2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Enhanced Stability Regions for Model Predictive Control of Nonlinear Process Systems *

Maaz Mahmood and Prashant Mhaskar[†]
Department of Chemical Engineering
McMaster University
Hamilton, ON L8S 4L7, Canada

Abstract— **This work considers the problem of predictive control of nonlinear process systems subject to input constraints. Lyapunov-based tools are used to develop control-law independent characterizations of the stability region and this characterization is exploited via the constraints handling capabilities of model predicative controllers to expand on the set of initial conditions for which closed–loop stability can be achieved. The utilization of this idea is first illustrated for the case of linear systems and a predictive controller is designed that achieves closed–loop stability for every initial condition in the null controllable region. For nonlinear process systems, constraints are formulated requiring the process to evolve within the region from where continued decay of the Lyapunov function value is achievable and incorporated in the predictive control design, thereby expanding on the set of initial conditions from where closed–loop stability can be achieved. The proposed method is illustrated using a chemical reactor example.**

Key words: Input constraints, model predictive control, Lyapunov-based control, stability region, feasibility region.

I. INTRODUCTION

The operation and control of chemical processes often encounters constraints that arise out of physical limitations on the control actuators. The constraints, if not accounted for in the control design, can cause performance deterioration or even instability in the closed–loop system. Specifically, the presence of constraints limits the set of initial conditions from where a process can be stabilized at a desired equilibrium point (the so-called null controllable region). A meaningful measure of how well the available control effort is being utilized by the control law can be obtained via a comparison of the stability region under a given control law with the null controllable region. Such a measure also provides assurance on the ability of the control law in recovering from the effect of disturbances that may temporarily drive the process away from the nominal operating point. These considerations have motivated extensive research on accounting for constraints via modifications in existing control approaches (e.g., anti-windup designs [1]) as well as fostered the development of controllers that explicitly account for the presence of constraints via Lyapunov-based (see, for example, [2], [3], [4], [5], [6], [7] and [8], [9] for excellent reviews) and model-predictive control designs (see, for example, [10], [11], [12], [13], [14], [15] and the survey paper, [16]).

Given that process dynamics are sometimes identified or approximated by linear process systems, extensive research work has focused on designing and analyzing controllers that utilize a linear process description in computing the control action. Characterization of the null controllable region for linear process systems, while being difficult, is a tractable problem and has been the focus of several research efforts [17], [18]. Furthermore, several controller designs have been proposed that allow the possibility of turning any given *subset* of the null controllable region into the stability region of a proposed controller design [19], [20]. For some classes of linear systems (systems with real eigenvalues, low order systems with complex eigenvalues), explicit expressions for the boundary of the null controllable region have recently been characterized [18]. The work in [18], however does not consider the problem of determining the control law that can stabilize all initial conditions in the null controllable region.

For nonlinear processes, the problem of explicitly characterizing the null controllable region remains intractable. Lyapunov-based control designs address the problem of explicit characterizations of the stability region (see, e.g., [2], [5], [6]) under given control laws. The stability regions, however, are limited to (possibly conservative estimates of) invariant subsets (Ω) of the set of states for which the Lyapunov function (V) can be made to decay (Π). In [21], [22] (see [9] for further results and references), the stability properties of auxiliary Lyapunov-based controllers of [2], [6] were utilized in formulating stability constraints in the optimization problem in a way that the predictive controllers of [21], [22] mimic the stability region of the auxiliary control designs. The predictive controllers of [21], [22], however, do not fully utilize the constraint handling properties of the predictive controller approach to expand on the set of initial conditions from where closed–loop stability can be achieved.

II. PRELIMINARIES

A. Process description

We consider nonlinear processes with input constraints, described by:

$$\dot{x} = f(x) + G(x)u(t);\ u \in \mathcal{U} \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the vector of state variables, $u \in \mathbb{R}^m$ denotes the manipulated inputs taking values in

*Partial financial support by NSERC and McMaster Advanced Control Consortium is gratefully acknowledged.

[†]Corresponding author: mhaskar@mcmaster.ca

a nonempty convex subset $\mathcal{U}$ of $\mathbb{R}^m$, where $\mathcal{U} = \{u \in \mathbb{R}^m : u_{min} \le u \le u_{max}\}$, $u_{min} \in \mathbb{R}^m$ and $u_{max} \in \mathbb{R}^m$ denote the lower and upper bounds on the manipulated input, $u^{norm} > 0$ is such that $\|u\| \le u^{norm}$ implies $u \in \mathcal{U}$, where $\|\cdot\|$ is the Euclidean norm of a vector, and $f(0) = 0$. The vector function $f(x)$ and the matrix $G(x) = [g_1(x) \cdots g_m(x)]$ are assumed to be sufficiently smooth on their domains of definition. The notation $L_f h$ denotes the standard Lie derivative of a scalar function $h(\cdot)$ with respect to the vector function $f(\cdot)$, ∂X denotes the boundary of a set X and $x(T^+)$ is used to denote the limit of the trajectory $x(t)$ as T is approached from the right, i.e., $x(T^+) = \lim_{t \to T^+} x(t)$. Throughout the manuscript, we assume that for any $u \in \mathcal{U}$ the solution of the system of Eq.1 exists and is continuous for all t.

III. Enhancing the stability region estimates using model predictive control

The stability region estimates of existing Lyapunov-based predictive controllers are limited (and dependent upon) stability region estimates obtained using the auxiliary control approaches. Such controllers do not fully utilize the constraint handling capabilities of the predictive control approach, and suffer from the same possible conservatism as the auxiliary control designs. In this section, we present a predictive control design wherein constraints are formulated that, by better utilizing Lyapunov-based analysis tools, enhance the set of initial conditions from where closed–loop stability is achieved. To clearly explain the key idea, we first consider linear systems subject to constraints and design a predictive controller that guarantees stabilization from all initial conditions for which closed–loop stability can be achieved subject to constraints. Generalization of this idea for nonlinear process systems is subsequently presented.

A. Linear systems subject to constraints

Linear descriptions of the process dynamics are often utilized in controller design for chemical processes. While extensive results exist on constructing control designs that guarantee stability from any given subset of the null controllable region (see, e.g., [17], [23], [24], [18], [25], [19], [20], [10], [19]), the computational complexity of the control design typically renders the control implementation impractical as larger and larger stability regions are desired. Furthermore, there exists a lack of results that guarantee stability for any initial condition in the entire null controllable region. In this section, we show how the characterization of the null controllable region, developed in [18], can be utilized within the predictive control approach in achieving stability for all initial conditions in the null controllable region. To this end, consider processes whose dynamics can be described by

$$\dot{x}(t) = Ax(t) + Bu(t), \ u \in \mathcal{U} \tag{2}$$

where A and B are constant $n \times n$ and $n \times m$ matrices respectively. A summary of characterization of the null controllable region is described below [18].

1) Null controllable region for linear systems: A state x_0 is said to be null controllable if there exists a $T \in [0, \infty)$ and an admissible control $u(t)$ such that the state trajectory $x(t)$ of the system of Eq.2 satisfies $x(0) = x_0$ and $x(T) = 0$, and the union of all null controllable sets is called the null controllable region of the system which we denote by X^{max}. The null controllable region characterized as (see [18])

$$X^{max} = \bigcup_{T \in [0,\infty)} \{x = -\int_0^T e^{-A\tau} Bu(\tau) d\tau : u(\tau) \in \mathcal{U}\}$$

can be shown to be a bounded convex open set containing the origin if A is unstable. It can be shown that the null controllable region of the multi-input system of Eq.2 is the Minkowski sum of the single input subsystems

$$\dot{x}(t) = Ax(t) + b_i u_i(t), \ u_i(t) \in \mathcal{U}_i \tag{3}$$

where $B = [b_1 \ b_2 \ \dots b_m]$ and u_i denotes the ith component of the vector u. Specifically, let X_i^{max} denote the null controllable region of the subsystem of Eq.3 then $X^{max} = \sum_{i=1}^{m} X_i^{max} = \{x_1 + x_2 + \cdots + x_m : x_i \in X_i^{max}, i = 1, \dots, m\}$. For systems with real eigenvalues (see [18] for computing the null controllable region for low dimensional systems with complex eigenvalues), the boundary of the null controllable region can be computed as [18]

$$\partial X_i^{max} = \pm[\sum_{j=1}^{n-1} 2(-1)^j e^{-A(t-t_j)} + -1^n I] A^{-1} b_i u_i^{norm} : \\ 0 = t_1 \le t_2 \le \dots \le t_{n-1} \le t \le \infty \tag{4}$$

Eq.4 can be used to verify whether a state lies within the null controllable region and, more importantly, can be used to compute, for a given state, the unique value of u_i^* such that the state resides on the boundary of the null controllable region of a system of the form of Eq.3 with a constraint of u_i^* on the manipulated input u_i. Utilizing these properties, for a given state x_0 we define a function $\bar{u}_i^*(x_0)$ as the unique positive number u_i^* for which $x_0 \in \partial X_i^{max}(u_i^*)$. Essentially, for a given state x_0, Eq.4 is solved to yield t_i, $i = 2 \dots n-1$, t and u_i^{norm}. In the next subsection, we show how the predictive control approach can utilize such a characterization in enabling stabilization from all points within the null controllable region.

2) Predictive control design with the null controllable region as the stability region: The key idea in the predictive control design is as follows: for any given value of the state, the value u_i^* represents the minimum control action required to stabilize the system. A meaningful control action therefore would be one that drives the process in a way that the minimum control action required to stabilize the system decreases. This intuitive idea is formulated mathematically in Theorem 2 below. To this end, consider the system of Eq.2 and an $x_0 \in X^{max}$. Let $x_{i,0} \in X_i^{max}(u_i^*)$, $i = 1, \dots, m$ be such that $x_0 = \sum_{i=1}^m x_{i,0}$, with $u_i^* \le u_i^{norm}$.

The predictive controller that guarantees stabilization from all initial conditions in X^{max} takes the form:

$$u_{i,MPC} = argmin\{J(x,t,u(\cdot))|u(\cdot) \in U,\ x(0) = x_{i,0}\} \tag{5}$$

$$s.t.\ \dot{x} \ = \ Ax + b_i u_i \tag{6}$$

$$\dot{\bar{u}}_i^*(x(t)) \leq 0 \tag{7}$$

Eq.6 is the linear model describing the time evolution of the state x, due to the ith manipulated input. The performance index is given by

$$J(x,t,u(\cdot)) \ = \ \dot{\bar{u}}^*(x_i(t)) \tag{8}$$

The minimizing controls $u_i^0(\cdot)$ are then applied to the plant and the procedure is repeated indefinitely. Note that the above formulation is a continuous time version of the MPC, and assumes instantaneous evaluation and implementation of the computed control value. The result under continuous implementation is presented in Theorem 2 below, and the 'implement and hold' approach demonstrated and discussed in the simulation example for linear systems and addressed explicitly in the predictive control design for nonlinear process systems in Theorem 3.

Theorem 2: *Consider the constrained system of Eq.2 under the MPC law of Eqs.5–8. Then, given any $x_0 \in X^{max}$, the optimization problem of Eq.5-8 is feasible for all times, and $\lim_{t\to\infty} x(t) = 0$.*

Proof of Theorem 2: We first prove the results for a single input system, and then illustrate the generalization to multi-input systems. In the proof, the key things to show are guaranteed feasibility of the optimization problem and the optimal solution leading to closed–loop stability.

Single input system: In this part of the proof, we will drop the subscript on the input with the understanding that a single input system is being analyzed. Consider an $x_0 \in X^{max}$, for which $\bar{u}^*(x_0) = u_0^* < u^{norm}$. In part 1, we show feasibility of the optimization problem, and in part 2, the implementation of the optimal solution resulting in closed–loop stability.

Part 1: Since $x_0 \in X^{max}(u^{norm})$, there exists at least one input trajectory $u(t)$ with $|u(t)| \leq u^{norm}$ such that $\lim_{t\to\infty} x(t) = 0$. Out of all such possible trajectories (for which $\lim_{t\to\infty} x(t) = 0$) let

$$u_1^* = \min_{|u_1(t)| \leq u^{norm}} \max_{t,\ x(0)=x_0} \bar{u}^*(x_{u_1}(t)) \tag{9}$$

where $x_{u_1}(t)$ denotes the state profile corresponding to an input profile of $u_1(t)$. Thus u_1^* represents the minimum (over all possible stabilizing trajectories) of the maximum (over time) value that the function $\bar{u}^*(\cdot)$ takes. Note that if $u_1^* \geq u^{norm}$ then an x_1^* such that $\bar{u}^*(x_1^*) = u_1^*$ will be such that $x_1^* \in X^{max}(u^{norm})$ (in other words, it would mean that the process starting from a state outside the null controllable region is actually stabilized) which leads to a contradiction, we therefore have that

$$u_1^* < u^{norm} \tag{10}$$

Let $u_1^* = u_0^* + \gamma$ with $\gamma > 0$. Since $x_0 \in \partial X^{max}(u_0^*)$, this implies that $x_0 \in X^{max}(u_0^* + \gamma/2)$. Denoting

$$u_2^* = \min_{|u_2(t)| \leq u_0^* + \gamma/2} \max_{t,\ x(0)=x_0} \bar{u}^*(x_{u_2}(t)) \tag{11}$$

and invoking Eq.10 again with $u_0^* + \gamma/2 = u^{norm}$, we get that $u_2^* < u_0^* + \gamma/2$. Furthermore, noting that the minimizations of Eq.9 and Eq.11 are exactly the same, albeit with a larger constraint in Eq.9 compared to Eq.11, we get that $u_1^* = u_0^* + \gamma \leq u_2^* < u_0^* + \gamma/2$, which once again leads to a contradiction, implying γ cannot be a positive real number. This finally leads to the conclusion that for any $x_0 \in X^{max}(u^{norm})$, there exists a manipulated input profile and corresponding state trajectory such that $\bar{u}^*(x(t+\delta t)) \leq \bar{u}^*(x(t))$ for all $\delta t > 0$. This implies that along such a trajectory the function $\bar{u}^*(x(\cdot))$ is non-increasing, implying the feasibility of the constraint $\dot{\bar{u}}^*(x(t)) \leq 0$.

Part 2: Having established the feasibility of the optimization problem in Part 1 above, consider now an x_0 in X^{max} for which $J^*(x_0, t, u(\cdot)) = \min \dot{\bar{u}}^*(x_0(t)) = 0$. This implies that for this x_0, the minimizing u_{MPC} is such that the vector $Ax_0 + bu$ (which represents the current direction of the state trajectory) is on the tangent plane to the surface defining $\partial X^{max}(\bar{u}^*(x_0))$. This would further imply that the vectors Ax_0 and bu_{MPC} must themselves be co-planar (if they were not, a different allowable value for u_{MPC} could have been chosen to point the vector $Ax_0 + bu$ away from the tangent plane to the surface defining $\partial X^{max}(\bar{u}^*(x_0))$, resulting in a $J^*(x_0(t)) < 0$). Upon implementation of such a u_{MPC}, the tangent to ∂X^{max} at $x(t^+)$ cannot remain in the same plane (due to the strict convexity of the boundary of the set X^{max}) as that of the vector b resulting in $\min \dot{\bar{u}}^*(x_0^+) < 0$. Therefore, for any x_0 for which the minimum of $\dot{\bar{u}}^*(x(0)) = 0$, the minimum of $\dot{\bar{u}}^*(x(0^+)) < 0$ ensuring convergence of $\bar{u}^*(x(t))$ to zero, in turn resulting in $\lim_{t\to\infty} x(t) = 0$.

Multiple input system: The result for the multiple input system is a direct generalization for the single input system. Having defined $x_0 = \sum_{i=1}^m x_{i,0}$, X^{max} and X_i^{max}, the evolution of the multiple input system is exactly the same as the sum of the multiple single input systems. Feasibility and stability of the subsystems yields stability for the original multi-input system.

Remark 2: The result achieving stabilization from the null controllable region can best be understood in light of the result using, say, a control Lyapunov function. Specifically, Lyapunov-based predictive controllers [21] do not guarantee stabilization from all initial conditions in the null controllable region due to the following reasons: (1) for a choice of a CLF V, $\dot{V}$ is not necessarily guaranteed to be negative for all initial conditions in X^{max}, (2) even if a certain choice of the CLF resulted in $\dot{V}$ being negative for all initial conditions in X^{max}, the level sets of a CLF may not necessarily coincide with the boundary of the null controllable region. The stability region estimate would therefore typically be a subset of the null controllable region. Note also that if it were always possible to obtain the global optimum to the optimization problem, then for the specific choice of the objective function, the constraint of Eq.7 would be redundant. Specifically, if a control action were to exist that would make $\dot{\tilde{u}}^*(x) \leq 0$ it would naturally be chosen over another control action for which $\dot{\tilde{u}}^*(x) > 0$ (due to the specified objective function). In implementing the control algorithm, however, the optimization problem may not always be able to compute the global optimum. The constraint of Eq.7 ensures that a local minima of $\dot{\tilde{u}}^*(x)$, for which $\dot{\tilde{u}}^*(x)$ may be greater than zero (and may lead to de-stabilization) is avoided and only a stabilizing solution is chosen.

B. Model predictive control of nonlinear systems

In contrast to linear systems, where an explicit characterization of the null controllable region is possible, for nonlinear process systems such a characterization remains an open problem. In [22], predictive controllers were designed that utilized auxiliary Lyapunov–based control design for estimating the feasibility and stability region. In the predictive control design of [22], the first layer of conservativeness stems from the estimation of Π which only captures initial conditions for which negative definiteness of $\dot{V}$ can be achieved by the auxiliary control law, instead of characterizing the set of initial conditions for which negative definiteness of $\dot{V}$ can be achieved independent of the control law (which we will characterize and denote by Π^+). Additionally, only requiring $\dot{V}$ to be negative allows stabilization from all initial conditions inside Ω but misses out on achieving stabilization from initial conditions outside Ω but inside Π.

1) Nonlinear model predictive controller: We utilize in this section the constraint handling capabilities of the predictive controller to expand on the set of initial conditions from where closed–loop stability can be achieved to alleviate the possible conservatism associated with Lyapunov-based control designs. To this end, we first characterize the set Π^+ for which negative definiteness of the Lyapunov function derivative can be achieved subject to manipulated input constraints (and independent of the control law) described by $\Pi^+ =$

$$\{x \in \mathbb{R}^n : L_f V(x) + \sum_{i=1}^{m} L_{G_i^{min}} V(x) u^i \leq -\epsilon^{**}\} \quad (12)$$

where $L_{G_i^{min}} V(x) u^i = L_{G_i} V(x) u^i_{max},\ if\ L_{G_i} V(x) \leq 0$ and $L_{G_i^{min}} V(x) u^i = L_{G_i} V(x) u^i_{min},\ if\ L_{G_i} V(x) > 0$ and ϵ^{**} is a positive number to be defined. The set Π^+ therefore denotes the entire set of initial conditions from where $\dot{V} < -\epsilon^{**}$ is achievable (and not just the set from where a specific control law can achieve $\dot{V} < 0$. The idea behind the expression in Eq.12 is as follows: each element of the vector $L_G V(x)$, denoted by $L_{g_i} V(x)$ captures the effect of the ith component of the manipulated input on the Lyapunov function derivative. The term $L_{G_i^{min}} V(x) u^i$ therefore captures the most that the ith manipulated input can contribute towards making $\dot{V}(x)$ negative. Alternatively, the expression can also be thought of as the set of states for which $\dot{V}(x)$ is negative under the 'bang-bang' control law given by $u_i(x) = -sgn(L_{g_i} V(x)) u_i^{norm}$ (for the case where $|u^i_{max}| = |u^i_{min}| = u_i^{norm}$) where $sgn(x) = 1$ if $x \geq 0$ and $sgn(x) = -1$ if $x < 0$. By accounting for the maximum control action available, the set Π^+ expands on the estimate Π. Subsequently, computation of the largest level set Ω^+, of the form

$$\Omega^+ \quad = \quad \{x \in \mathbb{R}^n : V(x) \leq c^{max^+}\} \quad (13)$$

completely contained in Π^+ improves upon the estimate Ω. Requiring $\dot{V} \leq -\epsilon^{**}$ instead of only requiring $\dot{V} < 0$ is formulated to ensure stabilization subject to implement and hold (similar to the result in Theorem 1). Having defined the sets Π^+ and Ω^+ the predictive controller enhancing the set of initial conditions from which stability is achieved (accounting specifically for initial conditions outside Ω^+ but inside Π^+) takes the form:

$$u \quad = \quad argmin\{J(x, t, u(\cdot)) | u(\cdot) \in S\} \quad (14)$$

$$s.t.\ \dot{x} \quad = \quad f(x) + G(x)u \quad (15)$$

$$\dot{V}(x(\tau)) \leq -\epsilon^* \ \ \forall\, \tau \in [t, t+\Delta)\ if\ \ V(x(t)) > \delta' \quad (16)$$

$$V(x(\tau)) \leq \delta' \ \forall\, \tau \in [t, t+\Delta)\ \ if\ \ V(x(t)) \leq \delta' \quad (17)$$

$$x(t+\tau) \quad \in \quad \Pi^+ \ \forall\, \tau \in [t, t+\Delta)\ if\ \ V(x(t)) > c^{max^+} \quad (18)$$

where $S = S(t, T)$ is the family of piecewise continuous functions (functions continuous from the right), with period Δ, mapping $[t, t+T]$ into U and T is the horizon. Eq.15 is the model describing the time evolution of the state x under continuous operation, V is the control Lyapunov function (CLF) and δ', $\epsilon^* > 0$ are parameters defined in Theorem 1. A control $u(\cdot)$ in S is characterized by the sequence $\{u[j]\}$ where $u[j] := u(j\Delta)$ and satisfies $u(\tau) = u[j]$ for all $\tau \in [t + j\Delta, t + (j+1)\Delta)$. The performance index is given by $J(x, t, u(\cdot)) =$

$$\int_t^{t+T} \left[\|x^u(s; x, t)\|_Q^2 + \|u(s)\|_R^2\right] ds + vV(x(t+\Delta)) \quad (19)$$

where Q is a positive semi-definite symmetric matrix, R is a strictly positive definite symmetric matrix and $v > 0$. $x^u(s; x, t)$ denotes the solution of Eq.1, due to control u, with initial state x at time t. The minimizing control $u^0(\cdot) \in S$ is then applied to the process over the interval $[t, t+\Delta)$ and the procedure is repeated indefinitely. The feasibility and stability properties of the predictive controller are formalized in Theorem 3 below:

Theorem 3: *Consider the constrained system of Eq.1 under the MPC law of Eqs.14–19. Then, given any $d > 0$, there exists a positive real number ϵ^{**} such that if $x_0 \in \Omega^+$, where Ω^+ was defined in Eq.13, then the optimization problem of Eq.14-19 is guaranteed to be feasible for all times, $x(t) \in \Omega^+$ for all $t \geq 0$ and $\limsup_{t\to\infty} \|x(t)\| \leq d$.*

Furthermore, for $x_0 \in \Pi^+ \backslash \Omega^+$ where Π^+ was defined in Eq.12, if the optimization problem of Eq.14-19 is successively feasible for all times, then $x(t) \in \Pi^+ \bigcup \Omega^+$ for all $t \geq 0$ and $\limsup_{t\to\infty} \|x(t)\| \leq d$.

Proof of Theorem 3: The proof of the theorem comprises of two parts. In part 1, we show the feasibility of the optimization problem for all $x \in \Omega^+$ and subsequent convergence to the desired neighborhood of the origin, while in part 2, for $x \notin \Omega^+$ we show convergence to the desired neighborhood of the origin upon assumption of feasibility of the optimization problem.

Part 1: From theorem 1 and the proof (see [22]) it follows that given d, there exist positive real numbers δ' and Δ^* such that if $\Delta \in (0, \Delta^*]$ then satisfaction of the constraints of Eqs.16-17 ensures convergence to the desired neighborhood of the origin. In the proof, we show the existence of the positive real number ϵ^{**} (yielding Ω^+) which ensures initial and continued satisfaction of the constraints of Eqs.16-17 for all $x_0 \in \Omega^+$. From the continuity of the functions $f(\cdot)$, $G(\cdot)$ $L_fV(\cdot)$ $L_GV(\cdot)$, the boundedness of u and by restricting the state x_0 to the set Ω^+, it follows that given ϵ^* and Δ^* there exists a positive real number ϵ^{**} such that if $L_fV(x_0) + L_GV(x_0)u_0 \leq -\epsilon^{**}$ then $L_fV(x(\tau)) + L_GV(x(\tau))u_0 \leq -\epsilon^* \ \forall \ \tau \in (0, \Delta^*]$, where ϵ^*, Δ^* were defined in Theorem 1. This ensures initial feasibility of the constraints of Eq.16 for all $x_0 \in \Omega^+$. Initial satisfaction of the constraints ensures that $V(x(t+\Delta)) \leq V(x(t))$, which in turn implies that $x(t+\Delta) \in \Omega^+$ for all $t \geq 0$, thereby yielding successive feasibility of the optimization problem. Successive feasibility of the optimization problem leads to convergence to the desired neighborhood of the origin.

Part 2: For all $x_0 \notin \Omega$, the assumption of initial and successive feasibility of the constraint of Eq.18 ensures that $x(t+\tau) \in \Pi^+$ for all $x(t) \notin \Omega^+$, $\tau \in (0, \Delta^*]$. Also, the satisfaction of the constraint of Eq.16 ensures that the value of the Lyapunov function continues to decrease, implying that the state trajectory eventually converges to the set Ω^+. Convergence to $\limsup_{t\to\infty} \|x(t)\| \leq d$ follow from part 1 above. This concludes the proof of Theorem 3.

Remark 3: For initial conditions within a level set of the Lyapunov function (Ω^+), successive decays in the Lyapunov function value is achievable and sufficient to drive the state to the desired neighborhood of the origin. For initial conditions outside the set Ω^+, the constraint of Eq.18 asks for the control action to be computed such that for the process state at the next time instant, negative definiteness of $\dot{V}$ can be successively achieved. This ensures that out of all possible control actions that can achieve negative definiteness of $\dot{V}$, one is chosen that ensures that the state trajectory stays within Π^+ from where continued decay of the Lyapunov function value is possible. A continued decay in the Lyapunov function value leads to convergence to the desired neighborhood of the origin. Note also that in contrast to the result on linear system, guaranteed feasibility for all initial conditions in the null controllable region simply cannot be achieved, yet Eq.18 represents a constraint that at least guides the control law to take some meaningful control action for initial conditions outside Ω^+. This constraint goes beyond (and does better than) simply requiring a decay in the value of the Lyapunov function and enables stabilization from a larger set of initial conditions (see the simulation example for a demonstration).

2) Illustrative chemical process example : Consider a continuous stirred tank reactor where an irreversible, first-order exothermic reaction of the form $A \xrightarrow{k} B$ takes place. The mathematical model for the process takes the form:

$$
\begin{aligned}
\dot{C_A} &= \frac{F}{V}(C_{A0} - C_A) - k_0 e^{\frac{-E}{RT_R}} C_A \\
\dot{T_R} &= \frac{F}{V}(T_{A0} - T_R) + \frac{(-\Delta H)}{\rho c_p} k_0 e^{\frac{-E}{RT_R}} C_A + \frac{Q}{\rho c_p V}
\end{aligned}
\tag{20}
$$

where C_A denotes the concentration of the species A, T_R denotes the temperature of the reactor, Q is the heat added to the reactor, V is the volume of the reactor, k_0, E, ΔH are the pre-exponential constant, the activation energy, and the enthalpy of the reaction and c_p and ρ are the heat capacity and fluid density in the reactor. The values of all process parameters can be found in [26]. The control objective is to stabilize the reactor at the unstable equilibrium point $(C_A^s,\ T_R^s) = \ (0.57\ Kmol/m^3, 395.3\ K)$ using the rate of heat input, Q, and change in inlet concentration of species A, $\Delta C_A = C_{A0} - C_{A0_s}$ as manipulated inputs with constraints: $|Q| \leq 32\ KJ/s$ and $|\Delta C_{A0}| \leq 1\ Kmol/m^3$. We first construct a Lyapunov-based predictive controller using a $V(x) = x'Px$ where $x = (C_A - C_A^s,\ T_R - T_R^s)$, $P = \begin{pmatrix} 0.983 & 0.025 \\ 0.025 & 0.001 \end{pmatrix}$ where the matrix P is computed by solving the Riccati inequality with the linearized system matrices. The parameters in the objective function of Eq.19 are chosen as $Q = qI$, with $q = 0.1$, and $R = \begin{pmatrix} 10.0 & 0.0 \\ 0.0 & 10000.0 \end{pmatrix}$. The set Π and the stability region estimate under the Lyapunov-based controller Ω are computed and shown in Fig.1. The constrained nonlinear optimization problem is solved using the MATLAB subroutine FMINCON, and the set of ODEs is integrated using the MATLAB solver ODE15s.

To illustrate the enhancement in the set of initial conditions from where closed–loop stability can be achieved using the proposed controller, we pick an initial condition $C_A(0),\ T_R(0) = 1.113\ kmol/m^3,\ 395.3\ K$ outside Ω^+ but inside Π^+. We first implement the Lyapunov-based predictive controller of Theorem 1 that only requires the value of the Lyapunov function to decrease. Since the initial condition is within the set Π^+, there exists a control action that can enforce negative definiteness of the Lyapunov function and the controller proceeds to implement such control action. However, enforcing negative definiteness of $\dot{V}$ (i.e., driving the trajectory to successively lower level curves of the Lyapunov function), is not sufficient to ensure that the trajectory remains within the set Π^+. At $t = 0.12$ min, the state trajectory escapes out of Π^+, and it is no longer possible to find a control action that enforces negative definiteness of $\dot{V}$. If the stability constraints are removed to allow feasibility of the optimization problem, the value of the Lyapunov function continues to increase and closed–loop stability is not achieved. In contrast, if the proposed predictive controller is implemented, it not only enforces negative definiteness of $\dot{V}$, but also ensures that the state trajectory does not escape Π^+. In other words, out of possible state trajectories along decreasing values of the level curves of $V(x)$, those are chosen (if they exist) that keep the state profile in Π^+. Closed-loop stability is thereby achieved, demonstrating an expansion on the set of initial conditions from where closed–loop stability can be achieved by better utilizing the constraint enforcing capabilities of the predictive control approach.

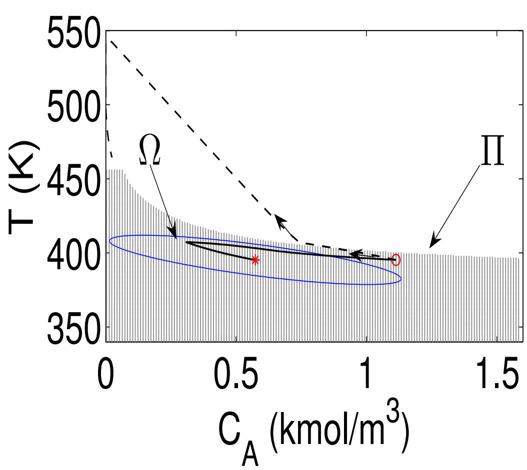

Fig. 1. Evolution of the state trajectory for the chemical reactor example under a Lyapunov-based predictive controller of [21] (dashed line) with a stability region Ω and under the proposed predictive controller (solid line) enabling stabilization from initial conditions outside Ω.

In conclusion, this work considered the problem of predictive control of nonlinear process systems subject to input constraints. A predictive controller for linear systems was first designed to achieve stability for every initial condition in the null controllable region without resorting to infinite horizons. For nonlinear process systems, predictive controllers were designed to expand on the set of initial conditions from where closed–loop stability is achievable. The proposed method was illustrated using a chemical reactor example.

REFERENCES

[1] N. Kapoor, A. R. Teel, and P. Daoutidis, "An anti-windup design for linear systems with input saturation," *Automatica*, vol. 34, pp. 559–574, 1998.

[2] Y. Lin and E. D. Sontag, "A universal formula for stabilization with bounded controls," *Syst. & Contr. Lett.*, vol. 16, pp. 393–397, 1991.

[3] S. Valluri and M. Soroush, "Analytical control of SISO nonlinear processes with input constraints," *AIChE J.*, vol. 44, pp. 116–130, 1998.

[4] N. Kapoor and P. Daoutidis, "Stabilization of nonlinear processes with input constraints," *Comp. & Chem. Eng.*, vol. 24, pp. 9–21, 2000.

[5] N. H. El-Farra and P. D. Christofides, "Integrating robustness, optimality, and constraints in control of nonlinear processes," *Chem. Eng. Sci.*, vol. 56, pp. 1841–1868, 2001.

[6] N. H. El-Farra and P. D. Christofides, "Bounded robust control of constrained multivariable nonlinear processes," *Chem. Eng. Sci.*, vol. 58, pp. 3025–3047, 2003.

[7] P. Mhaskar, N. H. El-Farra, and P. D. Christofides, "Hybrid predictive control of process systems," *AIChE J.*, vol. 50, pp. 1242–1259, 2004.

[8] W. B. Bequette, "Nonlinear control of chemical processes: A review," *Ind. & Eng. Chem. Res.*, vol. 30, pp. 1391–1413, 1991.

[9] P. D. Christofides and N. H. El-Farra, *Control of Nonlinear and Hybrid Process Systems: Designs for Uncertainty, Constraints and Time-Delays*. Berlin, Germany: Springer-Verlag, 2005.

[10] K. R. Muske and J. B. Rawlings, "Model predictive control with linear-models," *AIChE J.*, vol. 39, pp. 262–287, 1993.

[11] F. Allgower and H. Chen, "Nonlinear model predictive control schemes with guaranteed stability," in R. Berber and C. Kravaris (Eds.), *NATO ASI on Nonlinear Model Based Process Control*, (Kluwer Academic Publishers, Dordrecht, The Netherlands), pp. 465–494, 1998.

[12] P. O. M. Scokaert and J. B. Rawlings, "Feasibility issues in linear model predictive control," *AIChE J.*, vol. 45, pp. 1649–1259, 1999.

[13] J. A. Primbs, V. Nevistic, and J. C. Doyle, "A receding horizon generalization of pointwise min-norm controllers," *IEEE Trans. Automat. Contr.*, vol. 45, pp. 898–909, 2000.

[14] Z. Y. Wan and M. V. Kothare, "Efficient scheduled stabilizing model predictive control for constrained nonlinear systems," *Int. J. Rob. & Non. Contr.*, vol. 13, pp. 331–346, 2003.

[15] D. Limon, T. Alamo, F. Salas, and E. F. Camacho, "On the stability of constrained MPC without terminal constraint," *IEEE Trans. Automat. Contr.*, vol. 51, pp. 832–836, 2006.

[16] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert, "Constrained model predictive control: Stability and optimality," *Automatica*, vol. 36, pp. 789–814, 2000.

[17] J. LeMay, "Recoverable and reachable zones for control systems with linear plants and bounded controller outputs," *IEEE Trans. Automat. Contr.*, vol. 9, pp. 346–354, 1964.

[18] T. Hu, Z. Lin, and L. Qiu, "An explicit description of null controllable regions of linear systems with saturating actuators," *Sys. & Contr. Lett.*, vol. 47, pp. 65–78, 2002.

[19] D. Chmielewski and V. Manousiouthakis, "On constrained infinite-time linear quadratic optimal control," *Syst. Contr. &. Lett.*, vol. 29, pp. 121–129, 1996.

[20] R. Goebel, "Stabilizing a linear system with saturation through optimal control," *IEEE Trans. Automat. Contr.*, vol. 50, pp. 650–655, 2005.

[21] P. Mhaskar, N. H. El-Farra, and P. D. Christofides, "Predictive control of switched nonlinear systems with scheduled mode transitions," *IEEE Trans. Automat. Contr.*, vol. 50, pp. 1670–1680, 2005.

[22] P. Mhaskar, N. H. El-Farra, and P. D. Christofides, "Stabilization of nonlinear systems with state and control constraints using Lyapunov-based predictive control," *Syst. & Contr. Lett.*, vol. 55, pp. 650–659, 2006.

[23] P. O. Gutman and M. Cwikel, "Admissible-sets and feedback-control for discrete-time linear dynamic-systems with bounded controls and states," *IEEE Trans. Automat. Contr.*, vol. 31, pp. 373–376, 1986.

[24] J. Stephan, M. Bodson, and J. Lehoczky, "Calculation of recoverable sets for systems with input and state constraints," *Opt. Contr. Appl. & Meth.*, vol. 19, pp. 247–269, 1998.

[25] G. Bitsoris and M. Vassilaki, "Constrained regulation of linear systems," *Automatica*, vol. 31, pp. 223–227, 1995.

[26] M. Mahmood and P. Mhaskar, "Enhanced stability regions for model predictive control of nonlinear process systems," *AIChE J.,* accepted for publication, 2008.