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Recursive predictor design for linear systems with time delay

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Abstract

This paper presents a recursive control design method for multi-input, block-feedforward linear systems with delays in the input and in the interconnections between the state blocks. The controller is of predictor type, which means that it contains finite integrals over past state values. This design method is a generalization of the well known model reduction approach for systems with input delay. A recursive procedure replaces delay terms with non-delay ones step-by-step from the top of the cascade structure down. Controller gains are computed for the proxy system without delays, while the construction guarantees the same closed loop poles for the delay system and the proxy one.

1. Introduction

In this paper we develop a recursive method to design controllers for linear, block-feedforward systems with input and state delays. The method is a generalization of the well known approach to control systems with input, but no state delay of the form

$$\dot{x}(t) = Ax(t) + \sum_{i=0}^{l} B_i u(t - \tau_i) \tag{1.1}$$

where $x \in \Re^n$ and $u \in \Re^m$. For this class of systems, the problem can be reduced to control design for the system without delay,

$$\dot{x}(t) = Ax(t) + B_d u(t) \ , \quad B_d = \sum_{i=0}^{l} e^{-A\tau_i} B_i \tag{1.2}$$

The feedback control $u = -Kx$ for (1.2) can now be obtained by a control design method of choice, assuming that the pair (A, B_d) is stabilizable. The fact that the spectrum of $A - B_d K$ coincides with that of (1.1) with the control

$$u(t) = -K \left(x(t) + \sum_{i=0}^{l} \int_0^{\tau_i} e^{-A\theta} B_i u(t + \theta - \tau_i) d\theta \right) \tag{1.3}$$

provides the stabilizing feedback for (1.1) and a finite spectrum assignment for the closed loop system (see [5]). This method applies even if there is a distributed delay in u and/or the matrices A and B_i are time varying [5, 2]. Note that when there is only one delay τ, a simple manipulation shows that $u(t) = -K_A \tilde{x}(t + \tau | t)$, where $K_A = K e^{-A\tau}$ and $\tilde{x}(t+\tau|t)$ is the predicted value of the state x at time $t+\tau$, based on the information up to time t (values of u applied after t do not impact $x(t+\tau)$ because of the input delay). For this reason the control law of the form (1.3) is often referred to as "predictor-like" or "predictor-type" while the method is known as the model reduction (see [3], Section 4.2 and the references therein).

The model reduction technique does not work if state delay is also present. Indeed, control design for systems with state and input delays has been identified in [8] as one of the remaining largely open problems. In this paper, state delay is allowed under a structural constraint on delay dependent terms. That is, we consider multi-input systems with state and input delays having the following block-feedforward structure:

$$\dot{x} = \begin{bmatrix} A_1 & * & * & \dots & * \\ 0 & A_2 & * & \dots & * \\ 0 & 0 & A_3 & \dots & * \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \dots & A_{p-1} \end{bmatrix} x + \begin{bmatrix} * \\ * \\ * \\ \vdots \\ * \end{bmatrix} u \tag{1.4}$$

The entries "$*$" designate delayed terms. In other words, if we denote delay operators by μ_i (that is, $\mu_i x(t) = x(t - \tau_i), i = 0, \dots, l$), a "$*$" in (1.4) denotes a term of the form $\sum_{i=0}^{l} D_i \mu_i$ for some matrices D_i of appropriate dimensions. The method of this paper also works for distributed delays, but, to reduce notational complexity, only discrete delays are considered. Just like the model reduction method, our generalized predictor imposes no restrictions on the matrices A_i (i.e. they are allowed to have unstable modes) or on the delays τ_i (i.e. they need not be commensurate).

To design a controller for (1.4), we propose a recursive method, based on the spectrum equivalence result (observation) from [4]. In this paper the result is reinterpreted in a form that allows removal of delays from subsystems that grow larger at each recursion step. The delays are replaced by a matrix exponent factor in a fashion similar to the model reduction technique. The matrix exponent at each step depends on the matrix exponents from previous steps. After $p - 1$ steps, all the delays will have been removed and a "proxy" non-delay system obtained. A controller for the proxy system can be designed using one of the standard techniques (pole placement, LQR, H_∞, etc). The spectra for the orig-

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inal delay system and the delay-free proxy system are made the same by augmenting the proxy controller with predictor integrals similar to those in (1.3). Certain robustness properties of the delay and the proxy systems turned out to be equivalent as shown in Remark 2 and Section 4.

The paper is organized as follows. Section 2 provides the spectrum equivalence result that will be used in the recursive method. The recursive predictor design is presented in Section 3. Section 4 contains an example with simulation results.

2. Spectral equivalence result

The method proposed in this paper is based in the spectrum equivalence result of [4]. In its original form the result applies to linear systems of the form

$$\begin{aligned}\dot{x}(t) &= Fx(t) + \sum_{i=0}^{l} H_i\xi(t-\tau_i)\\ \dot{\xi}(t) &= A\xi(t) + Bu(t)\end{aligned} \tag{2.1}$$

where $x \in R^{n_x}$, $\xi \in R^{n_\xi}$, $u \in R^m$ and $0 = \tau_0 < \tau_1 < \ldots < \tau_l$. In [4] it was shown that any stabilizing control $v_0 = -K_x x - K_\xi \xi$ for the cascade system with no delay

$$\begin{aligned}\dot{x} &= Fx + \textstyle\sum_0^l H_i e^{-F\tau_i}\xi\\ \dot{\xi} &= A\xi + Bv_0\end{aligned} \tag{2.2}$$

provides a control law

$$u = -K_x\left(x + \sum_{i=0}^{l}\int_0^{\tau_i} e^{-F\theta}H_i\xi(t+\theta-\tau_i)d\theta\right) - K_\xi\xi \tag{2.3}$$

that stabilizes (2.1). Moreover, the two systems – (2.2) with the control v_0 and (2.1) with the control (2.3) – have the same closed loop poles.

For this paper we need the result extended to the class of systems

$$\begin{aligned}\dot{x}(t) &= Fx(t) + \sum_{i=0}^{l} H_i\xi(t-\tau_i)\\ \dot{\xi}(t) &= \mathcal{A}\xi_d(t) + Bu(t)\end{aligned} \tag{2.4}$$

where $\mathcal{A}$ is a linear functional acting on the present and past values of ξ ($\xi_d(t)$ denotes the state trajectory over the interval $[t-r, t]$, $r \geq \tau_l$) of the form

$$\mathcal{A}\xi_d(t) = \sum_{i=0}^{l} A_i\xi(t-\tau_i) + \int_0^r \Lambda(\theta)\xi(t-\theta)d\theta$$

The control law we consider is also modified – instead of the simple gain matrix K_ξ in (2.3) we employ a functional $\mathcal{K}_\xi$ having the same form as $\mathcal{A}$:

$$\mathcal{K}_\xi\xi_d(t) = \sum_{i=0}^{l} K_{\xi i}\xi(t-\tau_i) + \int_0^r \Upsilon_\xi(\theta)\xi(t-\theta)d\theta$$

With these changes, the closed loop system takes the form

$$\begin{aligned}\dot{x}(t) &= Fx(t) + \sum_{i=0}^{l} H_i\xi(t-\tau_i)\\ \dot{\xi}(t) &= -BK_x\left(x(t) + \sum_{i=0}^{l}\int_0^{\tau_i} e^{-F\theta}H_i\xi(t+\theta-\tau_i)d\theta\right)\\ &\quad +(\mathcal{A} - B\mathcal{K}_\xi)\xi_d(t)\end{aligned} \tag{2.5}$$

With the Laplace transform of $\int_0^{\tau_i} e^{-F\theta}H_i\xi(t+\theta-\tau_i)d\theta$ given by $(sI-F)^{-1}(e^{-F\tau_i} - e^{-s\tau_i}I)H_i\xi(s)$, the characteristic polynomial of (2.5) is

$$\chi(s) = \det\begin{bmatrix} sI - F & -\sum_0^l H_i e^{-s\tau_i}\\ BK_x & \begin{matrix} sI - \mathcal{A}(s) + B\mathcal{K}_\xi(s) + BK_x\\ \times(sI-F)^{-1}\sum_0^l (e^{-F\tau_i} - e^{-s\tau_i}I)H_i\end{matrix}\end{bmatrix} \tag{2.6}$$

where $\mathcal{A}(s) = \sum_{i=0}^{l} A_i e^{-\tau_i s} + \int_0^r \Lambda(\theta)e^{-\theta s}d\theta$ and $\mathcal{K}_\xi(s) = \sum_{i=0}^{l} K_{\xi i}e^{-\tau_i s} + \int_0^r \Upsilon(\theta)e^{-\theta s}d\theta$. Using the well known identity for determinants of block matrices

$$\det\begin{bmatrix} A & B\\ C & D\end{bmatrix} = \det A\ \det(D - CA^{-1}B)$$

we find that the terms with $e^{-s\tau_i}H_i$ in $\chi(s)$ cancel out and we have

$$\begin{aligned}\chi(s) &= \det(sI-F)\ \det(sI - \mathcal{A}(s) + B\mathcal{K}_\xi(s)\\ &\quad + BK_x(sI-F)^{-1}\textstyle\sum_0^l e^{-F\tau_i}H_i) =\\ &= \det\begin{bmatrix} sI - F & -\sum_0^l e^{-F\tau_i}H_i\\ BK_x & sI - \mathcal{A}(s) + B\mathcal{K}_\xi(s)\end{bmatrix}\end{aligned} \tag{2.7}$$

In other words, the "predictor-type" feedback term $K_x\left(x + \sum_0^l \int_0^{\tau_i} e^{-F\theta}H_i\xi(t+\theta-\tau_i)d\theta\right)$ compensates for the effect of delays in the x-dynamics and the design can proceed as if the interconnection term is $\sum_0^l e^{-F\tau_i}H_i\xi$, that is, without time delay. The method that handles delays in $\mathcal{A}(s)$ and $\mathcal{K}_\xi(s)$ will be described in the next section.

3. Recursive predictor design

Starting with the system given by (1.4), we augment the state vector by adding an integrator at each input to convert input delays into state delays. The approach also works without state augmentation, but by adding integrators we avoid a major source of sensitivity for the subsequently designed closed loop system (see Remark 1 below). Thus, we consider a block feedforward structure with delays in the interconnections between blocks:

$$\begin{aligned}
\dot{z}_1 &= A_1 z_1 + \textstyle\sum_{i=0}^{l} D_{1i}\bar{z}_2(t-\tau_i)\\
\dot{z}_2 &= A_2 z_2 + \textstyle\sum_{i=0}^{l} D_{2i}\bar{z}_3(t-\tau_i)\\
&\vdots\\
\dot{z}_{p-1} &= A_{p-1} z_{p-1} + \textstyle\sum_{i=0}^{l} D_{p-1i}\bar{z}_p(t-\tau_i)\\
\dot{z}_p &= A_p z_p + Bv
\end{aligned} \tag{3.1}$$

Each block state z_j belongs to $\Re^{n_j}$, $\bar{z}_j^T = [z_j^T, \ldots, z_p^T]^T$ and, as before, the delays $0 = \tau_0 < \tau_1 < \ldots < \tau_l$ are not assumed to be commensurate. State augmentation with integrators results in $A_p = 0$ and $B = I$, but we consider the general case because the control design method makes no use of the special structure of the z_p-subsystem.

Now we apply the spectrum equivalence result of Section 2 to construct a feedback that achieves a finite spectrum for the closed loop system. The design process consists of several stages. First, starting at the top (z_1 block), the delays τ_i are removed recursively from the top down and replaced with $e^{-F_j\tau_i}$ factors (index j corresponds to the step number and the matrices F_j are constructed recursively). When all delays are removed/replaced, the second stage consists of designing a control law for the proxy (non-delay) system. In the third stage, this linear control law is augmented with a set of predictor integrals using the matrices F_j generated by the recursive procedure.

To start the recursion we compare (2.4) to (3.1) and note that they are in the same form if we equate z_1 with the state x, $\bar{z}_2 = (z_2, \ldots, z_p)$ with ξ, $F_1 = A_1$, and $H_{1i} = D_{1i}$. Now, following the spectral equivalence approach, we replace the delays in the z_1 dynamics with $e^{-F_1\tau_i}$:

$$\begin{aligned}
\dot{x}_1 &= F_1 x_1 + \textstyle\sum_{i=0}^{l} e^{-F_1\tau_i} H_{1i}\bar{z}_2\\
\dot{z}_2 &= A_2 z_2 + \textstyle\sum_{i=0}^{l} D_{2i}\bar{z}_3(t-\tau_i)\\
&\vdots\\
\dot{z}_p &= A_p z_p + Bv
\end{aligned} \tag{3.2}$$

Note that now the block of states z_2 does not appear delayed any more in the dynamics of the system. In the subsequent steps we shell remove the delays from the blocks z_3, z_4, etc, down to z_p.

In the second step, the pair (z_1, z_2) is equated to the top of the cascade structure in (2.4), $x_2 = (z_1, z_2)$, and $\xi = \bar{z}_3$. With the partition of $\sum_0^l e^{-F_1\tau_i} H_{1i} = [E_{12} \ldots E_{1p}]$ corresponding to that of $\bar{z}_2$ we define

$$F_2 = \begin{bmatrix} F_1 & E_{12} \\ 0 & A_2 \end{bmatrix} \tag{3.3}$$

$$H_{20} = \begin{bmatrix} E_{13} \ldots E_{1p} \\ D_{20} \end{bmatrix} \quad \text{and} \quad H_{2i} = \begin{bmatrix} 0 \\ D_{2i} \end{bmatrix} \quad 1 \le i \le l \tag{3.4}$$

Again, we replace the delay terms from the second block with matrix exponents to obtain

$$\begin{aligned}
\dot{x}_2 &= F_2 x_2 + \textstyle\sum_{i=0}^{l} e^{-F_2\tau_i} H_{2i}\bar{z}_3\\
\dot{z}_3 &= A_3 z_3 + \textstyle\sum_{i=0}^{l} D_{3i}\bar{z}_3(t-\tau_i)\\
&\vdots\\
\dot{z}_p &= A_p z_p + Bv
\end{aligned} \tag{3.5}$$

Following this procedure we recursively construct matrices F_j and H_{ji} and replace all delays by matrices $e^{-F_j\tau_i}$. After $p-1$ steps we obtain a system with no delays:

$$\begin{aligned}
\dot{x}_{p-1} &= F_{p-1} x_{p-1} + \textstyle\sum_{i=0}^{l} e^{-F_{p-1}\tau_i} H_{p-1i}\bar{z}_p\\
\dot{z}_p &= A_p z_p + Bv
\end{aligned} \tag{3.6}$$

If the pair of matrices

$$F_p = \begin{bmatrix} F_{p-1} & \sum_{i=0}^{l} e^{-F_{p-1}\tau_i} H_{p-1i} \\ 0 & A_p \end{bmatrix}, \quad H_p = \begin{bmatrix} 0 \\ B \end{bmatrix}$$

is controllable (or at least stabilizable), a control law

$$v_p = -Kz = -K_1 z_1 - K_2 z_2 - \ldots - K_p z_p \tag{3.7}$$

that stabilizes the system (3.6) can be found using any of the well established methods for linear (non-delay) systems. Using the feedback gains from (3.7) we directly obtain the control law for (3.1) by using the formula

$$\begin{aligned}
v = -\sum_{j=1}^{p} K_j z_j - \sum_{j=1}^{p-1} \hat{K}_j \sum_{i=0}^{l} \int_0^{\tau_i} e^{-F_j\theta}&\\
\times H_{ji}\, \bar{z}_{j+1}(t+\theta-\tau_i) d\theta&
\end{aligned} \tag{3.8}$$

where $\hat{K}_j = [K_1 \; K_2 \; \ldots \; K_j]$. The stability of the closed loop system is assured by the following result.

Proposition 1 The closed loop system (3.1), (3.8) has the same finite spectrum as the non-delay system (3.6) with the control (3.7).

Proof: The spectrum of the closed loop system (3.6), (3.7) is given by the roots of the characteristic polynomial $\chi(s)$:

$$\chi(s) = \det \begin{bmatrix} sI - F_{p-1} & -\sum_{i=0}^{l} e^{-F_{p-1}\tau_i} H_{p-1i} \\ B\hat{K}_{p-1} & sI - A_p + BK_p \end{bmatrix} \tag{3.9}$$

Now we go back from the bottom to the top, in each step restoring delay terms (instead of $e^{-F_j\tau_i}$ factors) and augmenting the control law with predictor terms. Thus, in the first bottom-up step we replace $e^{-F_{p-1}\tau_i} H_{p-1i} z_p$ with $H_{p-1i} z_p(t-\tau_i)$ and augment the control law v_p,

$$v_{p-1} = v_p - \hat{K}_{p-1} \sum_{i=0}^{l} \int_0^{\tau_i} e^{-F_{p-1}\theta} H_{p-1i} z_p(t+\theta-\tau_i) d\theta$$

to obtain

$$\begin{aligned}\dot{x}_{p-1} &= F_{p-1}x_{p-1} + \textstyle\sum_{i=0}^{l} H_{p-1i}z_p(t-\tau_i) \\ \dot{z}_p &= -B\hat{K}_{p-1}\Big(x_{p-1} + \textstyle\sum_{i=0}^{l}\int_0^{\tau_1} e^{-F_{p-1}\theta} \\ &\quad \times H_{p-1i}z_p(t+\theta-\tau_i)d\theta) + A_{pc}z_p\end{aligned} \tag{3.10}$$

where $A_{pc} = A_p - BK_p$. The characteristic polynomial of (3.10) is

$$\chi_{p-1}(s) = \det\begin{bmatrix} sI - F_{p-1} & -\sum_0^l H_{p-1i}e^{-s\tau_i} \\ B\hat{K}_{p-1} & \begin{matrix} sI - A_{pc} + B\hat{K}_{p-1}(sI-F_{p-1})^{-1} \\ \times \sum_0^l (e^{-F_{p-1}\tau_i} - e^{-s\tau_i}I)H_{p-1i}\end{matrix}\end{bmatrix} \tag{3.11}$$

Direct application of the spectral equivalence result provides $\chi_{p-1}(s) = \chi(s)$.

For the second bottom-up step note that delayed z_p states only affect the lowest z_{p-1} block of the x_{p-1}-dynamics in (3.10) (see (3.4) for the form of matrices H_{ji} and note that H_{j0} multiplies states with no delay). Based on this observation we rewrite (3.10) as

$$\begin{aligned}\dot{x}_{p-2} &= F_{p-2}x_{p-2} + \textstyle\sum_{i=0}^{l} e^{-F_{p-1}\tau_i}H_{p-2i}\bar{z}_{p-1} \\ \dot{\bar{z}}_{p-1} &= -B\hat{K}_{p-2}x_{p-2} + \mathcal{A}_{p-1}\bar{z}_{p-1d}\end{aligned} \tag{3.12}$$

where

$$\mathcal{A}_{p-1}\bar{z}_{p-1d} = \begin{bmatrix} A_{p-1}z_{p-1} + \sum_{i=0}^{l} D_{p-1i}z_p(t-\tau_i) \\ (A_p - BK_p)z_p - BK_{p-1}z_{p-1} - B\hat{K}_{p-1} \\ \times\sum_0^l\int_0^{\tau_i} e^{-F_{p-1}\theta}H_{p-1i}z_p(t+\theta-\tau_i)d\theta\end{bmatrix}$$

Thus, as in the (bottom-up) step one, we replace $e^{-F_{p-1}\tau_i}H_{p-2i}\bar{z}_{p-1}$ with $H_{p-2i}\bar{z}_{p-1}(t-\tau_i)$ and augment the control law with

$$v_{p-2} = v_{p-1} + \hat{K}_{p-2}\sum_{i=0}^{l}\int_0^{\tau_i} e^{-F_{p-2}\theta}H_{p-2i}\bar{z}_{p-1}(t+\theta-\tau_i)d\theta \tag{3.13}$$

The characteristic polynomial of this new system is

$$\chi_{p-2}(s) = \det\begin{bmatrix} sI - F_{p-2} & -\sum_0^l H_{p-2i}e^{-s\tau_i} \\ B\hat{K}_{p-2} & \begin{matrix} sI - \mathcal{A}_{p-1}(s) + B\hat{K}_{p-2}(sI-F_{p-2})^{-1} \\ \times \sum_0^l (e^{-F_{p-2}\tau_i} - e^{-s\tau_i}I)H_{p-2i}\end{matrix}\end{bmatrix}$$

Using the spectral equivalence result we have that $\chi_{p-2}(s)$ is equal to the characteristic polynomial of (3.12). Then, because (3.12) is just a repackaged version of (3.10), $\chi_{p-2}(s) = \chi_{p-1}(s) = \chi(s)$. Continuing with the bottom up procedure we obtain that the characteristic polynomial of the closed loop system (3.1), (3.8) is equal to $\chi(s)$ and, thus, the system is stable and has a finite spectrum. This completes the proof of the proposition.

Remark 1 The approach described above can be applied directly (without the addition of integrators at each input) by using [5] Proposition 2.1. The set of determinants $(\chi(s), \chi_{p-1}(s), \ldots)$ would have the term "sI" in the (p,p) block entry replaced by "I". The spectral equivalence results remains unaffected by this change.

However, without the integrators, the model reduction approach may be very sensitive to round-off errors in the implementation (see [8], Section 6.2 and the references therein). The source of this sensitivity is the additional dynamics created by dependence of u on the integrals of past values of u (see equation (1.3)). Addition of integrators (input filters [6]) avoids this problem for the model reduction and for this method.

Remark 2 In the special case when each block and the input u in (3.1) are one-dimensional, it is easy to establish a relationship between robustness properties for the closed loop system (3.1), (3.8) and its non-delay counterpart (3.6), (3.7).

One familiar robustness measure is given in terms of the gain and phase margins obtained by cutting the loop at the plant input and considering the transfer function from the plant input to the controller output (without the minus sign). For the proxy system (3.6), (3.7) the (open) loop transfer function is given by

$$G_p(s) = k(sI - F_p)^{-1}H_p \tag{3.14}$$

where F_p and $H_p = [0\;0\ldots b]^T$ are the state matrix and the input vector in (3.6). The loop transfer function for the delay system (3.1), (3.8) is

$$G_d(s) = \kappa(s)(sI - \mathcal{A}(s))^{-1}H_p \tag{3.15}$$

where $\mathcal{A}(s)$ is the Laplace transform of the state matrix in (3.1), H_p is the input vector (the same as in (3.14)), and $\kappa(s)$ is the Laplace transform of the feedback functional (3.8). Because of the upper triangular system structure $\det(sI - F_p) = \det(sI - \mathcal{A}(s)) = \prod_{j=1}^{p}(sI - A_j)$ and, from the main result of this paper, $\det(sI - F_p + H_pk) = \det(sI - \mathcal{A}(s) + H_p\kappa(s))$. Thus, we can write

$$\begin{aligned}G_d(s) &= \det(sI - \mathcal{A}(s))^{-1}\ \kappa(s)\ \mathrm{adj}(sI - \mathcal{A}(s))H_p \\ &= \frac{\det(sI - \mathcal{A}(s) + H_p\kappa(s)) - \prod_{j=1}^{p}(sI - A_j)}{\prod_{j=1}^{p}(sI - A_j)} \\ &= \det(sI - F_p)^{-1}\left[\det(sI - F_p + H_pk) - \prod_{j=1}^{p}(sI - A_j)\right] \\ &= G_p(s)\end{aligned}$$

where the transformations also exploit the fact that only the last (p-th) entry of the vector H_p is non-zero. We conclude that the delay system inherits the loop robustness properties from the non delay system. In particular, if the LQR design is used for the latter, there is a robustness guarantee for both systems that includes $(\frac{1}{2}, \infty)$ gain margin and $\pm 60°$ phase margin (see [1], Section 5.4).

4. An illustrative example

We have selected a simple feedforward system with two delays to illustrate the recursive predictor design method:

$$\begin{aligned} \dot{z}_1(t) &= z_2(t-\tau_1) \\ \dot{z}_2(t) &= az_2(t) + z_3(t-\tau_2) \\ \dot{z}_3(t) &= u(t) \end{aligned} \tag{4.1}$$

The (nominal) parameters assumed for controller design are $\tau_1 = 0.65$, $\tau_2 = 0.4$, and $a = 1$. Hence, the open loop system is unstable due to a pair or integrators and an unstable pole at 1.

In the first step of the recursion, comparing (4.1) to (3.1) provides $A_1 = F_1 = 0$ and $H_{11} = [1\ 0], H_{12} = [0\ 0]$. Thus, $e^{-F_1\tau_1} = 1$, and we obtain:

$$\begin{aligned} \dot{x}_1(t) &= z_2(t) \\ \dot{z}_2(t) &= az_2(t) + z_3(t-\tau_2) \\ \dot{z}_3(t) &= u(t) \end{aligned} \tag{4.2}$$

For the second step, the prediction matrix F_2 and the interconnection matrix H_{22} (H_{21} matrix, corresponding to τ_1 is 0 at this step) are obtained from the top two equations:

$$F_2 = \begin{bmatrix} 0 & 1 \\ 0 & a \end{bmatrix}, \ H_{22} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Replacing z_2 with x_2 and using

$$e^{-F_2\tau_2}H_{22} = \begin{bmatrix} \dfrac{e^{-a\tau_2}-1}{a} & e^{-a\tau_2} \end{bmatrix}^T$$

the system (4.2) is transformed into

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) + \frac{e^{-a\tau_2}-1}{a} z_3(t) \\ \dot{x}_2(t) &= ax_2(t) + e^{-a\tau_2} z_3(t) \\ \dot{z}_3(t) &= u(t) \end{aligned} \tag{4.3}$$

which is delay free and for which a preferred linear control design method can be applied (it is easy to check that (4.3) is controllable for all a, τ_2).

From a control law

$$v_3 = -k_1x_1 - k_2x_2 - k_3z_3 \tag{4.4}$$

for (4.3), we reconstruct the control for the original system in two bottom-up steps[1]. First, to stabilize (4.2) we would use

$$v_2 = -[k_1\ k_2]\left(\begin{bmatrix} x_1 \\ z_2 \end{bmatrix} + \int_0^{\tau_2} e^{-F_2\theta} \times H_{22}z_3(t+\theta-\tau_2)\,d\theta\right) - k_3z_3$$

where we have changed x_2 into z_2 to remind ourselves that we have returned back to the original equation for the second state. In the second (and the last) bottom-up step we need to go from a stabilizing control for (4.2) to a stabilizing control for (4.1). Thus, the final control law is obtained by introducing a predictor for the z_1 state via

$$u = v_1 = v_2 - k_1 \int_0^{\tau_1} e^{-F_1\theta} H_{11} \begin{bmatrix} z_2(t+\theta-\tau_1) \\ z_3(t+\theta-\tau_1) \end{bmatrix} d\theta$$

Using the particular forms of the matrices F_j and H_{ji} we rewrite the control law as

$$\begin{aligned} u = -k_1 \Bigg(z_1 &+ \int_0^{\tau_1} z_2(t+\theta-\tau_1)d\theta \\ &+ \int_0^{\tau_2} \frac{e^{-a\theta}-1}{a} z_3(t+\theta-\tau_2)d\theta \Bigg) \\ -k_2 &\left(z_2 + \int_0^{\tau_2} e^{-a\theta} z_3(t+\theta-\tau_2)d\theta \right) - k_3z_3 \end{aligned} \tag{4.5}$$

For simulations, the integrals have been implemented by the trapezoidal rule with $\Delta t = 0.05$.

To find the controller gains for the non-delay, proxy system (4.3), we have used the standard LQR method to minimize

$$J = \int_0^{\infty} x^T(t)Qx(t) + u^2(t)dt$$

with $Q = \text{diag}\{15, 10, 10\}$. The approach produced the controller gains $k_1 = 3.9,\ k_2 = 22.1,\ k_3 = 6.1$ and the closed loop poles at $p_1 = -3.1$ and $p_{2/3} = -1 \pm j0.5$. According to the main result of this paper, the system with delay (4.1), (4.5) has the same closed loop poles.

The blue bold trace in Figure 1 shows the response of the (nominal) closed loop system to set point command for z_1. Note that set point regulation for z_1 has to overcome two delays τ_1 and τ_2, $\tau_1 + \tau_2 = 1.05$. With ω_B denoting the closed loop system bandwidth, we have $\omega_B(\tau_1+\tau_2) > 1$. Hence, the controller tuning can be considered aggressive and it is of interest to look at the robustness properties of the controller. Figure 1 also shows the closed loop system response with delay uncertainty: green-dash trace shows the case in which both delays are multiplied by 1.5 (in the system, but not in the controller), and red-dash-dot trace shows the response with both delays multiplied by 0.5.

Another aspect of the closed loop system robustness, its gain and phase margins, has been addresses in Remark 2 and is revisited for this example. The loop gain for the system in Figure 2 is $C(s)P(s)$, with $P(s)$ being the transfer function of the plant (system) and $C(s)$ the transfer function of the controller. The transfer function from the disturbance d to the controller output y is

$$G_{dy}(s) = \frac{C(s)P(s)}{1 + C(s)P(s)}$$

When $d \equiv 0$, the system in Figure 2 may represent the system (4.1) with the controller (4.5) or the proxy systen(4.3) with the controller (4.4). A sequence of step

[1] One can just use the formula (3.8), but we show the bottom-up reconstruction to illuminate the main idea.

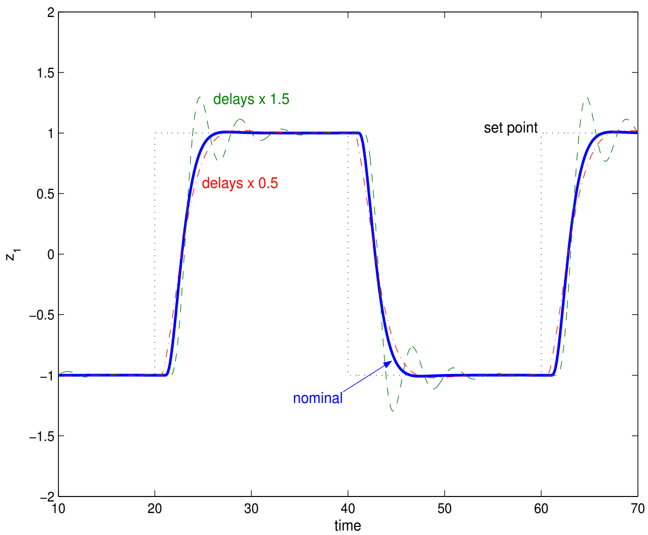

Figure 1: Response of the closed loop system to step input: nominal case and the effect of delay uncertainty.

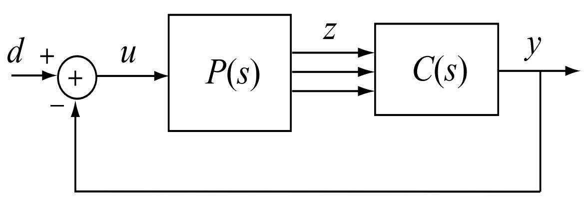

Figure 2: The configuration used for loop transfer function comparison of the time delay system and the non-delay proxy system.

changes for d is used to compare the responses of the two closed loop systems. Figure 3 shows that they are indeed the same (occasional small difference are probably caused by the discrete approximation of predictor integrals in (4.5)). We conclude that, not only are the poles of the delay and proxy closed loop system the same, but also that the zeros of the loop transfer functions are too. This confirms that the gain and phase margins for the two systems are the same and also implies equivalent robustness to input unmodeled dynamics and to a class of parametric uncertainties (see [7], Section 3.2).

For this example, we have also noticed a correlation between robustness properties of the delay and the proxy systems to τ_1 and τ_2 (note that they are real state delays for the former and just parameters for the latter). The proxy system stability is independent of τ_1 while the delay system remains stable over a (wide) range of τ_1 between 0 and 6 (recall, the nominal value used for the controller design is 0.65). Similarly, the proxy system is stable for the parameter τ_2 between 0 and 0.87 while the delay system is stable for the delay τ_2 between 0 and 0.63 (the nominal value is 0.4).

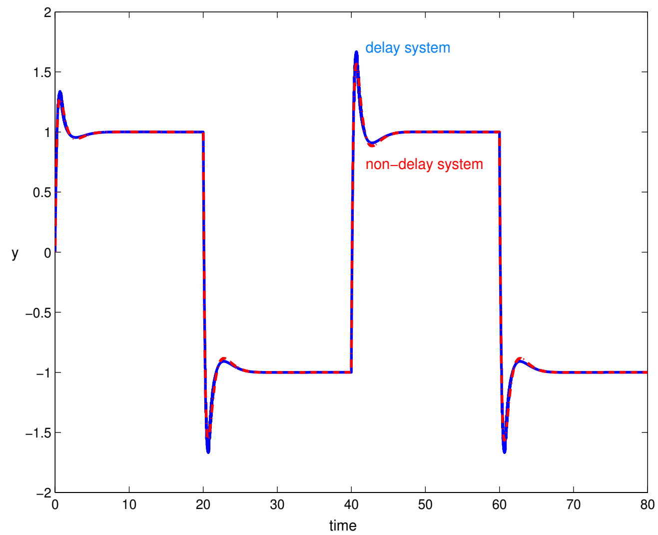

Figure 3: The response of the two loops (delay and proxy) to step changes in the disturbance d.

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