (25), (26). The upper and lower graphs show the time history of the error norm for the plant delays $\tau = 5$ and $\tau = 10$, respectively.

Figure1 was generated by the plant model (48) and the controller (50), (51) with

$$r(t) = \begin{bmatrix} sin(t) + 2sin(3t) + 3sin(0.5)t \\ cos(0.5t) + 2cos(2t) + 3cos(0.3t) \end{bmatrix}$$

The parameter values of the controller are $\mathrm{sign}(d_1) = -1$, $\mathrm{sign}(d_2) = 1$, $\sigma_1 = 5$, $\sigma_2 = 10$, $\gamma_{I1} = -5$ and $\gamma_{I2} = 10$.

VII. Concluding Remarks

A simple output feedback adaptive control is developed for a class of linear multi-input multi-output (MIMO) systems with time-varying state delay. An effective controller structure, based on a new error equation parametrization, is proposed to achieve tracking of reference signals with asymptotical zero error. To achieve robust properties both with respect to an *unknown plant delay* and to an external disturbance vector with *unknown bounds*, we introduce a new controller parametrization with a new integral control component and a diagonal constant gain matrix. The proposed adaptive control law constructions make economical use of known results from model reference adaptive control to the considered class of delayed system. Adaptive laws are developed using a suitable Lyapunov-Krasovskii type functional. To the authors' knowledge, this is the first work that applies output MRAC design to MIMO systems with unknown constant or time-varying delay.

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