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Planning a finite time transition from a non-stationary to a stationary point without overshoot

Johann Reger*

Abstract— **The problem of planning a finite time transition of a trajectory from a non-stationary point to a stationary set-point is addressed. As opposed to standard approaches, where the transition functions are polynomials, the specific choice of non-analytic function proposed may easily be trimmed to show no overshoot and only an adjustable undershoot during transition. The main result is a recursive formula for a simple parametrization of the transition function, as needed in tracking problems. Two examples underscore the ease of the approach.**

I. INTRODUCTION

Consider the problem of planning the transition of a time function $y(t)$ on a finite time interval $t \in [t_1, t_2]$. Let $t = t_1$ be the time associated with r left boundary conditions (BC)

$$y^{(i)}(t_1) := \left.\frac{d^i y}{dt^i}\right|_{t=t_1} = \underline{y}_i, \quad i = 0, 1, \ldots, r-1\,. \tag{1}$$

Correspondingly, at $t = t_2$ let $y(t)$ satisfy r right BC

$$y^{(i)}(t_2) := \left.\frac{d^i y}{dt^i}\right|_{t=t_2} = \overline{y}_i, \quad i = 0, 1, \ldots, r-1\,. \tag{2}$$

A straight-forward idea for tackling this problem is to use polynomials for meeting the $2r$ BC of (1) and (2). The least degree polynomial shows degree $2r-1$ and is uniquely determined by the BC. Polynomials with degrees larger than $2r-1$ that meet the BC may be found, as well. Various approaches of polynomial kind are exposed in [8], for example. An approximate optimization-based approach presents [4]. For the planning of transitions between stationary points see [5], where a simple to use formula for the transition polynomial is given. An input shaping approach by means of polynomials with additional exponential decay is derived in [7]. Recently, trajectory generation received attention within the different inversion-based approaches to the control of systems with internal dynamics [1], [2], [6], [3].

In this paper, let the task be confined to the planning of the transition from a non-stationary point of $y(t)$, as specified in (1), to a stationary point as given by the right BC

$$y(t_2) = \overline{y}_0, \quad y^{(i)}(t_2) = 0,\ i = 1, \ldots, r-1\,, \tag{3}$$

being a special case of the right BC (2).

It turns out that in the case of planning from non-stationary to stationary points there are decisive drawbacks when employing polynomials: Primarily, there is no a priori criterion to decide whether the transition polynomial resulting from the BC in (1) and (2) will show an overshoot or undershoot. Standard methods as calculating the set of zeroes with respect to the polynomial's first time derivative give a posteriori insight, only. Secondly, it is a well-known fact that large absolute values of $\underline{y}_i$ and $\overline{y}_i$ give rise to polynomials with very large degree, accompanied by the problem of a wavy transition in course of time.

Hence, the proposal of this paper is to refrain from polynomials, and rather employ a particular non-analytic function. A formula for the recursive parametrization of this function is provided that may easily be trimmed to show no overshoot and just a reduced undershoot when adjusting one single parameter.

The paper is organized as follows: Section II contains the derivation of the parametrization of a non-analytic function, adequate for solving the above-stated transition problem on a unity time interval. Section III provides the main result that holds for arbitrary time intervals. The paper ends with a discussion and some examples in Section IV.

II. PARAMETRIZATION OF A NON-ANALYTIC TRANSITION FUNCTION

Consider the transition function

$$y(t) = \left(c_0 + c_1 t + \ldots + c_{r-1} t^{r-1}\right) e^{\frac{-1}{(t-1)^n}} + \overline{y}_0 \tag{4}$$

with even exponent $n \in \{2, 4, 6, \ldots\}$ and real coefficients $c_i,\ i = 0, 1, \ldots, r-1$. It is not difficult to show that the ansatz (4) satisfies the stationary right BC (3) in a limit sense

$$\lim_{t\to 1} y(t) = \overline{y}_0 \quad \text{and} \quad \lim_{t\to 1} y^{(i)}(t) = 0,\ i = 1, 2, \ldots \tag{5}$$

which implies that $y(t)$ given in (4) is non-analytic at $t = 1$.

The coefficients c_i serve to satisfy the left BC (1) at the time instant $t_1 = 0$. In a next step, the result to be obtained at time instants $t_1 = 0$ and $t_2 = 1$ may then be generalized to arbitrary instants of time $t_1 < t_2,\ t_1, t_2 \in \mathbb{R}$.

In view of the left BC (1), the coefficients c_i may be determined by equating $y^{(i)}(0) = \underline{y}_i,\ i = 0, 1, \ldots, r-1$.

In the first place, observe that

$$y(0) = c_0\, e^{\frac{-1}{(-1)^n}} + \overline{y}_0 \overset{!}{=} \underline{y}_0 \ \Rightarrow\ c_0 = (\underline{y}_0 - \overline{y}_0)\, e\,. \tag{6}$$

Thereafter, for $i = 1, 2, \ldots$ determine the i-th time derivative

$$\begin{aligned} y^{(i)}(t) &= \sum_{\nu=0}^{i} \binom{i}{\nu} \left(\frac{d^{i-\nu}}{dt^{i-\nu}} \sum_{\mu=0}^{r-1} c_\mu t^\mu\right) \left(\frac{d^\nu}{dt^\nu} e^{\frac{-1}{(t-1)^n}}\right) \\ &= \sum_{\nu=0}^{i} \binom{i}{\nu} \left(\frac{d^\nu}{dt^\nu} e^{\frac{-1}{(t-1)^n}}\right) \sum_{\mu=i-\nu}^{r-1} c_\mu \frac{d^{i-\nu}}{dt^{i-\nu}} t^\mu \\ &= \sum_{\nu=0}^{i} \binom{i}{\nu} \left(\frac{d^\nu}{dt^\nu} e^{\frac{-1}{(t-1)^n}}\right) \sum_{\mu=i-\nu}^{r-1} \frac{\mu!}{(\mu+\nu-i)!}\, c_\mu\, t^{\mu+\nu-i} \end{aligned} \tag{7}$$

* J. Reger is postdoc with the Systems and Control Theory Group, Max Planck Institute for Dynamics of Complex Technical Systems, Sandtorstr. 1, D-39106 Magdeburg, Germany (email: reger@ieee.org)

and for adaption to the left BC (1), for any $i = 1, 2, \ldots, r-1$ at $t_1 = 0$ we have to require that

$$\begin{aligned}
\underline{y}_i &\stackrel{!}{=} \lim_{t\to 0} y^{(i)}(t) = \sum_{\nu=0}^{i} \binom{i}{\nu} \left(\lim_{t\to 0} \frac{d^\nu}{dt^\nu} e^{\frac{-1}{(t-1)^n}} \right) \\
&\sum_{\mu=i-\nu}^{r-1} \frac{\mu!}{(\mu+\nu-i)!}\, c_\mu \left(\lim_{t\to 0} t^{\mu+\nu-i} \right) \\
&= \sum_{\nu=0}^{i} \binom{i}{\nu} \left(\lim_{t\to 0} \frac{d^\nu}{dt^\nu} e^{\frac{-1}{(t-1)^n}} \right) (i-\nu)!\; c_{i-\nu} \\
&= \sum_{\nu=0}^{i} \frac{i!}{\nu!}\, c_{i-\nu} \left(\lim_{t\to -1} \frac{d^\nu}{dt^\nu} e^{\frac{-1}{t^n}} \right)
\end{aligned} \tag{8}$$

which together with equation (6) is a linear system of equations that allows to solve for the r unknown coefficients c_i in terms of the BC $\underline{y}_i$, $i = 0, 1, \ldots, r-1$, in a unique way.

The triangular structure of equation (8) suggests to exploit a simple recurrence scheme. Indeed, rewriting (8) yields

$$\begin{aligned}
\underline{y}_i &= i!\, c_i \left(\lim_{t\to -1} e^{\frac{-1}{t^n}} \right) + \sum_{\nu=1}^{i} \frac{i!}{\nu!}\, c_{i-\nu} \left(\lim_{t\to -1} \frac{d^\nu}{dt^\nu} e^{\frac{-1}{t^n}} \right) \\
&= \frac{i!}{e}\, c_i + \sum_{\nu=1}^{i} \frac{i!}{\nu!}\, c_{i-\nu} \left(\lim_{t\to -1} \frac{d^\nu}{dt^\nu} e^{\frac{-1}{t^n}} \right) .
\end{aligned} \tag{9}$$

Thus, with (6) we derive the recurrence ($i = 0, 1, \ldots, r-1$)

$$c_i = e \left(\frac{\underline{y}_i}{i!} - \sum_{\nu=1}^{i} \frac{c_{i-\nu}}{\nu!} \lim_{t\to -1} \frac{d^\nu}{dt^\nu} e^{\frac{-1}{t^n}} \right), \; c_0 = (\underline{y}_0 - \overline{y}_0)\, e\,. \tag{10}$$

The derivatives on the right hand side of (10) may be evaluated further. To this end, use the chain rule

$$\frac{df(t)}{dt} = f(t) \frac{dg(t)}{dt}, \quad f(t) = e^{g(t)},\; g(t) = \frac{-1}{t^n}\,. \tag{11}$$

In doing so, we may refer to Leibniz' rule for differentiating products again, hence

$$f^{(\nu+1)}(t) = \sum_{i=0}^{\nu} \binom{\nu}{i} \left(\frac{d^{\nu-i}}{dt^{\nu-i}} f(t) \right) \left(\frac{d^{i+1}}{dt^{i+1}} g(t) \right) \tag{12}$$

and shifting $\nu \to \nu - 1$ it follows that

$$f^{(\nu)}(t) = \sum_{i=0}^{\nu-1} \binom{\nu-1}{i} f^{(\nu-i-1)}(t)\, g^{(i+1)}(t), \quad \nu = 1, 2, \ldots \tag{13}$$

where the ν-th time derivative of $f(t)$ is expressed in terms of lower order derivatives in form of a recurrence. Finally, recalling (11) it remains to evaluate

$$\begin{aligned}
g^{(i+1)}(t) &= \frac{d^{i+1}}{dt^{i+1}} \left(\frac{-1}{t^n} \right) = (-1) \frac{d^{i+1}}{dt^{i+1}} t^{-n} \\
&= (-1)(-n)(-n-1)(-n-2)\cdots(-n-i) \frac{1}{t^{n+i+1}} \\
&= (-1)^i \frac{(n+i)!}{(n-1)!} \frac{1}{t^{n+i+1}}\,.
\end{aligned} \tag{14}$$

A consequence is the recurrence

$$\begin{aligned}
f^{(\nu)}(t) &= \sum_{i=0}^{\nu-1} \binom{\nu-1}{i} \frac{(n+i)!}{(n-1)!}\; \frac{(-1)^i}{t^{n+i+1}}\; f^{(\nu-i-1)}(t) \\
f^{(0)}(t) &= e^{\frac{-1}{t^n}}
\end{aligned} \tag{15}$$

which at $t = -1$ yields

$$\begin{aligned}
f^{(\nu)}(-1) &= \sum_{i=0}^{\nu-1} \binom{\nu-1}{i} \frac{(n+i)!}{(n-1)!} (-1)^{n+1} f^{(\nu-i-1)}(-1) \\
f^{(0)}(-1) &= 1/e
\end{aligned} \tag{16}$$

to be solved until index $\nu = r - 1$, as indicated by (10).

III. Main Result

Simple steps of manipulation show that a possible transition function, which satisfies the $2r$ BC of (1) and (3) at arbitrary instants of time t_1 and t_2, reads

$$y(t) = \underline{y}_0 + (1/e)^{\left(\frac{t_2 - t_1}{t_2 - t}\right)^n} \sum_{i=0}^{r-1} c_i \left(\frac{t - t_1}{t_2 - t_1} \right)^i \tag{17}$$

with coefficients c_i that result from the recurrence

$$\begin{aligned}
c_i &= e \left(\frac{\underline{y}_i\, (t_2 - t_1)^i}{i!} - \sum_{\nu=1}^{i} \frac{c_{i-\nu}}{\nu!} f^{(\nu)}(-1) \right) && (18) \\
c_0 &= (\underline{y}_0 - \overline{y}_0)\, e\,. && (19)
\end{aligned}$$

where the values of $f^{(\nu)}(-1)$ follow from (16).

IV. Discussion and Examples

In order to find a minimal parameter $n = n_{\min}$ subject to which no overshoot occurs for $t \in (t_1, t_2)$, note that with the coefficients c_i determined as above, the necessary condition $\frac{d}{dt} y(t) = 0$ for an extremal point may be written as

$$\begin{aligned}
&\sum_{i=0}^{r-1} c_i\, (t - t_1)^i\, (t_2 - t_1)^{r-1-i} \times \\
&\qquad \left(i\, (t_2 - t)^{n+1} - n\, (t_2 - t_1)^n\, (t - t_1) \right) = 0\,.
\end{aligned} \tag{20}$$

In the main, two cases need to be distinguished:

1) When increasing n starting from 2, given the bottom-up-transition $\underline{y}_0 < \overline{y}_0$ and $\underline{y}_1 > 0$ (top-down-transition $\underline{y}_0 > \overline{y}_0$ and $\underline{y}_1 < 0$), then $n_{\min}$ is the first number for which the polynomial in (20) shows no zeroes in (t_1, t_2). Thus, the overshoot as depicted in the plots of Figure 1 can be avoided by increasing n.
2) When increasing n starting from 2, given the bottom-up-transition $\underline{y}_0 < \overline{y}_0$ and $\underline{y}_1 < 0$ (top-down-transition $\underline{y}_0 > \overline{y}_0$ and $\underline{y}_1 > 0$), then $n_{\min}$ is the first number for which the polynomial in (20) shows one single zero in (t_1, t_2). In this case, besides avoiding an overshoot, one may additionally reduce the undershoot by a further increase of the parameter n until the undershoot falls below a specified bound, as shown in the lower plots of Figure 1 (see arrows).

TABLE I

LEFT BC FOR THE PARAMETRIZATION OF THE TRANSITION FUNCTION (17) AS DEPICTED IN FIGURE 2 AND FIGURE 3

left BC	$\underline{y}_0$	$\underline{y}_1$	$\underline{y}_2$	$\underline{y}_3$	$\underline{y}_4$
Figure 1	0	20	30	60	40
Figure 2	0	-15	200	100	40

Either of these cases is demonstrated resorting to an example transition from $t_1 = 0$ to $t_2 = 2$ subject to $r = 5$ non-stationary BC at $t = t_1$ (see Table I). A stationary value of $\overline{y}_0 = 10$ shall be reached for both transitions at $t = t_2$.

Case 1 is illustrated in Figure 2: A calculation of the corresponding zeroes of (20) for $n = 2, 4, 6, \ldots, 16$ yields that $n_{\min} = 8$, where no overshoot takes place, anymore. An increase of n further accelerates the response.

Case 2 is illustrated in Figure 3: A calculation of the corresponding zeroes of (20) for $n = 2, 4, 6, \ldots, 16$ yields that $n_{\min} = 6$, where no overshoot takes place, anymore. A further increase of n helps accelerate the response and further reduces the undershoot. Such transitions resemble behaviors that are typical within the tracking of non-minimum phase systems.

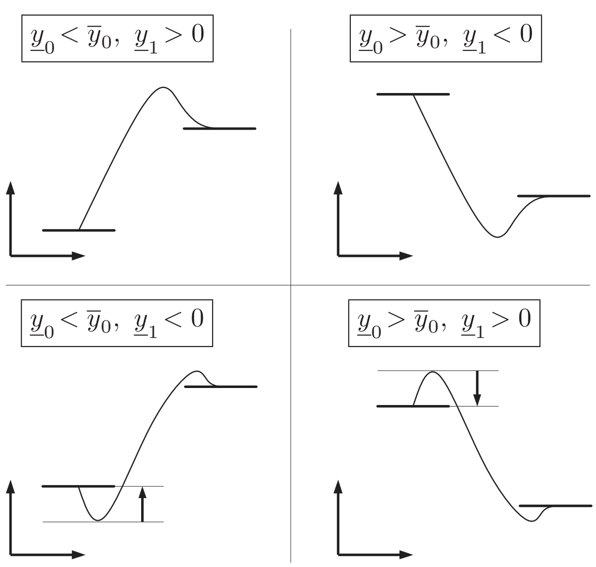

Fig. 1. Dependency of overshoot and undershoot on left side boundary conditions–by increasing n in the non-analytic transition (17) the depicted overshoots may be avoided and the undershoots (marked with arrow) are reduced

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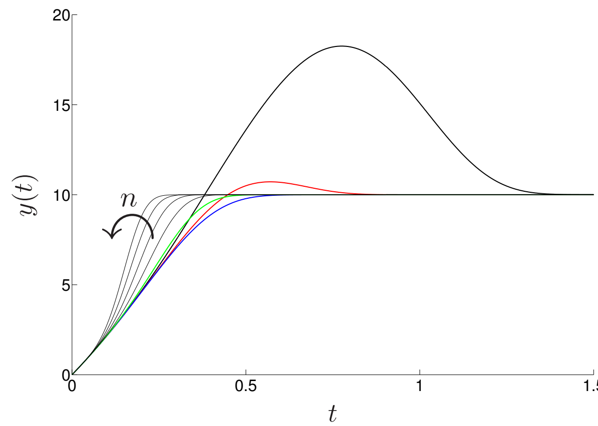

Fig. 2. Case 1: transition function (17) for left BC according to Table I; $n = 2$ (black line), $n = 4$ (red line), $n = 6$ (blue line); for $n = 8$ (green line) no overshoot occurs anymore; thin black lines show faster response when increasing n (plotted are $n = 10, 12, 14, 16$)

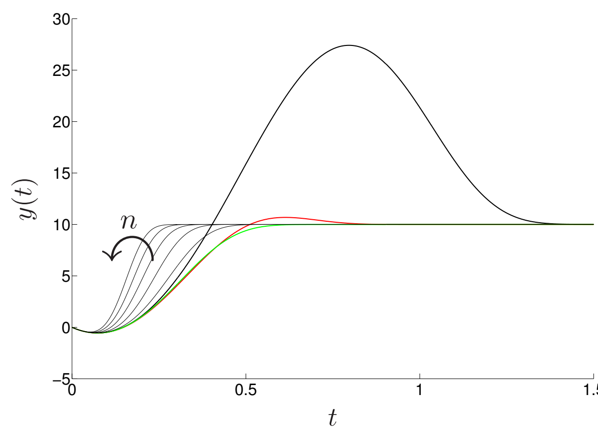

Fig. 3. Case 2: transition function (17) for left BC according to Table I; $n = 2$ (black line), $n = 4$ (red line); for $n = 6$ (green line) no overshoot occurs anymore; increasing n helps reduce the undershoot further (plotted are $n = 8, 10, 12, 14, 16$)

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