2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Optimal Life Insurance, Consumption and Portfolio: A Dynamic Programming Approach

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Abstract— **A continuous-time model of optimal life insurance, consumption and portfolio is examined by dynamic programming technique. The Hamilton-Jacobi-Bellman (HJB in short) equation with the absorbing boundary condition is derived. Then explicit solutions for Constant Relative Risk Aversion (CRRA in short) utilities with subsistence levels are obtained. Asymptotic analysis is used to analyze the model.**

Key words: life insurance, consumption/investment, HJB equation, absorbing boundary condition, CRRA utilities with subsistence levels, asymptotic analysis.

I. Introduction

This paper considers the optimal life insurance purchase, consumption and portfolio management strategies for a wage earner subject to mortality risk in a continuous time economy. Decisions are made continuously about these three strategies for all time $t \in [0, T]$, where the fixed planning horizon T can be interpreted as the retirement time of the wage earner.

The wage earner receives his income at rate $i(t)$ continuously, but this is terminated by the wage earner's death or retirement, whichever happens first. We use a random variable to model the wage earner's lifetime. The wage earner needs to buy life insurance to protect his family due to his premature death. Aside from consumption and life insurance purchase, the wage earner also has the opportunity to invest in the capital market which consists of a riskless security and a risky security.

Yarri [8] provided the key idea for research on life insurance, consumption and/or portfolio decisions under an uncertain lifetime, that is, the problem under the random horizon can be converted to one under the fixed horizon. the uncertainty of lifetime is the sole source of uncertainty in his model, and he used a nonnegative bounded random variable to model uncertain lifetime. Leung [3] pointed out that Yaari's model cannot have an interior solution which lasts until the maximum lifetime for the optimal consumption. Ye [9] showed that using a nonnegative bounded random variavle to model uncertain lifetime will produce abnormal behavior of the model from the perspective of dynamic programming. Merton [5] briefly considered consumption/portfolio under an uncertain life using the dynamic programming approach, but he ignored life insurance. Richard [7] used Yaari's setting for an uncertain lifetime and dynamic programming to consider a life-cycle life insurance and consumption/investment problem. Richard introduced the concept of a continuous-time life insurance market where the wage earner continuously buys term life insurance, the term being infinitesimally short. But the terminal condition for his model is in question. Recently, Pliska and Ye [6] used a comparative technique to study the demands of life insurance. Ye [10] summarized the results from the martingale approach in Ye [9]. Zhu [11] studied individual consumption, life insurance, and portfolio decisions in one-period environment.

This paper is organized as follows. The next section describes the intemporal model proposed in Ye [9]. Section 3 derives the HJB equation with the absorbing boundary condition which is important in numerical research, and then derives the optimal feedback control. In Section 4 we obtain explicit solutions for the family of CRRA utilities with subsistence requirements, and asymptotic analysis is used to analyze the model. We conclude with some remarks in Section 5.

II. The Model

Let $W(t)$ be a standard 1-dimensional Brownian motion defined on a given probability space $(\Omega, \mathcal{F}, P)$. Let $T < \infty$ be a fixed planning horizon, here interpreted as the wage earner's retirement time. Let $\mathbb{F} = \{\mathcal{F}_t, t \in [0, T]\}$ be the P-augmentation of the filtration $\sigma\{W(s), s \leq t\}, t \in [0, T]$, so $\mathcal{F}_t$ represents the information at time t.

The continuous-time economy consists of a frictionless financial market and an insurance market. We are going to describe their details separately in the following.

We assume that there is a risk-free security in the financial market whose time-t price is denoted by $S_0(t)$. It evolves according to

$$\frac{dS_0(t)}{S_0(t)} = r(t)dt, \tag{1}$$

where $r(\cdot)$ is a function of time t satisfying $\int_0^T |r(t)|dt < \infty$. This condition ensures the above equation is well-defined.

There is a risky security in the financial market. It evolves according to the linear stochastic differential

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978-1-4244-2079-7/08/$25.00 ©2008 AACC.

equation

$$\frac{dS_1(t)}{S_1(t)} = \mu(t)dt + \sigma(t)dW(t), \tag{2}$$

where $\mu(\cdot)$ and $\sigma(\cdot)$ are functions of time t satisfying $\int_0^T |(\mu(t) - r(t))/\sigma(t)| < \infty$. This condition ensures the financial market is complete (see Chapter 1, Karatzas and Shreve [2]).

We suppose the wage earner is alive at time $t = 0$ and his lifetime is denoted by a random variable τ. The hazard rate function of τ is denoted by $\lambda(t), t \in [0, T]$. According to Collett [1], the conditional probability density function, $f(s, t)$, for the death at time s conditional upon the wage earner being alive at time $t \leq s$ is given by

$$f(s,t) \triangleq \lambda(s) \exp\left\{ -\int_t^s \lambda(u)du \right\}. \tag{3}$$

Suppose life insurance is offered continuously and the wage earner enters a life insurance contract by paying premiums at the rate $p(t)$ at each point in time t. In compensation, if the wage earner dies at time t when the premium payment rate is $p(t)$, then the insurance company pays an insurance amount $p(t)/\eta(t)$. Here $\eta(t)$ is called *the insurance premium-payout ratio*.

Suppose the wage earner is endowed with the initial wealth x and will receive the income at rate $i(t)$ during the period $[0, \min\{T, \tau\}]$, that is, during a period which will be terminated by the wage earner's death or retirement at T, whichever happens first.

We now define some processes describing the wage earner's decisions:

- $c(t) \triangleq$ Consumption rate at time t, which is an $\{\mathcal{F}_t\}$-progressively measurable, nonnegative process satisfying $\int_0^T c(t)dt < \infty$ almost surely.
- $p(t) \triangleq$ Premium rate (e.g., dollars per annum) at time t, which is an $\{\mathcal{F}_t\}$-predictable process satisfying $\int_0^T |p(t)|dt < \infty$ almost surely.
- $\theta(t) \triangleq$ Dollar amount in the risky security at time t, which is an $\{\mathcal{F}_t\}$-progressively measurable process satisfying $\int_0^T \sigma^2(t)\theta^2(t)dt < \infty$ almost surely.

For a wage earner's decision, (c, p, θ), the wealth process $X(t)$ on $[0, \min\{T, \tau\}]$ is defined by

$$\begin{aligned} X(t) &= x - \int_0^t c(s)ds - \int_0^t p(s)ds + \int_0^t i(s)ds \\ &\quad + \int_0^t \frac{X(s) - \theta(s)}{S_0(s)} dS_0(s) + \int_0^t \frac{\theta(s)}{S_1(s)} dS_1(s). \end{aligned} \tag{4}$$

Using (1) and (2), we write (4) as the stochastic differential equation

$$\begin{aligned} dX(t) &= r(t)X(t)dt - c(t)dt - p(t)dt + i(t)dt \\ &\quad + \theta(t)[(\mu(t) - r(t))dt + \sigma(t)dW(t)]. \end{aligned} \tag{5}$$

If the wage earner dies at time t, $0 < t \leq T$, the estate will get the insurance amount $p(t)/\eta(t)$. Then the wage earner's total bequest when he dies at time t with wealth $X(t)$ is

$$Z(t) = X(t) + \frac{p(t)}{\eta(t)} \quad \text{on} \quad \{\tau = t\}. \tag{6}$$

We denote by $\mathcal{A}(x)$ the control set of all 3-tuples (c, p, θ) such that $X(t) + b(t) \geq 0$ and $Z(t) \geq 0, \forall t \in [0, T]$, where $b(t)$ is defined as

$$b(t) = \int_t^T i(s) \exp\left\{ -\int_t^s [r(v) + \eta(v)]dv \right\} ds. \tag{7}$$

A $(c, p, \theta) \in \mathcal{A}(x)$ is called as an admissible decision.

Remark 2.1:

- Here we give an economic meaning for $b(t)$. Suppose the wage earner wants to borrow money from a financial institution using his future income as a mortgage. The question is how much the wage earner can borrow from the financial institution. We analyze this problem as follows. The financial institution issues a loan to the wage earner at time t and the wage earner transfers all of his future income $i(s), t \leq s \leq T$, to the financial institution. However, the wage earner's life is uncertain, so the financial institution buys life insurance for the wage earner in order to prevent a loss due to the wage earner's premature death before he pays off the loan. Hence the financial institution uses the wage earner's future income to pay the loan and pay the life insurance premiums. If the wage earner dies at time s, where $t < s < T$, then the insured amount must be enough to pay off the loan, and if he is alive at the time T, his wage accumulating from time t to time T must be enough to pay off the insurance premiums accumulating from time t to time T and pay off the loan which is continuously compounded by the interest rate $r(\cdot)$ (in general, the loan rate is higher than the risk-free rate in reality. The methodology in this item of this remark can still be applied if you use the actual loan rate.). Let $l(s), t < s \leq T$, be the time s principal balance for the loan process described above, so

$$\begin{cases} l(s) = \int_t^s r(u)l(u)du \\ \qquad + \int_t^s (i(u) - \tilde{p}(u))du \quad on \quad \{\tau \geq s\}, \\ l(s) + \frac{\tilde{p}(s)}{\eta(s)} \geq 0 \quad on \quad \{\tau = s\}, \\ l(T) \geq 0 \quad on \quad \{\tau > T\}, \end{cases}$$

where $\tilde{p}(s)$ is the insurance premiums paid at time s by the financial institution to hedge the wage earner's mortality risk. Rewriting the above equation,

$$\begin{cases} l(s) \leq \int_t^s (r(u)+\eta(u))l(u)du \\ \qquad + \int_t^s i(u)du \quad on \quad \{\tau \geq s\}, \\ l(T) \geq 0 \quad on \quad \{\tau > T\}. \end{cases}$$

Introduce the variable substitution $w = T - s$, use Grönwall's inequality, and do some algebra, then we have

$$\begin{aligned} l(t) &\geq -\int_t^T i(u) \exp\left\{-\int_t^u [r(v)+\eta(v)]dv\right\} du \\ &= -b(t). \end{aligned}$$

Hence the wage earner can borrow $b(t)$ at most from the financial institution using the future income as the mortgage. Now it is clear that the function $b(t)$ represents the value at time t of the wage earner's future income from time t to time T.

- We interpret $X(t)+b(t)$ as the total wealth at time t, so it is reasonable that we require $X(t)+b(t)$ to be nonnegative. In fact, we can show once the wage earner's wealth $X(t)$ reaches $-b(t)$, then no further consumption for the wage earner can take place. Moreover, if he dies between time t and time T with $X(t) = -b(t)$ at time t, his bequest is 0 almost surely, and his terminal wealth $X(T)$ is 0 almost surely (see Ye [9]). In other words, $X(t) = -b(t)$ is an absorbing state for the wealth process $X(\cdot)$ when $(c, p, \theta) \in \mathcal{A}(x)$. Hence $\mathcal{A}(x)$ is an empty set when $x < -b(t)$.

Suppose that the wage earner's preference structure is given by (U_1, U_2, U_3). $U_1(\cdot, t)$ is a utility function for the consumption with the subsistence consumption $\bar{c}(t)$ assumed to be a nonnegative function of time t, $U_2(\cdot, t)$ is a utility function for the bequest with the subsistence bequest $\bar{Z}(t)$ assumed to be a nonnegative function of time t, and $U_3(\cdot)$ is a utility function for the terminal wealther with the subsistence terminal wealth $\bar{X}$ assumed to be a nonnegative number.

Remark 2.2: One refers to Ye [9], [10] for the mathematical definition of a preference structure. The subsistence levels $\bar{c}(t)$ and $\bar{Z}(t)$ at time t mean the wage earner does not want his consumption $c(t)$ and the bequest $Z(t)$ lower than $\bar{c}(t)$ and $\bar{Z}(t)$ at time t, respectively, and the subsistence level $\bar{X}$ means he does not want his terminal wealth $X(T)$ lower than $\bar{X}$. From the perspective of mathematics, the subsistence levels impose implicit constraints on the wage earner's decisions, viz., $c(t) \geq \bar{c}(t)$, $\forall t \in [0, T]$, $Z(t) \geq \bar{Z}(t)$, $\forall t \in [0, T]$, and $X(T) \geq \bar{X}$. As shown in Ye [9], these implicit constraints can be satisfied only if

$$X(t) + b(t) \geq \bar{b}(t), \quad \forall t \in [0, T], \tag{8}$$

where $\bar{b}(t)$ is defined as

$$\begin{aligned} &\bar{b}(t) \\ &= \int_t^T \left(\bar{c}(s) + \eta(s)\bar{Z}(s)\right) \exp\left\{-\int_t^s [r(v)+\eta(v)]dv\right\} ds \\ &\quad + \bar{X} \exp\left\{-\int_t^T [r(s)+\eta(s)]\, ds\right\}. \end{aligned} \tag{9}$$

Furthermore, once the wage earner's wealth $X(t)$ reaches $\bar{b}(t) - b(t)$, then $c(s) = \bar{c}(s)$ and $Z(s) = \bar{Z}(s)$ for any s satisfying $t \leq s \leq T$, and $X(T) = \bar{X}$. Hence we interpret $X(t) + b(t) - \bar{b}(t)$ as the total available wealth at time t while $x(t) + b(t)$ is the total wealth at time t. Note that $\bar{c}(\cdot)$, $\bar{Z}(\cdot)$, and $\bar{X}$ are nonnegative, then the subsistence levels impose a more restrictive constraint on the wealth process than admissible decisions do.

The wage earner's problem is to choose life insurance purchase and consumption/portfolio investment strategies so as to maximize the expected utility

$$\begin{aligned} V(x) \triangleq \sup_{(c,p,\theta)\in\mathcal{A}_1(x)} E\Bigg[&\int_0^{T\wedge\tau} U_1(c(s), s)ds \\ &+ U_2(Z(\tau), \tau)1_{\{\tau\leq T\}} \\ &+ U_3(X(T))1_{\{\tau>T\}}\Bigg] \end{aligned} \tag{10}$$

where $T \wedge \tau \triangleq \min\{T, \tau\}$, and where

$$\begin{aligned} \mathcal{A}_1(x) \triangleq \Bigg\{&(c, p, \theta) \in \mathcal{A}(x); \\ E\Bigg[&\int_0^{T\wedge\tau} U_1^-(s, c(s))ds + U_2^-(\tau, Z(\tau))1_{\{\tau\leq T\}} \\ &+ U_3^-(X(T))1_{\{\tau>T\}}\Bigg] > -\infty\Bigg\} \end{aligned}$$

where $U_i^- \triangleq -\min\{0, U_i\}$, for $i = 1, 2, 3$.

The function of $\mathcal{A}_1(x)$ is to pick out every admissible control which satisfies the subsistence requirements (see Ye [9], [10]).

III. Stochastic Dynamic Programming

In this section we use the stochastic dynamic programming technique to derive the HJB equation, and then derive the optimal feedback control from the HJB equation. We restate (10) in a dynamic programming form. For any $(c, p, \theta) \in \mathcal{A}_1(t, x)$, where the definition of $\mathcal{A}_1(t, x)$ is similar to the definition $\mathcal{A}_1(x)$ except that the starting time is time t and the wealth at time t is x, define

$$\begin{aligned} &J(t, x; c, p, \theta) \\ &\triangleq E\Bigg[\int_t^{T\wedge\tau} U_1(c(s), s)ds + U_2(Z(\tau), \tau)1_{\{\tau\leq T\}} \\ &\qquad + U_3(X(T))1_{\{\tau>T\}}\Bigg|\tau > t, \mathcal{F}_t\Bigg] \end{aligned} \tag{11}$$

and

$$V(t,x) \triangleq \sup_{\{c,p,\theta\}\in\mathcal{A}_1(t,x)} J(t,x;c,p,\theta). \quad (12)$$

The above value function is nontraditional due to the random horizon. Using Fubini-Tonelli theorem (see Ye [9]), we can $J(t,x;c,p,\theta)$ as follows:

Lemma 3.1: If the death time τ is independent of the filtration $\mathbb{F}$. For each $(c,p,\theta)\in\mathcal{A}_1(x,i)$,

$$\begin{aligned} &J(t,x;c,p,\theta)\\ &= E\Bigg[\int_t^T [\bar{F}(u,t)U_1(c(u),u) + f(u,t)U_2(Z(u),u)]du \\ &\quad + \bar{F}(T,t)U_3(X(T))\Big|\mathcal{F}_t\Bigg], \end{aligned} \quad (13)$$

where $\bar{F}(u,t)\triangleq \exp\left\{-\int_t^s \lambda(u)du\right\}$ and $f(u,t)$ is given by (3).

From Lemma 3.1, we know that the wage earner who faces unpredictable death acts as if he will live at least until time T, but with a subjective rate of time preferences equal to his "force of mortality" for his consumption and terminal wealth. From the mathematical point of view, this lemma enables us to convert the optimization problem with a random terminal time to a problem with a fixed terminal time.

According to Ye [9], that is, set up the optimality principal and use Itô's lemma, we derive so-called HJB equation

$$\begin{cases} V_t(t,x) - \lambda(t)V(t,x) \\ +\sup_{(c\ge\bar{c}(t),p\ge\eta(t)(\bar{Z}(t)-x),\theta)}\Psi(t,x;c,p,\theta) = 0, \\ V(T,x) = U_3(x), \end{cases} \quad (14)$$

on the domain $D\triangleq\{(t,x)\in[0,T]\times(-\infty,+\infty), x > \bar{b}(t)-b(t)\}$. Where

$$\begin{aligned} \Psi(t,x;c,p,\theta) \triangleq\ & (r(t)x + \theta(\mu(t)-r(t)) + i(t) \\ & -c-p)V_x(t,x) + \frac{1}{2}\sigma^2(t)\theta^2 V_{xx}(t,x) \\ & +U_1(c,t) + \lambda(t)U_2\left(x+p/\eta(t),t\right). \end{aligned} \quad (15)$$

Moreover, V satisfies the absorbing boundary condition (see (8))

$$\begin{aligned} &V(t,\bar{b}(t)-b(t)) \\ &= \int_t^T [\bar{F}(u,t)U_1(\bar{c}(u),u) + f(u,t)U_2(\bar{Z}(u),u)]du \\ &\quad + \bar{F}(T,t)U_3(\bar{X}). \end{aligned} \quad (16)$$

The boundary condition (16) for V follows from Remark 2.2 and Lemma 3.1. The first-order conditions for a regular interior maximum to (15) are

$$0 = \Psi_c(t,x;c^*,p^*,\theta^*) = -V_x(t,x) + U_{1,c}(c^*,t), \quad (17)$$

$$\begin{aligned} 0 &= \Psi_p(t,x;c^*,p^*,\theta^*) \\ &= -V_x(t,x) + \frac{\lambda(t)}{\eta(t)}U_{2,Z}\left(x+p^*/\eta(t),t\right), \end{aligned} \quad (18)$$

and

$$\begin{aligned} 0 &= \Psi_\theta(t,x;c^*,p^*,\theta^*) \\ &= (\mu(t)-r(t))V_x(t,x) + \sigma^2(t)\theta^* V_{xx}(t,x). \end{aligned} \quad (19)$$

A set of sufficient conditions for a regular interior maximum is

$$\Psi_{cc} = U_{1,cc}(c^*,t) < 0, \quad \Psi_{pp} = \frac{\lambda(t)}{\eta^2(t)}U_{2,ZZ}(Z^*,t) < 0,$$

$$\Psi_{\theta\theta} = \sigma^2(t)V_{xx}(t,x) < 0.$$

Note that the first two conditions are automatically satisfied according to the definition of utility functions. Thus a sufficient condition for a maximum is $V_{xx}(t,x) < 0$.

IV. The Case of Constant Relative Risk Aversion

In this section we derive explicit solutions for the case where the wage earner has the same constant relative risk aversion for the consumption, the bequest and the terminal wealth. Assume for $\gamma < 1$, $\rho > 0$, $a_i(t) > 0$, $\forall t\in[0,T]$, $i=1,2$, and $a_3 > 0$ that

$$U_1(c,t) = \frac{e^{-\rho t}}{\gamma}a_1(t)(c-\bar{c}(t))^\gamma,$$

$$U_2(Z,t) = \frac{e^{-\rho t}}{\gamma}a_2(t)(Z-\bar{Z}(t))^\gamma,$$

and

$$U_3(x) = \frac{e^{-\rho T}}{\gamma}a_3(x-\bar{X})^\gamma,$$

where $c\ge\bar{c}(t)$ and $Z\ge\bar{Z}(t)$ for all $t\in[0,T]$, and $x\ge\bar{X}$. Here $\gamma=0$ represents the logarithmic utility functions.

From (17), (18), and (19), we have that

$$c^*(t) = \bar{c}(t) + \left(\frac{a_1(t)}{V_x e^{\rho t}}\right)^{1/(1-\gamma)}, \quad (20)$$

$$x + \frac{p^*(t)}{\eta(t)} = \bar{Z}(t) + \left(\frac{a_2(t)}{V_x e^{\rho t}}\frac{\lambda(t)}{\eta(t)}\right)^{1/(1-\gamma)}, \quad (21)$$

$$\theta^*(t) = -\frac{(\mu(t)-r(t))V_x}{\sigma^2(t)V_{xx}}. \quad (22)$$

We now plug (20)-(22) in (14) and take as a trial solution

$$V(t,x) = \frac{a(t)}{\gamma}(x+b(t)-\bar{b}(t))^\gamma, \quad x\ge\bar{b}(t)-b(t), \quad (23)$$

where $a(\cdot)$ is a function to be determined. Then $a(t)$ must satisfy the following ordinary differential equation:

$$\begin{aligned} &\frac{da(t)}{dt} \\ &= \left(\lambda(t) - \frac{\gamma}{2(1-\gamma)}\left(\frac{\mu(t)-r(t)}{\sigma(t)}\right)^2 - \gamma(r(t)+\eta(t))\right)a(t) \\ &\quad -(1-\gamma)e^{-\rho t/(1-\gamma)}K(t)(a(t))^{-\gamma/(1-\gamma)} \end{aligned}$$

with $a(T) = a_3 e^{-\rho T}$, where

$$K(t) \triangleq (a_1(t))^{1/(1-\gamma)} + (a_2(t))^{1/(1-\gamma)} \frac{(\lambda(t))^{1/(1-\gamma)}}{(\eta(t))^{\gamma/(1-\gamma)}}. \quad (24)$$

To solve for $a(t)$, suppose for some new function $m(\cdot)$ that $a(t)$ has the form:

$$a(t) = e^{-\rho t}(m(t))^{1-\gamma}, \quad (25)$$

and define

$$H(t) \triangleq \frac{\lambda(t)+\rho}{1-\gamma} - \frac{1}{2}\gamma\left(\frac{\mu(t)-r(t)}{(1-\gamma)\sigma(t)}\right)^2 - \frac{\gamma}{1-\gamma}(r(t)+\eta(t)). \quad (26)$$

Then

$$\frac{dm(t)}{dt} - H(t)m(t) + K(t) = 0, \qquad m(T) = a_3^{1/(1-\gamma)}. \quad (27)$$

Solving (27) by an integrating factor, we get that

$$\begin{aligned} m(t) &= a_3^{1/(1-\gamma)} \exp\left\{-\int_t^T H(v)dv\right\} \\ &\quad + \int_t^T \exp\left\{-\int_t^s H(v)dv\right\} K(s)ds. \quad (28) \end{aligned}$$

Hence

$$\begin{aligned} a(t) &= e^{-\rho t}\Bigg[a_3^{1/(1-\gamma)} \exp\left\{-\int_t^T H(v)dv\right\} \\ &\quad + \int_t^T \exp\left\{-\int_t^s H(v)dv\right\} K(s)ds\Bigg]^{1-\gamma}. \quad (29) \end{aligned}$$

From (20)-(22), (23) and (25) the optimal consumption, life insurance and portfolio rules can be explicitly written in feedback form as

$$c^*(t) = \bar{c}(t) + \frac{(a_1(t))^{1/(1-\gamma)}}{m(t)}(x + b(t) - \bar{b}(t)), \quad (30)$$

$$\begin{aligned} Z^*(t) &= x + \frac{p^*(t)}{\eta(t)} \\ &= \bar{Z}(t) + \left(\frac{\lambda(t)}{\eta(t)}\right)^{1/(1-\gamma)} \frac{(a_2(t))^{1/(1-\gamma)}}{m(t)}(x + b(t) - \bar{b}(t)), \quad (31) \end{aligned}$$

and

$$\theta^*(t) = \frac{\mu(t)-r(t)}{(1-\gamma)\sigma^2(t)}(x + b(t) - \bar{b}(t)). \quad (32)$$

The above formulas for c^* and θ^* are consistent with the classical results in Merton [4]. In particular, the optimal portfolio fraction $\frac{\mu(t)-r(t)}{(1-\gamma)\sigma^2(t)}$ is the same as Merton's. This means, under the assumption of independence between the mortality risk and stock return risk, the mortality risk will not affect the risky investment.

From (30) and (31), the wage earner's optimal decisions c^* and Z^* consist of a "compulsory" part and "free choice" part. The "free choice" part depends on his spare money $x + b(t) - \bar{b}(t)$. The wage earner's optimal portfolio decisions is made based on his spare money $x + b(t) - \bar{b}(t)$. We study the following cases using asymptotic analysis:

- $Z^*(t) \to \bar{Z}(t)$ as $\lambda(t) \to 0$ for any $t \in [0, T]$. This means that a long-lived wage earner will just maintain the subsistence level of the bequest.
- $c^*(t) \to \bar{c}(t)$ as $a_1(t) \to 0$ for any $t \in [0, T]$. This means that the consumption becomes unimportant to the wage earner as the weight a_1 approaches to zero, then the wage earner will just maintain the subsistence level of consumption.
- $Z^*(t) \to \bar{Z}(t)$ as $a_2(t) \to 0$ for any $t \in [0, T]$. This means that the bequest becomes unimportant to the wage earner as the weight a_2 approaches to zero, then the wage earner will just maintain the subsistence level of the bequest.
- $X^*(T) \to \bar{X}$ as $a_3 \to 0$. This means that saving more for the after-retirement life becomes unimportant, then the wage earner will just maintain the subsistence level of the terminal wealth. In this case, setting $\bar{c}(t) = 0$ and $\bar{Z}(t) = 0$, for all $t \in [0, T]$, and $\bar{X} = 0$, the solutions are the same as Richard's [7] if some errors in his solutions are corrected, although the model is an intemporal model while Richard's is a life-cycle model.

From (31), the policy of insurance premiums is given by

$$\begin{aligned} p^*(t) &= \eta(t)\Bigg\{\left[\left(\frac{\lambda(t)}{\eta(t)}\right)^{1/(1-\gamma)} \frac{(a_2(t))^{1/(1-\gamma)}}{m(t)} - 1\right]x \\ &\quad + \left(\frac{\lambda(t)}{\eta(t)}\right)^{1/(1-\gamma)} (a_2(t))^{1/(1-\gamma)} \frac{b(t)-\bar{b}(t)}{m(t)} + \bar{Z}(t)\Bigg\}. \quad (33) \end{aligned}$$

From the above formula, the future income has a positive effect on life insurance purchase as we expect. The subsistence levels of consumption and terminal wealth have a negative effect on life insurance purchase, this means the wage earner tends to buy less life insurance as these two subsistence levels increase. As we carefully examine the effect of the subsistence level of the bequest on life insurance purchase, we find that the current subsistence level of the bequest has a positive effect, while the future subsistence level of the bequest has a negative effect.

Now let's consider a wage earner who starts to work at age 25 (the initial time), his expected retirement time is age 65 (T=65-25=40), and his initial wage at age 25 is $\$50,000$, growing at the rate 3% every year. His risk aversion parameter $\gamma = -3$, the utility discounted rate $\rho = 0.03$, all utility weights are 1, and all subsistence levels are zero. The hazard rate for him is $1/200 + 9/8000t$. The market parameters are given by Table I.

Figure 1 was computed via (30) and shows the optimal consumption proportion, viz., $c^*(t)/(x + b(t))$, as a function of age and the total overall wealth $x + b(t)$. We

TABLE I

THE PARAMETERS

Parameters	r	μ	σ	$\eta(t)$
Value	0.04	0.09	0.18	$\frac{1}{200} + \frac{9}{8000}t$

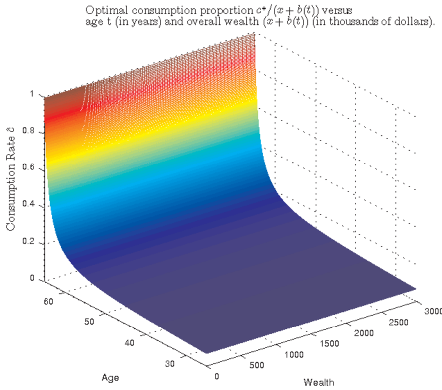

Fig. 1. Optimal consumption proportion without constraints using exact solution

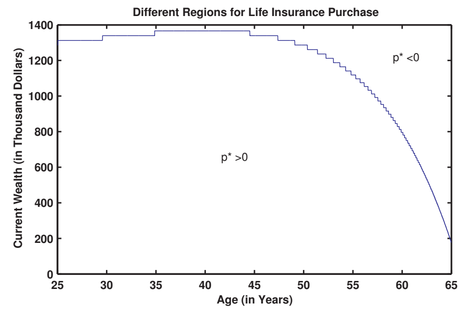

Fig. 3. The critical level curve

see this proportion is constant with respect to overall wealth, it is relatively small in the early years, and it rises with respect to age.

We used formula (33) to produce Figure 2, which shows the optimal life insurance purchase amount $p^*(t)$ in terms of age and the total overall wealth. We see that for small values of overall wealth the optimal insurance payment is increasing with respect to age up to a certain point, and then the payment declines as the wage earner approaches retirement. Moreover, with age fixed the optimal insurance payment is a decreasing function of the overall wealth. Finally, for large values of overall wealth we see that the optimal insurance payment is actually negative. In particular, when the total overall wealth exceeds a critical level which varies with the wage earner's age, it becomes optimal for the wage earner to sell a life insurance policy on his own life. We plot this critical level in terms of the overall wealth and age in Figure 3. When the overall wealth is below the critical level the wage earner buys life insurance, but when the overall wealth is above the critical level the wage earner sells life insurance. In particular, the wage earner will buy life insurance in the early years since the critical level of the wealth is very high at that time. But the wage earner might find it optimal to sell life insurance close to retirement time.

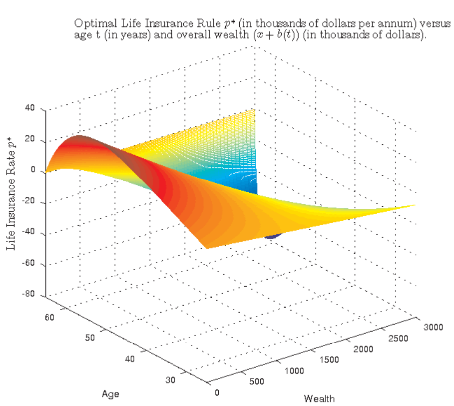

Fig. 2. Optimal life insurance rule without constraints using exact solution

V. DISCUSSION

We derived HJB equation with the absorbing boundary condition for the model. Explicit solutions were found for a rich family of CRRA utilities with subsistence levels. Several economic implications were understood via interpreting the model setting and the solutions. We also used asymptotic analysis to interpret the model.

One point worth mentioning is that the asymptotic results in Section 4 provides an unified perspective for investigating variants of the objective functional (10). For example, if we let $a_2(t) \to 0$ for each $t \in [0, T]$, $\bar{Z}(t) = 0$ for each $t \in [0, T]$, $a_3 \to 0$, and $\bar{X} = 0$, then the asymptotic results correspond to maximizing the expected utility from consumption.

We considered the financial market which consists of a riskless and a risky securities and the insurance market which is allowed to sell life insurance. This combination of the financial market and the insurance market is complete in that the wage earner's any reasonable financial plan $(c, Z(\cdot), X(T))$, $X(T)$ can be viewed as the pension plan, can be replicated in these two markets (see Ye [9]). It is not technically difficult to include multiple risky securities in the model assuming the financial market is complete. However, either incompleteness of the financial market or prohibition of selling life insurance makes the combination incomplete. The incomplete financial market has been extensively studied in the literature. The numerical method has been carried out in Ye [9] to deal with the constraint on life insurance.

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