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Perturbed Feedback Linearization of Attitude Dynamics

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Abstract— **The paper introduces a theoretical foundation of the perturbed feedback linearization methodology for realization of linear spacecraft attitude deviation dynamics. The approach is based on nonuniqueness representation of underdetermined linear algebraic equations solution via nullspace parametrization. A prescribed stable linear second order time-invariant ordinary differential equation in a spacecraft attitude deviation norm measure is evaluated along the solution trajectories of the spacecraft equations of motion, yielding a linear relation in the control variables. Generalized inversion of the relation results in a control law that consists of particular and auxiliary parts. The particular part acts on the range space of the controls coefficient row vector, and it works to drive the spacecraft attitude variables in order to nullify the attitude deviation norm measure. The auxiliary part acts on the complementary orthogonal subspace, and provides the necessary spacecraft internal stability. The null-control vector in the auxiliary part is projected onto the controls coefficient nullspace by a nullprojection matrix, and is designed to yield perturbed feedback linearization of the spacecraft internal dynamics. The feedback control design utilizes the concept of damped generalized inverse to limit the growth of the Moore-Penrose generalized inverse. The control law yields globally uniformly ultimately bounded trajectory tracking errors, and it reveals a tradeoff between trajectory tracking accuracy and damped generalized inverse stability.**

I. INTRODUCTION

Among the interesting and most systematic methodologies applied to the spacecraft control problem are those based on feedback linearizing transformations. Despite the simplicity and richness of linear control theory gained from feedback linearization, the methodology has its disadvantages, among which is the need for mathematical model inversion to obtain the required control forces, which is usually subjected to simplifications and approximations that adversely affect mathematical modeling fidelity.

For the above mentioned reason, it is desirable to find control methodologies that benefit from the elegant linear control theory, and avoid inverting the mathematical model. The controls coefficient generalized inversion-based feedback linearization has been introduced in [1] for this purpose, in the context of spacecraft control. The approach is based on casting the nonlinear spacecraft control problem in a pointwise-linear form, and utilizing a simple linear algebra relation to tackle the problem. The primary tool used is the Moore-Penrose generalized matrix inverse (MPGI) [2], [3].

The control design procedure begins by defining a norm measure function of the spacecraft's attitude variables deviations from their desired values, and prespecifying a stable second-order linear differential equation in the measure function, resembling the desired attitude deviation dynamics. The differential equation is then transformed to a relation that is linear in the control vector by differentiating the norm measure function along the trajectories defined by the solution of the spacecraft's state space mathematical model. The Greville formula [4] is utilized thereafter to invert this relation for the control law required to realize the desired stable linear attitude deviation dynamics.

The derived control law has a special structure. it consists of auxiliary and particular parts, residing in the nullspace of the *controls coefficient* row vector and the range space of its generalized inverse, respectively. The auxiliary part contains a free nullvector, named the *null-control vector*, and is being projected onto the controls coefficient nullspace by means of a nullprojection matrix. Therefore, the choice of the null-control vector does not affect the dynamics of the deviation measure function, and it parameterizes *all* control laws that are capable of realizing that dynamics.

The controls coefficient generalized inversion guarantees stable attitude dynamics feedback linearization. To fulfill internal stability requirement, and inspired by the control law's affinity in the null-control vector, the later has been chosen in [1] to be linear in the angular velocity vector, resulting in a stable perturbed feedback linearization of the spacecraft internal dynamics.

Generalized inversion stability robustness is achieved by modifying the structure of the controls coefficient MPGI by means of a damping factor that limits its growth as steady state response is approached. Depending on the amount of modification, this *damped controls coefficient generalized inverse* results in a tradeoff between trajectory tracking accuracy and generalized inversion stability. The methodology yields desired linear attitude deviation dynamics realization with uniformly ultimately bounded (UUB) trajectory tracking errors, and reveals a tradeoff between trajectory tracking accuracy and damped generalized inverse stability based on the size of the generalized inversion damping factor.

Based on the null-control vector design, this paper enhances the concept of perturbed feedback linearization originally introduced in [1]. The domain of attraction for internal stability can be arbitrarily enlarged by increasing the design controller gain. The analysis provides explicit sufficiency bound on the required gain for a prescribed stability region.

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II. Spacecraft Mathematical Model

The spacecraft mathematical model is given by

$$\dot{\rho} = G(\rho)\omega, \qquad \rho(0) = \rho_0 \tag{1}$$

$$\dot{\omega} = J^{-1}\omega^{\times}J\omega + \tau, \qquad \omega(0) = \omega_0 \tag{2}$$

where $\rho \in \mathbb{R}^{3\times 1}$ is the spacecraft vector of modified Rodrigues attitude parameters (MRPs) [5], $\omega \in \mathbb{R}^{3\times 1}$ is the vector of spacecraft angular velocity components in its body reference frame, $J \in \mathbb{R}^{3\times 3}$ is a diagonal matrix containing the spacecraft's body principal moments of inertia, and $\tau := J^{-1}u \in \mathbb{R}^{3\times 1}$ is the vector of scaled control torques, where $u \in \mathbb{R}^{3\times 1}$ contains the applied jet actuator torque components about the spacecraft's principal axes. The cross product matrix $x^{\times}$ which corresponds to a vector $x \in \mathbb{R}^{3\times 1}$ is skew symmetric of the form

$$x^{\times} = \begin{bmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{bmatrix}$$

and the matrix valued function $G(\rho) : \mathbb{R}^{3\times 1} \to \mathbb{R}^{3\times 3}$ is given by

$$G(\rho) = \frac{1}{2}\left(\frac{1-\rho^T\rho}{2}I_{3\times 3} - \rho^{\times} + \rho\rho^T\right). \tag{3}$$

III. Attitude Deviation Norm Measure Dynamics

Let $\rho_r(t) \in \mathbb{R}^{3\times 1}$ be a prescribed desired spacecraft attitude vector such that $\rho_r(t)$ is twice continuously differentiable in t. The spacecraft attitude deviation vector from $\rho_r(t)$ is defined as

$$z(\rho, t) := \rho - \rho_r(t). \tag{4}$$

Consequently, the scalar attitude deviation norm measure function $\phi : \mathbb{R}^{4\times 1} \to \mathbb{R}$ is defined to be the squared norm of $z(\rho, t)$

$$\phi = \| z(\rho, t) \|^2 = \| \rho - \rho_r(t) \|^2 . \tag{5}$$

The first two time derivatives of ϕ along the spacecraft trajectories given by the solution of (1) and (2) are

$$\dot{\phi} = \frac{\partial\phi}{\partial\rho}G(\rho)\omega + \frac{\partial\phi}{\partial t} \tag{6}$$

$$= 2z^T(\rho, t)\left[G(\rho)\omega - \dot{\rho}_r(t)\right] \tag{7}$$

and

$$\ddot{\phi} = 2\left[G(\rho)\omega - \dot{\rho}_r(t)\right]^T\left[G(\rho)\omega - \dot{\rho}_r(t)\right] + 2z^T(\rho, t)\left[\dot{G}(\rho, \omega)\omega + G(\rho)\left[J^{-1}\omega^{\times}J\omega + \tau\right] - \ddot{\rho}_r(t)\right] \tag{8}$$

where $\dot{G}(\rho, \omega)$ is the time derivative of $G(\rho)$ obtained by differentiating the individual elements of $G(\rho)$ along the kinematical subsystem given by equations (1). We prespecify a desired stable linear second-order dynamics of ϕ in the form

$$\ddot{\phi} + c_1\dot{\phi} + c_2\phi = 0, \quad c_1, c_2 > 0. \tag{9}$$

With ϕ, $\dot{\phi}$, and $\ddot{\phi}$ given by (5), (7), and (92), it is possible to write (9) in the pointwise-linear form

$$\mathcal{A}(\rho, t)\tau = \mathcal{B}(\rho, \omega, t), \tag{10}$$

where the vector valued function $\mathcal{A}(\rho, t) : \mathbb{R}^{4\times 1} \to \mathbb{R}^{1\times 3}$ is given by

$$\mathcal{A}(\rho, t) = 2z^T(\rho, t)G(\rho) \tag{11}$$

and the scalar valued function $\mathcal{B}(\rho, \omega, t) : \mathbb{R}^{7\times 1} \to \mathbb{R}$ is

$$\mathcal{B}(\rho, \omega, t) = -2\left[G(\rho)\omega - \dot{\rho}_r(t)\right]^T\left[G(\rho)\omega - \dot{\rho}_r(t)\right] - 2z^T(\rho, t)\left[\dot{G}(\rho, \omega)\omega + G(\rho)J^{-1}\omega^{\times}J\omega - \ddot{\rho}_r(t)\right] - 2c_1z^T(\rho, t)\left[G(\rho)\omega - \dot{\rho}_r(t)\right] - c_2\| z(\rho, t) \|^2 . \tag{12}$$

The row vector function $\mathcal{A}(\rho, t)$ is the controls coefficient of the attitude deviation norm measure dynamics given by (9) along the spacecraft trajectories, and the scalar function $\mathcal{B}(\rho, \omega, t)$ is the corresponding controls load.

IV. Reference Internal Dynamics

Invertibility of the matrix $G(\rho)$ makes it possible to solve explicitly for the angular velocity vector ω, which takes the form

$$\omega = G^{-1}(\rho)\dot{\rho}. \tag{13}$$

Therefore, a reference vector of dynamic variables $\omega_r(t)$ can be obtained from (1) by substituting the desired vector of kinematic variables $\rho_r(t)$ and its time derivative $\dot{\rho}_r(t)$ in place of ρ and $\dot{\rho}$, respectively, such that

$$\omega_r(t) = G^{-1}(\rho_r(t))\dot{\rho}_r(t). \tag{14}$$

Definition 1 (Realizability of linear attitude deviation norm measure dynamics)**.** For a given twice time continuously differentiable desired spacecraft attitude vector $\rho_d(t)$, the linear attitude deviation norm measure dynamics given by (9) is said to be realizable by the spacecraft equations of motion (1) and (2) at specific values of ρ and t if there exists a control vector τ that solves (10) for these values of ρ and t. If this is true for all ρ and t such that $z(\rho, t) \neq \mathbf{0}_{3\times 1}$, then the linear attitude deviation norm measure dynamics is said to be globally realizable by the spacecraft equations of motion.

V. Linearly Parameterized Attitude Control Laws

For proof of the following proposition, the reader is referred to [1].

Proposition 1 (Linearly parameterized attitude control laws)**.** *For any desired spacecraft attitude vector $\rho_r(t)$, the linear attitude deviation norm measure dynamics given by (9) is globally realizable by the spacecraft equations of motion (1) and (2). Furthermore, the infinite set of all control laws realizing that dynamics by the spacecraft equations of motion is parameterized by an arbitrarily chosen null-control vector $y \in \mathbb{R}^{3\times 1}$ as*

$$\tau = \mathcal{A}^{+}(\rho, t)\mathcal{B}(\rho, \omega, t) + \mathcal{P}(\rho, t)y \tag{15}$$

where "$\mathcal{A}^+$" stands for the MPGI *of the controls coefficient given by*

$$\mathcal{A}^+(\rho,t) = \frac{\mathcal{A}^T(\rho,t)}{\| \mathcal{A}(\rho,t) \|^2}, \quad \mathcal{A}(\rho,t) \neq \mathbf{0}_{1\times 3} \qquad (16)$$

and $\mathcal{P}(\rho,t) \in \mathbb{R}^{3\times 3}$ is the corresponding controls coefficient nullprojector (CCNP) given by

$$\mathcal{P}(\rho,t) = I_{3\times 3} - \mathcal{A}^+(\rho,t)\mathcal{A}(\rho,t). \qquad (17)$$

Any choice of the null-control vector y in the control law expression given by (15) yields a solution to (10). Therefore, the choice of y does not affect realizability of the linear attitude deviation norm measure dynamics given by (9). Nevertheless, the choice of y substantially affects the spacecraft internal state response [6]. In particular, an inadequate choice of y can destabilize the spacecraft internal dynamics given by (2) or causes unsatisfactory closed loop performance. Substituting the control laws expressions given by (15) in the spacecraft's equations of motion (1) and (2) yields the following parametrization of the infinite set of spacecraft closed loop systems equations that realize the dynamics given by (9)

$$\dot{\rho} = G(\rho)\omega \qquad (18)$$

$$\dot{\omega} = J^{-1}\omega^{\times}J\omega + \mathcal{A}^+(\rho,t)\mathcal{B}(\rho,\omega,t) + \mathcal{P}(\rho,t)y. \qquad (19)$$

VI. Perturbed Controls Coefficient Nullprojector

Definition 2 (Perturbed controls coefficient nullprojector)**.** The perturbed CCNP $\widetilde{\mathcal{P}}(\rho,\delta,t)$ is defined as

$$\widetilde{\mathcal{P}}(\rho,\delta,t) := I_{3\times 3} - h(\delta)\mathcal{A}^+(\rho,t)\mathcal{A}(\rho,t) \qquad (20)$$

where $h(\delta) : \mathbb{R}^{1\times 1} \to \mathbb{R}^{1\times 1}$ is any continuous function such that

$$h(\delta) = 1 \quad \text{if and only if} \quad \delta = 0.$$

Properties of the Perturbed Controls Coefficient Nullprojector

The following properties of the perturbed CCNP are utilized in the present development of the controls coefficient generalized inversion based attitude tracking control.

1) The perturbed CCNP $\widetilde{\mathcal{P}}(\rho,\delta,t)$ is of full rank for all $\delta \neq 0$.
2) The CCNP $\mathcal{P}(\rho,t)$ commutes with the perturbed CCNP $\widetilde{\mathcal{P}}(\rho,\delta,t)$ for all $\delta \in \mathbb{R}$. Furthermore, their matrix multiplication yields the CCNP itself, i.e.,

$$\mathcal{P}(\rho,t)\widetilde{\mathcal{P}}(\rho,\delta,t) = \widetilde{\mathcal{P}}(\rho,\delta,t)\mathcal{P}(\rho,t) = \mathcal{P}(\rho,t). \quad (21)$$

3) The CCNP $\mathcal{P}(\rho,t)$ commutes with its inverted perturbation $\widetilde{\mathcal{P}}^{-1}(\rho,\delta,t)$ for all $\delta \neq 0$. Furthermore, their matrix multiplication yields the CCNP itself, i.e.,

$$\widetilde{\mathcal{P}}^{-1}(\rho,\delta,t)\mathcal{P}(\rho,t) = \mathcal{P}(\rho,t)\widetilde{\mathcal{P}}^{-1}(\rho,\delta,t) = \mathcal{P}(\rho,t). \qquad (22)$$

Proofs of first and third properties are found in Ref. [1]. Second property is verified by direct evaluation of $\mathcal{P}(\rho,t)$ and $\widetilde{\mathcal{P}}(\rho,\delta,t)$ expressions given by (17) and (20).

VII. Damped Controls Coefficient Generalized Inverse

The expression of $\mathcal{A}^+(\rho,t)$ given by (16) implies that

$$\lim_{\mathcal{A}(\rho,t)\to\mathbf{0}_{1\times 3}} \mathcal{A}^+(\rho,t) = \infty_{3\times 1}. \qquad (23)$$

Implications of the controls coefficient singularity on closed loop stability is depicted by the following singularity analysis.

Controls Coefficient Singularity Analysis

The definition of $\mathcal{A}(\rho,t)$ given by (11) implies that

$$\lim_{z(\rho,t)\to\mathbf{0}_{3\times 1}} \mathcal{A}(\rho,t) = \mathbf{0}_{1\times 3} \qquad (24)$$

for all finite values of $\rho \in \mathbb{R}^3$. Therefore, a control law τ globally realizes the linear attitude deviation norm measure dynamics of (9) by the spacecraft equations of motion (1) and (2) only if

$$\lim_{t\to\infty} \mathcal{A}(\rho,t) = \mathbf{0}_{1\times 3} \qquad (25)$$

which implies from (23) that

$$\lim_{z(\rho,t)\to\mathbf{0}_{3\times 1}} \| \mathcal{A}^+(\rho,t) \| = \infty. \qquad (26)$$

However, a fundamental property of the matrix $G(\rho)$ is [7]

$$\sigma_{min}(G(\rho)) = \sigma_{max}(G(\rho)) = \sigma(G(\rho)) \geq \frac{1}{4}. \qquad (27)$$

Therefore,

$$\| G^T(\rho)z(\rho,t) \| = \sigma(G(\rho)) \| z(\rho,t) \|, \qquad (28)$$

and the definition of $\mathcal{A}(\rho,t)$ given by (11) implies that

$$\| \mathcal{A}^+(\rho,t) \| = \frac{1}{2\sigma(G(\rho)) \| z(\rho,t) \|} \qquad (29)$$

and that

$$\| \mathcal{A}^+(\rho,t)z^T(\rho,t) \| \leq \| \mathcal{A}^+(\rho,t) \| \| z(\rho,t) \| \qquad (30)$$

$$= \frac{1}{2\sigma(G(\rho)) \| z(\rho,t) \|} \| z(\rho,t) \|$$

$$= \frac{1}{2\sigma(G(\rho))} \leq 2 \qquad (31)$$

and that

$$\| \mathcal{A}^+(\rho,t)z^T(\rho,t)z(\rho,t) \| \leq \| \mathcal{A}^+(\rho,t)z^T(\rho,t) \| \| z(\rho,t) \| \leq 2 \| z(\rho,t) \| . \qquad (32)$$

Inequalities (31) and (32) imply that

$$\lim_{z(\rho,t)\to\mathbf{0}_{3\times 1}} \| \mathcal{A}^+(\rho,t)z^T(\rho,t) \| \leq 2 \qquad (33)$$

and

$$\lim_{z(\rho,t)\to\mathbf{0}_{3\times 1}} \| \mathcal{A}^+(\rho,t)z^T(\rho,t)z(\rho,t) \| = 0. \qquad (34)$$

Damped Controls Coefficient Generalized Inverse

For the purpose of controlling the growth of the CCGI $\mathcal{A}^+(\rho,t)$ in the control law given by (15), the damped CCGI $\mathcal{A}_d^+(\rho,\beta,t)$ is introduced.

Definition 3 (Damped controls coefficient generalized inverse)**.** The damped CCGI is defined as

$$\mathcal{A}_d^+(\rho,\beta,t) := \begin{cases} \frac{\mathcal{A}^T(\rho,t)}{\|\mathcal{A}(\rho,t)\|^2} & : \quad \| \mathcal{A}(\rho,t) \| \geq \beta \\ \frac{\mathcal{A}^T(\rho,t)}{\beta^2} & : \quad \| \mathcal{A}(\rho,t) \| < \beta \end{cases} \tag{35}$$

where the scalar β is a positive *generalized inverse damping factor*.

The above definition implies that

$$\|\mathcal{A}_d^+(\rho,\beta,t)\| \leq \frac{1}{\beta} \tag{36}$$

and that

$$\lim_{z(\rho,t)\to \mathbf{0}_{3\times 1}} \| \mathcal{A}_d^+(\rho,\beta,t) \| \tag{37}$$

$$= \lim_{z(\rho,t)\to \mathbf{0}_{3\times 1}} \frac{2}{\beta^2}\|G^T(\rho)z(\rho,t)\| = 0 \tag{38}$$

and that $\mathcal{A}_d^+(\rho,\beta,t)$ pointwise converges to $\mathcal{A}^+(\rho,t)$ as β vanishes.

Damped Controls Coefficient Nullprojector

The damped controls coefficient nullprojector is a modified controls coefficient nullprojector with vanishing dependency on the steady state attitude variables.

Definition 4 (Damped controls coefficient nullprojector)**.** The damped CCNP $\mathcal{P}_d(\rho,\beta,t)$ is defined as

$$\mathcal{P}_d(\rho,\beta,t) := I_{3\times 3} - \mathcal{A}_d^+(\rho,\beta,t)\mathcal{A}(\rho,t) \tag{39}$$

where $\mathcal{A}_d^+(\rho,\beta,t)$ is given by (35).

The above definition implies that

$$\lim_{z(\rho,t)\to \mathbf{0}_{3\times 1}} \mathcal{P}_d(\rho,\beta,t) = I_{3\times 3}. \tag{40}$$

Hence, the damped CCNP maps the null-control vector to itself during steady state phase of response, during which the auxiliary part of the control law converges to the null-control vector.

VIII. Controls Coefficient Generalized Inversion Control

For convenience, let the control law expression given by (15) be written as

$$\tau = \mathcal{H}_1(\rho,\omega,t)\omega + \mathcal{H}_2(\rho,t) + \mathcal{P}(\rho,t)y \tag{41}$$

where

$$\begin{aligned}\mathcal{H}_1(\rho,\omega,t) = &-2\mathcal{A}^+(\rho,t)z^T(\rho,t)\\ &\Big[\dot{G}(\rho,\omega) + G(\rho)J^{-1}\omega^\times J + c_1 G(\rho)\Big]\\ &- 2\mathcal{A}^+(\rho,t)\Big[G(\rho)\omega - \dot{\rho}_r(t)\Big]^T G(\rho)\end{aligned} \tag{42}$$

and

$$\begin{aligned}\mathcal{H}_2(\rho,t) = &-c_2\mathcal{A}^+(\rho,t)z^T(\rho,t)z(\rho,t)\\ &+ 2\mathcal{A}^+(\rho,t)z^T(\rho,t)\Big[\ddot{\rho}_r(t) + c_1\dot{\rho}_r(t)\Big]\\ &- 2\mathcal{A}^+(\rho,t) \parallel \dot{\rho}_r(t) \parallel_2^2 .\end{aligned} \tag{43}$$

To avoid closed loop instability due to the CCGI unstable dynamics described by (26), we define $\mathcal{H}_{1d}(\rho,\omega,\beta,t)$ and $\mathcal{H}_{2d}(\rho,\beta,t)$ by replacing $\mathcal{A}^+(\rho,t)$ in the last terms of the $\mathcal{H}_1(\rho,\omega,t)$ and $\mathcal{H}_2(\rho,t)$ expressions given by Eqs. (42) and (43) with the damped CCGI $\mathcal{A}_d^+(\rho,\beta,t)$, such that

$$\begin{aligned}\mathcal{H}_{1d}(\rho,\omega,\beta,t) = &-2\mathcal{A}^+(\rho,t)z^T(\rho,t)\\ &\Big[\dot{G}(\rho,\omega) + G(\rho)J^{-1}\omega^\times J + c_1 G(\rho)\Big]\\ &- 2\mathcal{A}_d^+(\rho,\beta,t)\Big[G(\rho)\omega - \dot{\rho}_r(t)\Big]^T G(\rho)\end{aligned} \tag{44}$$

and

$$\begin{aligned}\mathcal{H}_{2d}(\rho,\beta,t) = &-c_2\mathcal{A}^+(\rho,t)z^T(\rho,t)z(\rho,t)\\ &+ 2\mathcal{A}^+(\rho,t)z^T(\rho,t)\Big[\ddot{\rho}_r(t) + c_1\dot{\rho}_r(t)\Big]\\ &- 2\mathcal{A}_d^+(\rho,\beta,t) \parallel \dot{\rho}_r(t) \parallel^2 .\end{aligned} \tag{45}$$

The other terms in $\mathcal{H}_1(\rho,\omega,t)$ and $\mathcal{H}_2(\rho,t)$ involving the CCGI $\mathcal{A}^+(\rho,t)$ are left unaltered, because they remain bounded according to inequality (33) and Eq. (34) as the closed loop system reaches steady state.

The control vector τ_d is defined as

$$\tau_d = \mathcal{H}_{1d}(\rho,\omega,\beta,t)\omega + \mathcal{H}_{2d}(\rho,\beta,t) + \mathcal{P}_d(\rho,\beta,t)y. \tag{46}$$

Theorem 1 (Implication on attitude stability)**.** *Let the control law τ_d be given by (46), where the null-control vector y is arbitrary. Then the desired attitude deviation dynamics given by (9) is realized by the spacecraft equations of motion (1) and (2) for all values of ϕ in the domain D_ϕ given by*

$$D_\phi : \phi \geq \frac{\beta^2}{4\sigma^2(G(\rho))}, \tag{47}$$

and the resulting attitude trajectory tracking errors are UUB.

Proof: Let ϕ_d be a norm measure function of the attitude deviation obtained by applying the control law given by (46) to the spacecraft equations of motion (1) and (2), and let $\dot{\phi}_d$, $\ddot{\phi}_d$ be its first two time derivatives. Hence,

$$\phi_d \;:=\; \phi_d(\rho,t) = \phi(\rho,t) \tag{48}$$

$$\dot{\phi}_d \;:=\; \dot{\phi}_d(\rho,\omega,t) = \dot{\phi}(\rho,\omega,t) \tag{49}$$

$$\ddot{\phi}_d \;:=\; \ddot{\phi}_d(\rho,\omega,\tau_d,t) = \ddot{\phi}(\rho,\omega,\tau,t) + \mathcal{A}(\rho,t)\tau_d - \mathcal{A}(\rho,t)\tau \tag{50}$$

where τ and τ_d are given by (41) and (46), respectively. Adding $c_1\dot{\phi}_d + c_2\phi_d$ to both sides of (50) yields

$$\ddot{\phi}_d + c_1\dot{\phi}_d + c_2\phi_d = \ddot{\phi} + c_1\dot{\phi} + c_2\phi + \mathcal{A}(\rho,t)\tau_d - \mathcal{A}(\rho,t)\tau \tag{51}$$

$$= \mathcal{A}(\rho,t)[\tau_d - \tau]. \tag{52}$$

Let the state vector $\Phi_d \in \mathbb{R}^{2\times 1}$ be defined as

$$\Phi_d = \begin{bmatrix}\phi_d & \dot{\phi}_d\end{bmatrix}^T. \tag{53}$$

The attitude deviation norm measure closed loop dynamics can be written in the state space form

$$\dot{\Phi}_d = \Lambda_{11}\Phi_d + \Lambda_{12}(\rho,\omega,\beta,t)\omega + \Delta_1(\rho,\beta,t) \tag{54}$$

where the strictly stable system matrix $\Lambda_{11} \in \mathbb{R}^{2\times 2}$ is

$$\Lambda_{11} = \begin{bmatrix} 0 & 1 \\ -c_2 & -c_1 \end{bmatrix} \tag{55}$$

and the matrix valued function $\Lambda_{12}(\rho,\omega,\beta,t) : \mathbb{R}^{7\times 1} \to \mathbb{R}^{2\times 3}$ is

$$\Lambda_{12}(\rho,\omega,\beta,t) = \begin{bmatrix} \mathbf{0}_{1\times 3} \\ \mathcal{A}(\rho,t)\Big[\mathcal{H}_{1d}(\rho,\omega,\beta,t) - \mathcal{H}_1(\rho,\omega,t)\Big] \end{bmatrix}$$

and the matrix valued function $\Delta_1(\rho,\beta,t) : \mathbb{R}^{5\times 1} \to \mathbb{R}^{2\times 1}$ is

$$\Delta_1(\rho,\beta,t) = \begin{bmatrix} 0 \\ \mathcal{A}(\rho,t)\Big[\mathcal{H}_{2d}(\rho,\beta,t) - \mathcal{H}_2(\rho,t) \\ +\mathcal{P}_d(\rho,\beta,t) - \mathcal{P}(\rho,t)\Big] \end{bmatrix}. \tag{56}$$

The expression of $\mathcal{A}^{+}(\rho,t)$ given by (16) is identical to the expression of $\mathcal{A}_d^{+}(\rho,\beta,t)$ given by (35) for values of ρ and t that satisfy $\| \mathcal{A}(\rho,t) \| \geq \beta$. These values of ρ and t can be described in terms of ϕ by noticing that the singular values of the matrix $G(\rho)$ are repeated, so that

$$\|\mathcal{A}(\rho,t)\| = \|2z^T(\rho,t)G(\rho)\| = 2\sigma(G(\rho))\|z(\rho,t)\| \tag{57}$$

which implies from the definition of ϕ that the corresponding set D_ϕ of ϕ values is given by (47). Therefore, for values of ϕ in D_ϕ, $\Lambda_{12}(\rho,\omega,\beta,t) = \mathbf{0}_{2\times 3}$ and $\Delta_1(\rho,\beta,t) = \mathbf{0}_{2\times 1}$, the expressions (46) for τ_d and (41) for τ are identical, and the desired linear attitude deviation dynamics given by (9) is realized. Nevertheless, for values of ϕ in the bounded open complement set given by

$$D_{\phi_c} : \phi < \frac{\beta^2}{4\sigma^2(G(\rho))}, \tag{58}$$

the definition of $\mathcal{A}^{+}(\rho,t)$ is different from the definition of $\mathcal{A}_d^{+}(\rho,\beta,t)$, and the desired attitude dynamics is not realized. Instead, the dynamics given by (52) is the one that is realized over this bounded domain, resulting in uniformly ultimately bounded attitude trajectory tracking errors rather than in asymptotic attitude tracking.

Stability of internal dynamics is the most important factor to be considered in designing the null-control vector y. The structure of the control law τ_d has a special feature, namely the affinity of its auxiliary part in y, which provides pointwise-linear parametrization to the nonlinear control law.

A. Perturbed Feedback-Linearizing Null-Control Vector

Let the null-control vector y be chosen as

$$y = -J^{-1}\omega^\times J\omega - \mathcal{H}_{1d}(\rho,\omega,\beta,t)\omega - \mathcal{H}_{2d}(\rho,\beta,t) + k(\omega - \omega_r(t)) \tag{59}$$

where $k \in \mathbb{R}^{3\times 3}$ is a strictly stable constant matrix gain. Hence, another class of control laws that realize the attitude deviation norm measure dynamics given by (9) is obtained by substituting this choice of y in (46). The corresponding closed loop dynamical subsystems realizing the dynamics given by (9) is obtained by substituting this class of control laws in (2), resulting in

$$\dot{\omega} = [I - \mathcal{P}_d(\rho,\beta,t)][J^{-1}\omega^\times J\omega + \mathcal{H}_{1d}(\rho,\omega,\beta,t)\omega + \mathcal{H}_{2d}(\rho,\beta,t)] + \mathcal{P}_d(\rho,\beta,t)k(\omega - \omega_r(t)). \tag{60}$$

Nevertheless, the third property of the perturbed CCNP $\widetilde{\mathcal{P}}(\rho,\delta,t)$ given by (22) implies that over the domain given by (47), the closed loop dynamical subsystem given by (60) can be written as

$$\dot{\omega} = [I - \mathcal{P}_d(\rho,\beta,t)\widetilde{\mathcal{P}}^{-1}(\rho,\delta,t)] [J^{-1}\omega^\times J\omega + \mathcal{H}_{1d}(\rho,\omega,\beta,t)\omega + \mathcal{H}_{2d}(\rho,\beta,t)] + \mathcal{P}_d(\rho,\beta,t)\widetilde{\mathcal{P}}^{-1}(\rho,\delta,t)k(\omega - \omega_r(t)). \tag{61}$$

Therefore, the closed loop dynamical subsystem given by (60) is a perturbation from the linear system

$$\dot{\omega} = k(\omega - \omega_r(t)) \tag{62}$$

that is obtained by perturbing the CCNP $\mathcal{P}(\rho,t)$ in (61) in the manner given by (20). The dynamical system given by (62) together with the kinematics given by (9) constitute a feedback linearizing transformation of the global spacecraft closed loop dynamics over the domain given by $\mathcal{A}(\rho,t) \geq \beta$, realized up to a perturbation $\widetilde{\mathcal{P}}(\rho,\delta,t)$ from the CCNP.

Theorem 2 (Perturbed feedback linearization CCGI trajectory tracking control)**.** *Let the control law τ_d be given by (46), where the null-control vector y is given by (59). Then for any initial condition of the spacecraft system equations (1) and (2), there exists a constant gain matrix k such that closed loop dynamical subsystem given by (2) is UUB.*

Proof: The closed loop dynamical subsystem given by (60) can be rewritten as

$$\dot{\omega} = \eta_1(\rho,\omega,t)\omega + \eta_2(\rho,t)\omega + \Delta_2(\rho,\beta,t) \tag{63}$$

where

$$\eta_1(\rho,\omega,t) = [I - \mathcal{P}_d(\rho,\beta,t)][J^{-1}\omega^\times J + \mathcal{H}_{1d}(\rho,\omega,\beta,t)] + k \tag{64}$$

and

$$\eta_2(\rho,t) = -[I - \mathcal{P}_d(\rho,\beta,t)]k \tag{65}$$

and

$$\Delta_2(\rho,t) = [I - \mathcal{P}_d(\rho,\beta,t)]\mathcal{H}_{2d}(\rho,\beta,t) - \mathcal{P}_d(\rho,\beta,t)k\omega_r(t). \tag{66}$$

Stability of the first part of the system equations (63) given by

$$\dot{\omega} = \eta_1(\rho, \omega, t)\omega \tag{67}$$

can be analyzed by Lyapunov indirect method by verifying that its Jacobian at $\omega = \mathbf{0}_{3\times1}$ is given by

$$\frac{\partial[\eta_1(\rho,\omega,t)\omega]}{\partial\omega}|_{\omega=\mathbf{0}_{3\times1}} = \eta_1(\rho, \mathbf{0}_{3\times1}, t) = k, \tag{68}$$

which is strictly stable. Therefore, for any strictly stable $k \in \mathbb{R}^{3\times3}$, for any bounded trajectory $\rho(t)$ and $t > 0$, there exists a domain of attraction $D_\omega \subset \mathbb{R}^{3\times1}$ of $\omega = \mathbf{0}_{3\times1}$ such that the system given by (67) is locally exponentially stable. Moreover, it follows from Lyapunov direct method that for all $\omega \in D_\omega$ and for any positive definite constant matrix Q_ω, there exists a positive definite constant matrix $P_\omega \in \mathbb{R}^{3\times3}$, such that the following inequality is satisfied

$$\omega^T[\eta_1^T(\rho,\omega,t)P_\omega + P_\omega\eta_1(\rho,\omega,t)]\omega < -\omega^T Q_\omega \omega. \tag{69}$$

Next, we consider stability of the first and second parts of the system equations (63) given by

$$\dot{\omega} = \eta_1(\rho,\omega,t)\omega + \eta_2(\rho,t)\omega. \tag{70}$$

In viewing $\eta_2(\rho,t)$ as a perturbation from $\eta_1(\rho,\omega,t)$, (65) implies that

$$\|\eta_2(\rho,t)\omega\| \leq \|I - \mathcal{P}_d(\rho,\beta,t)\|\|k\|\|\omega\| = \sigma_{max}(k)\|\omega\|. \tag{71}$$

On the other hand, (40) implies that

$$\lim_{t\to0}\eta_2(\rho,t) = \mathbf{0}_{3\times3}. \tag{72}$$

The conditions given by (71) and (72) imply that the equilibrium point $\omega = \mathbf{0}_{3\times1}$ of the system given by (70) is exponentially stable over D_ω. Finally, stability of the forced closed loop dynamical subsystem of (63) and the corresponding spacecraft closed loop stability are analyzed by augmenting the dynamical subsystem with the kinematical state space model given by (54). The augmented state space model takes the form

$$\dot{\xi} = \mathcal{F}_1(\rho,\omega,t)\xi + \mathcal{F}_2(\rho,\omega,\beta,t)\xi + \Delta(\rho,\beta,t) \tag{73}$$

where the augmented state is

$$\xi = \begin{bmatrix}\Phi_d^T & \omega^T\end{bmatrix}^T \tag{74}$$

and the block-diagonal matrix $\mathcal{F}_1(\rho,\omega,t)$ is

$$\mathcal{F}_1(\rho,\omega,t) = \begin{bmatrix}\Lambda_{11} & \mathbf{0}_{2\times3} \\ \mathbf{0}_{3\times2} & \eta_1(\rho,\omega,t)+\eta_2(\rho,t)\end{bmatrix} \tag{75}$$

and the cross coupling matrix $\mathcal{F}_2(\rho,\omega,\beta,t)$ is

$$\mathcal{F}_2(\rho,\omega,\beta,t) = \begin{bmatrix}\mathbf{0}_{2\times2} & \Lambda_{12}(\rho,\omega,\beta,t) \\ \mathbf{0}_{3\times2} & \mathbf{0}_{3\times3}\end{bmatrix} \tag{76}$$

indicating a unidirectional dynamic coupling, i.e., the internal dynamics is independent of the attitude deviation dynamics, and the driving input vector $\Delta(\rho,\beta,t)$ is

$$\Delta(\rho,\beta,t) = \begin{bmatrix}\Delta_1(\rho,\beta,t) \\ \Delta_2(\rho,\beta,t)\end{bmatrix}. \tag{77}$$

The first part of the augmented state-space model is the decoupled system

$$\dot{\xi} = \mathcal{F}_1(\rho,\omega,t)\xi. \tag{78}$$

For values of the matrix gain k that are strictly stable, the equilibrium point $\xi = \mathbf{0}_{5\times1}$ of the system equations (78) is exponentially stable for all $\xi \in D_\xi$, where

$$D_\xi : \mathbb{R}^2 \times D_\omega. \tag{79}$$

Therefore, for any positive definite constant matrix $Q_\xi \in \mathbb{R}^{5\times5}$, for any bounded $\rho(t) \in \mathbb{R}^3$ and $t > 0$, and for all $\xi \in \mathbb{R}^{5\times1}$ in the domain of attraction D_ξ of $\xi = \mathbf{0}_{5\times1}$, there exists a Lyapunov function ([8], pp. 167)

$$V_\xi(\xi) = \xi^T P_\xi \xi \tag{80}$$

where $P_\xi \in \mathbb{R}^{5\times5}$ is a positive definite constant matrix, such that the time derivative of V_ξ along the trajectories of the system given by (78) is negative definite, resulting in

$$\xi^T[\mathcal{F}_1^T(\rho,\omega,t)P_\xi + P_\xi\mathcal{F}_1(\rho,\omega,t)]\xi < -\xi^T Q_\xi \xi \tag{81}$$

for all $\xi \in D_\xi$. The first and second parts of the augmented state-space model of (73) is the system

$$\dot{\xi} = \mathcal{F}_1(\rho,\omega,t)\xi + \mathcal{F}_2(\rho,\omega,\beta,t)\xi. \tag{82}$$

Evaluating the time derivative of V_ξ along the trajectories of the system given by (82) yields

$$\begin{aligned}\frac{\partial V_\xi(\xi)}{\partial\xi}&[\mathcal{F}_1(\rho,\omega,t)\xi + \mathcal{F}_2(\rho,\omega,\beta,t)\xi] = \\ &2\xi^T P_\xi[\mathcal{F}_1(\rho,\omega,t)\xi + \mathcal{F}_2(\rho,\omega,\beta,t)\xi] = \\ &\xi^T[\mathcal{F}_1^T(\rho,\omega,t)P_\xi + P_\xi\mathcal{F}_1(\rho,\omega,t)]\xi \\ &+ 2\xi^T P_\xi \mathcal{F}_2(\rho,\omega,\beta,t)\xi.\end{aligned} \tag{83}$$

A norm bound on the last term in (83) is

$$2\xi^T P_\xi\mathcal{F}_2(\rho,\omega,\beta,t)\xi \leq 2\lambda_{max}(P_\xi)\|\mathcal{F}_2(\rho,\omega,\beta,t)\|\|\xi\|^2 \tag{84}$$

where

$$\begin{aligned}&\|\mathcal{F}_2(\rho,\omega,\beta,t)\| = \|\Lambda_{12}(\rho,\beta,t)\| \\ &= \|\mathcal{A}(\rho,t)[\mathcal{H}_{1d}(\rho,\omega,\beta,t) - \mathcal{H}_1(\rho,\omega,t)]\| = \\ &2\|(\mathcal{A}(\rho,t)\mathcal{A}_d^+(\rho,\beta,t) - 1)[G(\rho)\omega - \dot{\rho}_r(t)]^T G(\rho)\|.\end{aligned} \tag{85}$$

Therefore, a bound on the expression of (83) is obtained from (81) and (84) as

$$\begin{aligned}&\frac{\partial V_\xi(\xi)}{\partial\xi}[\mathcal{F}_1(\rho,\omega,t)\xi + \mathcal{F}_2(\rho,\omega,\beta,t)\xi] \\ &< [-\lambda_{min}(Q_\xi) \\ &+2\lambda_{max}(P_\xi) \parallel \mathcal{F}_2(\rho,\omega,\beta,t)] \parallel \parallel \xi \parallel^2 \\ &= \big[-\lambda_{min}(Q_\xi) \\ &+4\lambda_{max}(P_\xi) \parallel (\mathcal{A}(\rho,t)\mathcal{A}_d^+(\rho,\beta,t) - 1) \\ &\times[G(\rho)\omega - \dot{\rho}_r(t)]^T G(\rho) \parallel \big] \parallel \xi \parallel^2 \leq \epsilon_\xi \parallel \xi \parallel^2\end{aligned} \tag{86}$$

where

$$\epsilon_\xi = -\lambda_{min}(Q_\xi) + 4\lambda_{max}(P_\xi) \times (\frac{2}{\beta} \parallel G^T(\rho) z(\rho,t) \parallel +1)(\sigma(G(\rho))\|\omega\| + \parallel \dot{\rho}_r(t) \parallel)\sigma(G(\rho)). \quad (87)$$

If Q_ξ is chosen such that

$$\lambda_{min}(Q_\xi) > 4\lambda_{max}(P_\xi)(\frac{2}{\beta} \parallel G^T(\rho) z(\rho,t) \parallel +1) \times (\sigma(G(\rho))\|\omega\| + \parallel \dot{\rho}_r(t) \parallel)\sigma(G(\rho)) \quad (88)$$

then ϵ_ξ is negative, which implies that the equilibrium point $\xi = \mathbf{0}_{5\times 1}$ of the nonlinear system of (82) is locally exponential stable over the domain of attraction D_ξ. An upper bound on Q_ξ is obtained from (81) by evaluating $\mathcal{F}_1(\rho,\omega,t)$ at $\omega = \mathbf{0}_{3\times 1}$, such that

$$\begin{aligned} \lambda_{min}(Q_\xi) &< 2\|\mathcal{F}_1^T(\rho,\omega=\mathbf{0}_{3\times 1},t)\|\|P_\xi\| \\ &= 2\|\Lambda_{11}\|\|\eta_1(\rho,\omega=\mathbf{0}_{3\times 1},t)+\eta_2(\rho,t)\|\|P_\xi\| \\ &\leq 4\sigma_{max}(\Lambda_{11})\sigma_{max}(k)\sigma_{max}(P_\xi) \\ &= 4\sigma_{max}(\Lambda_{11})\sigma_{max}(k)\lambda_{max}(P_\xi) \end{aligned} \quad (89)$$

Inequalities (88) and (89) are used to obtain the following sufficient condition on the matrix gain k for local exponential stability of the system given by (82) over D_ξ

$$\sigma_{max}(k) > \frac{\sigma(G(\bar{\rho}))}{\sigma_{max}(\Lambda_{11})}(\sigma(G(\bar{\rho}))\|\omega(0)\| + \sup_t \parallel \dot{\rho}_r(t) \parallel) \times (\frac{2}{\beta}\sigma(G(\bar{\rho}))\|z(\mathbf{0}_{3\times 1},0) \parallel +1) \quad (90)$$

where

$$\bar{\rho} = \arg(\max\{\|\rho(0)\|, \sup_t \|\rho_r(t)\|\}), \quad (91)$$

and $\omega(0) \in D_\omega$ according to (69). The time derivative of V_ξ along the trajectories of the augmented state space model given by (73) is

$$\begin{aligned} \dot{V}_\xi(\rho,\xi,\beta,t) &= 2\xi^T P_\xi[\mathcal{F}_1(\rho,\omega,t)\xi + \mathcal{F}_2(\rho,\omega,\beta,t)\xi] \\ &\quad +2\xi^T P_\xi \Delta(\rho,\beta,t). \end{aligned} \quad (92)$$

A norm bound on the second term in (92) is

$$2\xi^T P_\xi \Delta(\rho,\beta,t) \leq 2\lambda_{max}(P_\xi)\|\Delta(\rho,\beta,t)\|\|\xi\|. \quad (93)$$

Therefore, (86) and (93) imply that

$$\dot{V}_\xi(\rho,\xi,\beta,t) \leq \epsilon_\xi\|\xi\|^2 + 2\lambda_{max}(P_\xi)\|\Delta(\rho,\beta,t)\|\|\xi\| \quad (94)$$

for all $\xi \in D_\xi$. Rewriting ϵ_ξ as

$$\epsilon_\xi = (1-\theta)\epsilon_\xi + \theta\epsilon_\xi, \quad \theta \in (0,1) \quad (95)$$

then (94) can be written as

$$\dot{V}_\xi(\rho,\xi,\beta,t) \leq (1-\theta)\epsilon_\xi\|\xi\|^2 + \theta\epsilon_\xi\|\xi\|^2 + 2\lambda_{max}(P_\xi)\|\Delta(\rho,\beta,t)\|\|\xi\|. \quad (96)$$

Hence, if the stable matrix gain k is chosen such that (90) is satisfied, then for all ξ such that

$$\parallel \xi \parallel > \frac{2\lambda_{max}(P_\xi)}{-\theta\epsilon_\xi} \parallel \Delta(\rho,\beta,t) \parallel \quad (97)$$

the following inequality holds

$$\dot{V}(\rho,\omega,\beta,t) < (1-\theta)\epsilon_\xi \parallel \xi \parallel^2, \quad (98)$$

which implies that trajectories of the system given by (73) are UUB over D_ξ, the corresponding ξ trajectories enter the complement domain of the domain given by (97) in finite time, the complement domain becomes an invariant set for ξ, and uniform ultimate boundedness of ω follows. Norm bounds on the components of the driving vector $\Delta(\rho,\beta,t)$ given by (56) and (65) are

$$\|\Delta_1(\rho,\beta,t)\| \leq 2\| - \mathcal{A}(\rho,t)\mathcal{A}_d^+(\rho,\beta,t) + 1\|\|\dot{\rho}_d\|^2 + \|\mathcal{A}(\rho,t)\mathcal{P}_d(\rho,\beta,t)\| \quad (99)$$

and

$$\|\Delta_2(\rho,\beta,t)\| \leq 2c_2\|z(\rho,t)\| + 4\|\ddot{\rho}_r(t) + c_1\dot{\rho}_r(t)\| + \frac{2}{\beta}\|\dot{\rho}_r(t)\|^2 + \sigma_{max}(k)\|\omega_r(t)\|. \quad (100)$$

IX. Control System Design Procedure

The procedures for designing CCGI attitude tracking control systems are summarized in the following steps

1) A desired spacecraft attitude trajectory $\rho_r(t)$ is prescribed, where ρ_r is at least twice differentiable in t. The desired angular velocity vector $\omega_r(t)$ is accordingly defined by (14).
2) The coefficients c_1 and c_2 in the attitude deviation norm measure dynamics equation (9) are chosen such that the dynamics of ϕ is stable. This implies that both c_1 and c_2 are strictly positive.
3) The expressions given by (11) and (12) for $\mathcal{A}(\rho,t)$ and $\mathcal{B}(\rho,\omega,t)$ are obtained, where $G(\rho)$ and $z(\rho,t)$ are given by (3) and (4) respectively.
4) The CCGI $\mathcal{A}^+(\rho,t)$ given by (16) is modified in the manner of (35), and $\mathcal{A}_d^+(\rho,\beta,t)$ is used to define the expressions of $\mathcal{P}_d(\rho,\beta,t)$, $\mathcal{H}_{1d}(\rho,\omega,\beta,t)$, and $\mathcal{H}_{2d}(\rho,\beta,t)$ according to (39), (44), and (45), respectively.
5) The control law τ_d given by (46) is applied for perturbed feedback linearization. The null-control vector is given by (59), where the constant matrix gain $k \in \mathbb{R}^{3\times 3}$ is strictly stable and satisfies (90), and $\omega(0) \in D_\omega$. For a given value of the matrix gain k, a specific value of ω is in D_ω if (69) is satisfied for some positive definite constant matrices $P_\omega, Q_\omega \in \mathbb{R}^{3\times 3}$. The domain D_ω can be arbitrarily enlarged by decreasing $\lambda_{max}(k)$ so that (69) remains satisfied.
6) Integrate (1) and (2) to obtain the trajectories of $\rho(t)$ and $\omega(t)$, where $u = J\tau$. The resulting trajectory tracking errors are uniformly ultimately bounded according to (58).

The controls coefficient generalized inversion design structure is illustrated by the block diagram in Figure (1).

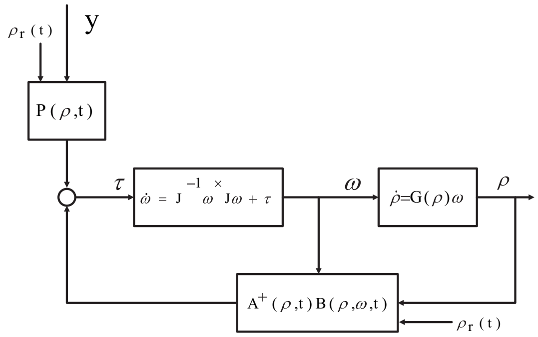

Fig. 1. Control coefficient generalized inversion design structure

X. Numerical Simulations

The spacecraft model used for numerical simulations has inertia parameters $I_1 = 200$ Kg-m^2, I_2 = 150 Kg-m^2, $I_3 =$ 175 Kg-m^2. The desired MRPs trajectories are chosen to be $\rho_{ri} = \cos 0.1t, \;\; i = 1, 2, 3$. Their initial values are given by the vector $\rho(0) = [-0.42 \quad 0.35 \quad -0.26]^T$, and the initial spacecraft body angular velocity vector is $\omega(0) = [0.30 \quad -0.61 \quad -0.30]^T$. Parameter value of $\beta = 0.3$ is chosen. Figures 2 , 3, and 4 show plots of the MRPs, the angular velocity components, and the scaled control variables versus time t, where constants $\delta = 0.1$, $c_1 = 3.5$, and $c_2 = 1.5$ are chosen. The constant matrix gain is taken to be $k = diag\{-5.5, -6.5, -6.0\}$, so that the condition given by (90) is satisfied. Altering the value of β reveals the tradeoff between trajectory tracking accuracy and damped generalized inverse stability [1].

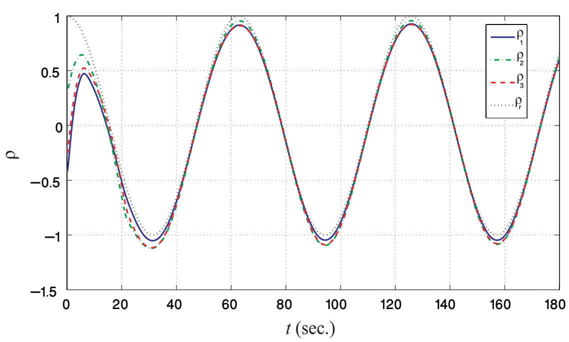

Fig. 2. MRPs: Perturbed feedback linearizing control

XI. Conclusion

A controls coefficient generalized inversion design methodology for spacecraft attitude trajectory tracking is presented, based on constructing the null-control vector in the generalized inversion-based Greville formula for the general solution of linear algebraic system equations. The closed loop attitude dynamics depends on a predetermined attitude deviation servo-constraint dynamics, and therefore it is invariant under the choice of the null-control vector. However, the internal dynamics is substantially affected by the null-control vector. Accordingly, the construction of the null-control vector is made in order to globally linearize internal dynamics, forming together with the predetermined linear attitude deviation dynamics a feedback linearization transformation of the spacecraft global dynamics, realized up to a perturbation from the controls coefficient nullprojector.

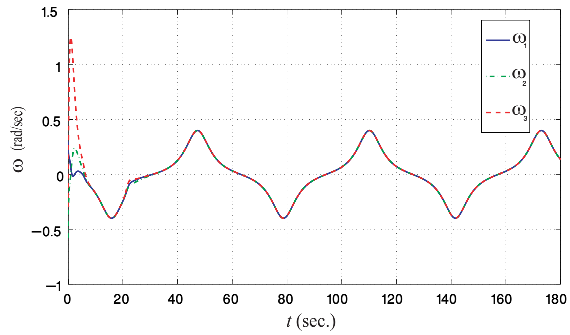

Fig. 3. Angular velocity components: Perturbed feedback linearizing control

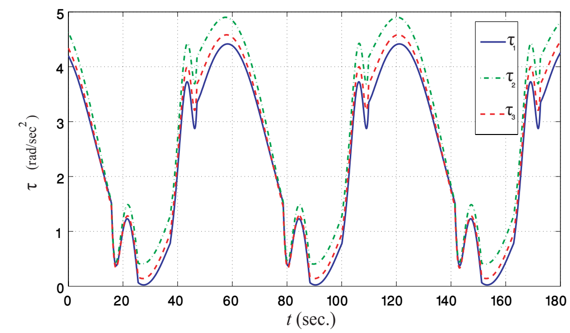

Fig. 4. Scaled control torque variables: Perturbed feedback linearizing control

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