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The Servomechanism Problem for Unknown MIMO LTI Positive Systems: Feedforward and Robust Tuning Regulators

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***Abstract*— In this paper we study the servomechanism problem for unknown MIMO positive LTI systems under constant reference signals and constant measurable disturbances. In particular, we show that in general the servomechanism problem for standard positive systems is not solvable. Thereafter, we show what subclass of reference and disturbance signals can be considered, and more importantly, we provide the controller structure that solves the problem for the feasible subclass of reference and disturbance signals.**

***Index Terms*— positive linear systems; tracking; regulation; servomechanism problem; tuning regulator; feedforward compensator**

I. Introduction

In this paper we consider the study of the servomechanism problem for multi-input multi-output (MIMO) linear time-invariant (LTI) positive systems. In particular, we present results on the tracking and regulation problem of nonnegative constant reference signals for unknown stable MIMO positive LTI systems with measurable constant disturbances. The assumption that no mathematical model of the system is known is a very practical assumption, since in "real world problems", the model of the system to be controlled is often unavailable.

Positive systems are of great practical importance, as the nonnegative property occurs quite frequently in numerous applications and in nature. Positive systems and their counterparts, compartmental systems, are visible in biology where they are used to describe the transportation, accumulation, and drainage processes of elements and compounds like hormones, glucose, insulin, metals, etc. Stocking and industrial systems which involve chemical reactions are also examples of positive systems [1]. Positive systems have been of great interest to numerous researchers over several decades; for various interesting citations refer to [1], [2].

The interest in positive systems spans several decades and numerous disciplines. In this paper, we will present results that will add to positive system theory by introducing the concept of tracking reference signals under measurable disturbances while maintaining nonnegativity of the states and outputs. Our interest will be to show that any unknown stable MIMO positive LTI system under a *subclass* of nonnegative constant disturbances can attain both tracking/regulation and nonnegativity of states and outputs for *a subclass* of nonnegative constant references.

The paper is organized as follows. Preliminaries are given first, where the terminology, positive systems and compartmental systems, tuning regulators, and feedforward controllers are discussed. The main discussion of the problems considered and the results are described in Section III. An example is provided in Section IV and final remarks follow.

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II. Preliminaries

A. Terminology

Let the set $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$, the set $\mathbb{R}_+^n := \{x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n \mid x_i \in \mathbb{R}_+,\ \forall i = 1, ..., n\}$. If exclusion of 0 from the sets will be necessary, then we will denote the sets in the standard way $\mathbb{R}_+ \setminus \{0\}$ ($\mathbb{R}_+^n \setminus \{0\}$). The set of eigenvalues of a matrix $\mathcal{A}$ will be denoted as $\sigma(\mathcal{A})$. The ij^{th} entry of a matrix $\mathcal{A}$ will be denoted as a_{ij}. A matrix $\mathcal{A} \in \mathbb{R}^{n\times n}$ is Hurwitz or stable when all the eigenvalues (λ) of $\mathcal{A}$ are in the open left half plane of the complex plane $\mathbb{C}$. A *nonnegative* matrix $\mathcal{A}$ has all of its entries greater or equal to 0, i.e. $a_{ij} \in \mathbb{R}_+$. A *Metzler* matrix $\mathcal{A}$ is a matrix for which all off-diagonal elements of $\mathcal{A}$ are nonnegative, i.e. $a_{ij} \in \mathbb{R}_+$ for all $i \neq j$. A *compartmental* matrix $\mathcal{A}$ is a matrix that is Metzler, where the sum of the components within a column is less than or equal to zero, i.e. $\sum_{i=1}^{n} a_{ij} \leq 0$ for all $j = 1, 2, ..., n$. A *permutation matrix* is a square ($n \times n$) matrix that has been obtained by permuting the rows of an identity matrix according to some permutation of the numbers 1 to n. A *monomial matrix* is a matrix that can be expressed as a product of a diagonal matrix and a permutation matrix, i.e. there is exactly one nonzero entry in each row and each column.

B. Positive Linear Systems and Compartmental Systems

In this section we give an overview of both *positive linear systems* [1], [3], [4], and a very important subset of positive linear systems known as *compartmental systems* [1], [5]. The inclusion of compartmental systems within this subsection will be made because in general compartmental systems are stable, a property of great significance throughout the paper.

We first define a positive linear system [1] in the traditional sense.

Definition 2.1: A linear system

$$\begin{array}{rcl} \dot{x} & = & Ax + Bu \\ y & = & Cx + Du \end{array} \qquad (1)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{r \times n}$, and $D \in \mathbb{R}^{r \times m}$ is considered to be a *positive linear system* if for every nonnegative initial state and for every nonnegative input the state of the system and the output remain nonnegative.

It turns out that Definition 2.1 has a very nice interpretation in terms of the matrix quadruple (A, B, C, D).

Theorem 2.1 ([1]): A linear system (1) is positive if and only if the matrix A is a Metzler matrix, and B, C, and D are nonnegative matrices.

An interesting subset of positive systems is that of compartmental systems. A compartmental system consists of n interconnected compartments [5] or reservoirs. The main mathematical distinction, for LTI systems, between a positive system and a compartmental system is that a positive system's A matrix is Metzler, while a compartmental system's A matrix is compartmental. For a more complete study and interesting results on compartmental systems see [5] and references therein.

C. Tuning Regulators and Feedforward Control

In this section we describe two controllers, the tuning regulator and the feedforward compensator, which solve the tracking problem for *unknown*[1] stable LTI systems under constant disturbances. The results of this section can be found in their entirety and in their general form in [6], [7]. The tuning regulator described within this subsection is nothing more but a generalization of the classical "on-line tuning" controller [8].

Consider the plant

$$\begin{array}{rcl} \dot{x} & = & Ax + Bu + E\omega \\ y & = & Cx + Du + F\omega \\ e & := & y_{ref} - y \end{array} \qquad (2)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^r$, the disturbance vector $\omega \in \mathbb{R}^{\tilde{\Omega}}$, and $y_{ref} \in \mathbb{R}^r$ is a desired tracking signal. Assume that the output y and the disturbance ω are measurable with $m = r$, and that the matrix A is Hurwitz.

In the case of constant disturbances (ω) and constant tracking (y_{ref}) signals, the tuning regulator that solves the "robust servomechanism problem" [6], i.e. such that

(i) the closed loop system is stable,
(ii) for all tracking signals and disturbances $e \to 0$ as $t \to \infty$, and
(iii) property (ii) occurs for all plant perturbations which maintain closed loop stability,

is given by:

$$\begin{array}{rcl} \dot{\eta} & = & \epsilon e \\ u & = & (D - CA^{-1}B)^{-1}\eta, \end{array} \qquad (3)$$

[1]by unknown we mean that there is no knowledge of (A, B, C, D)

where $\epsilon \in (0, \epsilon^*]$, $\epsilon^* \in \mathbb{R}_+ \setminus \{0\}$, and we assume that the initial condition $\eta_0 = 0$.

On the other hand, in the case of constant disturbances and constant tracking signals, the feedforward compensator that solves the "servomechanism problem", i.e. such that

(i) the closed loop system is stable, and
(ii) for all tracking signals and disturbances $e \to 0$ as $t \to \infty$.

is given by

$$u = K_r y_{ref} + K_d \omega \qquad (4)$$

with $K_r = (D - CA^{-1}B)^{-1}$ and $K_d = -(D - CA^{-1}B)^{-1}(F - CA^{-1}E)$.

We summarize the above discussion by a Theorem for the case of MIMO LTI systems.

Theorem 2.2 ([6]): Consider the system (2), under the assumption that $y_{ref} \in \mathbb{R}^r$ and $\omega \in \mathbb{R}^{\tilde{\Omega}}$ are constants. Then there exists an ϵ^* such that $\forall \epsilon \in (0, \epsilon^*]$ the tuning regulator (3) solves the "robust servomechanism problem" and the feedforward compansator (4) solves the "servomechanism problem" if and only if $rank(D - CA^{-1}B) = r$.

Throughout the paper it is emphasized that no model of the system exists, thus it is worth pointing out an algorithm which will supply us with the gain matrix $(D - CA^{-1}B)$ without the knowledge of the actual values of (A, B, C, D). We present the algorithm next.

Algorithm 2.1: It is assumed that the outputs of the system are measurable and the inputs are excitable with no disturbances acting on the plant, i.e. $\omega = 0$.

1) Apply an input vector $u = [0 \ ... \ 0 \ \overline{u}_i \ 0 \ ... \ 0]^T$ to (2), $\forall i = 1, ..., m$, with $\overline{u}_i$ having a non-zero steady-state value.
2) Measure the corresponding steady-state value of the output vectors $y = \overline{y}_i \in \mathbb{R}^r$, $\forall i = 1, ..., m$, where $\overline{y}_i = [\overline{y}_i^1 \ \overline{y}_i^2 \ ... \ \overline{y}_i^r]^T$.
3) Solve the equation:

$$K_1 \begin{bmatrix} \overline{u}_1 & 0 & \ldots & 0 \\ 0 & \overline{u}_2 & \ldots & 0 \\ & & \ddots & \\ 0 & 0 & \ldots & \overline{u}_m \end{bmatrix} = \begin{bmatrix} \overline{y}_1^1 & \overline{y}_2^1 & \ldots & \overline{y}_m^1 \\ \overline{y}_1^2 & \overline{y}_2^2 & \ldots & \overline{y}_m^2 \\ & & \ddots & \\ \overline{y}_1^r & \overline{y}_2^r & \ldots & \overline{y}_m^r \end{bmatrix}$$

for $K_1 = (D - CA^{-1}B)$.

Note, in the case of a tuning regulator and the feedforward compensator we need the inverse of K_1; this is easily obtained once we know that the existence condition $rank(K_1) = r$ is satisfied.

Next we present an algorithm to obtain the gain matrix $(F - CA^{-1}E)$ without any knowledge of the system plant.

Algorithm 2.2: It is assumed that the outputs of the system are measurable and the disturbances are excitable with the inputs set to zero, i.e. $u = 0$.

1) Apply a disturbance vector $\omega = [0 \ ... \ 0 \ \overline{\omega}_i \ 0 \ ... \ 0]^T$ to (2), $\forall i = 1, ..., \tilde{\Omega}$, with $\overline{\omega}_i$ having a non-zero steady-state value.
2) Measure the corresponding steady-state value of the output vectors $y = \overline{y}_i \in \mathbb{R}^r$, $\forall i = 1, ..., \tilde{\Omega}$, where $\overline{y}_i = [\overline{y}_i^1 \ \overline{y}_i^2 \ ... \ \overline{y}_i^r]^T$.

3) Solve the equation:

$$K_2 \begin{bmatrix} \overline{\omega}_1 & 0 & ... & 0 \\ 0 & \overline{\omega}_2 & ... & 0 \\ & & \ddots & \\ 0 & 0 & ... & \overline{\omega}_{\tilde{\Omega}} \end{bmatrix} = \begin{bmatrix} \overline{y}_1^1 & \overline{y}_2^1 & ... & \overline{y}_{\tilde{\Omega}}^1 \\ \overline{y}_1^2 & \overline{y}_2^2 & ... & \overline{y}_{\tilde{\Omega}}^2 \\ & & \ddots & \\ \overline{y}_1^r & \overline{y}_2^r & ... & \overline{y}_{\tilde{\Omega}}^r \end{bmatrix}$$

for $K_2 = (F - CA^{-1}E)$.

In the above experiment it is assumed that the measurable disturbance can be excited in $\tilde{\Omega}$ independent ways. This is often the case in practice [6], e.g. in the control of distillation columns, the input feed composition is a measurable disturbance which can be excited; in a system of water tanks, an additional flow of water can be added to the various tanks, etc. Occasionally, however, it may be possible to measure a disturbance, but not excite it. In these cases, the operating records of the measurable disturbances can be monitored, e.g. in commercial heat exchangers [9].

Both algorithms will be illustrated in Section IV.

Remark 2.1: Note that both K_r and K_d can be found via Algorithm 2.1 and Algorithm 2.2 with no knowledge of the system. In the case of the tuning regulator (3) the choice of $\epsilon > 0$ to use is found by "on-line tuning", i.e. the controller (3) is applied to the system to be controlled, and a 1-dimensional search on $\epsilon > 0$ is carried out to obtain the *best output response for the system*; Theorem 2.2 guarantees that such a stabilizing controller can always be found.

III. Main Discussion and Results

In this section we discuss the main ideas of the paper. First, we introduce the plant and the two problems of interest. Second, we point out that for positive systems one cannot solve the tracking problem in general with nonnegative control inputs; with the latter result in mind, we provide necessary and sufficient conditions when nonnegative control inputs can solve the tracking and disturbance rejection problem for unknown stable MIMO positive LTI systems. Moreover, we provide the type of controller for the problem in mind.

Plant 3.1: Throughout the rest of the paper we consider the plant (2) with the assumption that A is stable and Metzler, the matrices (B, C, D, E, F) are nonnegative, and the input u, disturbance ω, and tracking signal y_{ref} are nonnegative. Moreover, we assume that the condition of Theorem 2.2 holds true, i.e. $rank(D - CA^{-1}B) = r$. With the above assumptions, we are strictly interested in positive systems, as per Definition 2.1.

Next, we introduce two problems of interest. The first problem considers unknown plants, which do not experience any perturbations. The second problem considers unknown plants that may experience perturbations.

Problem 3.1: Consider Plant 3.1. Find a controller $u \in \mathbb{R}_+^m$ that solves the *servomechanism problem* for all reference tracking signals $y_{ref} \in \mathbb{R}_+^r$ and for all disturbance signals $\omega \in \mathbb{R}_+^{\overline{\Omega}}$, i.e. find a controller u such that

(a) closed loop stability is maintained;
(b) nonnegativity of states x and outputs y occurs for all time;
(c) tracking of reference signals occurs, i.e. $e = y - y_{ref} \to 0$, as $t \to \infty$, $\forall y_{ref} \in \mathbb{R}_+^r$ and $\forall \omega \in \mathbb{R}_+^{\overline{\Omega}}$.

The second problem considers plants that may undergo some perturbations.

Problem 3.2: Consider Plant 3.1. Find a controller $u \in \mathbb{R}_+^m$ that solves the *robust servomechanism problem* for all reference tracking signals $y_{ref} \in \mathbb{R}_+^r$ and for all disturbance signals $\omega \in \mathbb{R}_+^{\overline{\Omega}}$, i.e. find a controller u such that conditions (a), (b), (c) of Problem 3.1 are satisfied, and in addition

(d) assume that an LTI controller has been found so that conditions (a), (b), (c) are satisfied; then for all perturbations of the nominal plant modal which maintain properties (a) and (b) of Problem 3.1, it is desired that the controller can still achieve asymptotic tracking and regulation, i.e. property (c) of Problem 3.1 still holds.

all hold true.

We now illustrate that both Problem 3.1 and Problem 3.2 are unattainable. The first step in showing the latter is the presentation of two key results from matrix theory.

Lemma 3.1 ([10]): If a nonnegative matrix $\mathcal{A}$ is square and nonsingular, then its inverse $\mathcal{A}^{-1}$ is also nonnegative if and only if $\mathcal{A}$ is a monomial matrix.

Lemma 3.2 ([4]): Let $\mathcal{A}$ be a Metzler matrix; then $-A^{-1}$ exists and is a nonnegative matrix if and only if A is Hurwitz.

We are now ready to state the first major result.

Theorem 3.1: There does not exist a solution to Problems 3.1 and 3.2 for almost all positive systems 3.1.

Proof: A necessary result for Problem 3.1 and Problem 3.2 to hold is the need for the steady-state value of the input u_{ss} to be nonnegative. We now show that this in general does not hold. The steady-state can be calculated via K_r and K_d [6], i.e.

$$u_{ss} = K_r \overline{y}_{ref} + K_d \overline{\omega},$$

for any $\overline{y}_{ref}$ and $\overline{\omega}$. However, $K_r = (D - CA^{-1}B)^{-1}$ and $K_d = (D - CA^{-1}B)^{-1}(F - CA^{-1}E)$ and since we are dealing with positive systems, i.e. the system matrices (B, C, D, E, F) are nonnegative, and since $-A^{-1}$ is also nonnegative by Lemma 3.2, we can conclude that the matrices

$$(D - CA^{-1}B) \text{ and } (F - CA^{-1}E)$$

are also nonnegative. In order to have a nonnegative u_{ss} we need the inverse of $(D - CA^{-1}B)$ to be nonnegative if $\omega = 0$, since $\overline{y}_{ref}$ is nonnegative. Assume now that $r > 1$ and notice that by Lemma 3.1 this holds if and only if $(D - CA^{-1}B)$ is a monomial matrix, which generically is not the case, i.e. $(D - CA^{-1}B)$ is a monomial matrix if and only if $(D - CA^{-1}B)^{-1}$ is a monomial matrix which is true if and only if

$$[0 \quad I] \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} [0 \quad I]^T$$

is monomial, or alternatively if and only if

$$[0 \quad I] \frac{adj \begin{pmatrix} A & B \\ C & D \end{pmatrix}}{det \begin{pmatrix} A & B \\ C & D \end{pmatrix}} [0 \quad I]^T \tag{5}$$

is monomial, where

$$det\begin{pmatrix} A & B \\ C & D \end{pmatrix} \neq 0$$

follows from the assumption that $rank(D - CA^{-1}B) = r$. It directly follows now that the class of parameters of (C, A, B, D) which results in (5) being monomial is a hypersurface in the parameter space of (C, A, B, D) [11].

Also, notice that if $\overline{y}_{ref} = 0$, then $u_{ss} = -(D - CA^{-1}B)^{-1}(F - CA^{-1}E)\omega$ and there never exists a case when all entries of $-(D - CA^{-1}B)^{-1}(F - CA^{-1}E)$ are positive, since $-(D - CA^{-1}B)^{-1} \leq 0$ entry-wise, by the above argument.

Assume now that $r = 1$, and that $\overline{y}_{ref} = 0$. In this case, $u_{ss} = -(D - CA^{-1}B)^{-1}(F_C A^{-1}E)\omega$ is negative for all $\omega > 0$, which implies that there exists no solution to Problems 3.1 and 3.2. ∎

The above result leads us to two new restricted problems in which we want to find the largest subclass of signals $y_{ref} \in Y_{ref} \subset \mathbb{R}_+^r$ and $\omega \in \Omega \subset \mathbb{R}_+^{\overline{\Omega}}$ such that the latter problems can be solved.

Problem 3.3: Obtain the largest subclass of tracking signals $y_{ref} \in Y_{ref} \subset \mathbb{R}_+^r$ and disturbance signals $\omega \in \Omega \subset \mathbb{R}_+^{\overline{\Omega}}$ such that Problem 3.1 is solvable.

In similar fashion, we define an analogous problem for Problem 3.2.

Problem 3.4: Obtain the largest subclass of tracking signals $y_{ref} \in Y_{ref} \subset \mathbb{R}_+^r$ and disturbance signals $\omega \in \Omega \subset \mathbb{R}_+^{\overline{\Omega}}$ such that Problem 3.2 is solvable.

Through the rest of this paper, we shift our attention to the solution of the latter two problems. Next, we will show that Problem 3.3 can be solved by using a feedforward compensator, and thereafter, we show that Problem 3.4 can be solved by using a combination of a feedforward compensator and a tuning regulator.

Theorem 3.2: Problem 3.3 is solvable if and only if

$$(y_{ref}, \omega) \in Y_{ref} \times \Omega := \{(\overline{y}_{ref}, \overline{\omega}) \in \mathbb{R}_+^r \times \mathbb{R}_+^{\overline{\Omega}} \mid K_r\overline{y}_{ref} \geq -K_d\overline{\omega} \text{ component-wise}\}. \quad (6)$$

Moreover, it suffices to use the feedforward compensator (4) as the control input u.

Proof:

($\Rightarrow$) Since Problem 3.3 is solvable, then

$$\begin{aligned} u_{ss} &= K_r y_{ref} + K_d\omega \text{ which implies component-wise:} \\ 0 &\leq K_r y_{ref} + K_d\omega \text{ or that:} \\ K_r y_{ref} &\geq -K_d\omega \end{aligned}$$

($\Leftarrow$) Conditions (a) and (c) of Problem 3.3 hold by the results of [6]. Condition (b) can be guaranteed if the control input $u \geq 0$ component-wise for all time. However, the feedforward compensator (4) has the property that:

$$u = u_{ss} \geq 0,$$

which will guarantee nonnegativity of the states and outputs, thus solving Problem 3.3 with $Y_{ref} \times \Omega$ defined by (6). ∎

Feedforward compensators are an effective theoretical tool to solve the tracking and regulation problem. However in practice, due to possible changes to the parameters of the plant, feedforward controllers in general may lead to unsatisfactory tracking/regulation. Thus in our next result, we tackle Problem 3.4, which takes perturbation of the plant parameters into account.

Theorem 3.3: Problem 3.4 is solvable if and only if

$$(y_{ref}, \omega) \in Y_{ref} \times \Omega := \{(\overline{y}_{ref}, \overline{\omega}) \in \mathbb{R}_+^r \times \mathbb{R}_+^{\overline{\Omega}} \mid K_r\overline{y}_{ref} > -K_d\overline{\omega} \text{ component-wise}\}. \quad (7)$$

Moreover, it suffices to use the feedforward compensator (4) and the tuning regulator control (3) as the control input u, i.e.

$$u = u_{ff} + u_{tr}, \quad (8)$$

where u_{ff} is the feedforward compensator (4) and u_{tr} is the tuning regulator (3), where an $\epsilon^* > 0$ exists such that the closed loop system is stable $\forall\epsilon \in (0, \epsilon^*]$.

Proof:

($\Rightarrow$) Follows the same argument as in the proof of Theorem 3.2.

($\Leftarrow$) Conditions (a) and (b) follow from [6]. Condition (c) will hold if $u \geq 0$ for all time. Condition (d) is guaranteed by [6] and if $u \geq 0$. Thus, we now illustrate that there exists an ϵ^* so that $u \geq 0$ for all $\epsilon \in (0, \epsilon^*]$, for all time. We first implement the feedforward compensator (4) and obtain an input $u_{ff} > 0$ by (7). Our next step then is to add the tuning regulator and show that $\forall\delta \in \mathbb{R}_+^m \setminus \{0\}$, there exists an $\epsilon^* > 0$ such that

$$u_{tr} \geq -\delta, \quad \forall t \in [0, \infty),\ \forall\epsilon \in (0, \epsilon^*].$$

In particular, we would like to choose $\delta \in \mathbb{R}_+^m \setminus \{0\}$ so that $u_{ff} \geq \delta$, component-wise. Now, if such an ϵ^* exists, then

$$\begin{aligned} u &= u_{ff} + u_{tr} \\ &\geq u_{ff} - \delta \\ &\geq 0, \end{aligned}$$

i.e. u is nonnegative for all time, which is the result needed. Given $\delta \in \mathbb{R}_+^m \setminus \{0\}$, in order to show that $u_{tr} \geq -\delta$ we will use the results of singular perturbation, which the reader can find an overview of in the Appendix, where various terms and definitions of variables used in the remainder of the proof have been introduced and defined.

In order to prove the above, we can break down the proof into three parts, i.e. we will show that

(1) there exists an ϵ_1 and a time $t_{O(\epsilon_1)}$ such that for all $t \geq t_{O(\epsilon_1)}$

$$\|u(t) - \overline{u}(t)\| = O(\epsilon) \quad \Rightarrow \quad |u_i(t) - \overline{u}_i(t)| \leq \delta_i,$$

$\forall i = 1, ..., m$, where $\overline{u}$ monotonically approaches a nonnegative value for $\epsilon \in (0, \epsilon_1]$; and

(2) that there exists an ϵ_2 such that for all time $t \in [0, t_{O(\epsilon_1)}]$

$$u_i(t) \geq -\delta_i, \quad \forall i = 1, ..., n$$

(3) finally by choosing $\epsilon_\delta = min[\epsilon_1 \ \epsilon_2]$ the result follows for any given δ.

Point (2) and (3) are clear, with (2) being a consequence of continuity; thus, it only leaves (1) to prove. In order to show (1), we will use singular perturbation results ([12], Chapter 11). The closed loop system after applying the feedforward compensator and the tuning regulator is of the form:

$$\begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A & BK_r \\ -\epsilon C & -\epsilon DK_r \end{bmatrix} \begin{bmatrix} x \\ \eta \end{bmatrix} + \begin{bmatrix} E + K_d & K_r \\ -\epsilon F & \epsilon I \end{bmatrix} \begin{bmatrix} \omega \\ y_{ref} \end{bmatrix}.$$

Now, since the equilibrium (x_{ss}, η_{ss}) is independent of ϵ (trivial to show), we can transform the system as needed:

$$\begin{bmatrix} z \\ q \end{bmatrix} := \begin{bmatrix} x \\ \eta \end{bmatrix} - \begin{bmatrix} x_{ss} \\ \eta_{ss} \end{bmatrix}$$

resulting in the new system

$$\begin{bmatrix} \dot{q} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\epsilon DK_r & -\epsilon C \\ BK_r & A \end{bmatrix} \begin{bmatrix} q \\ z \end{bmatrix}. \tag{9}$$

Next, let us scale the derivatives (i.e. scaling of time, $t_0 = 0$) by

$$\epsilon dt = d\tau$$

resulting in the transformed system

$$\begin{bmatrix} \overset{\odot}{q} \\ \epsilon \overset{\odot}{z} \end{bmatrix} = \begin{bmatrix} -DK & -C \\ BK & A \end{bmatrix} \begin{bmatrix} q \\ z \end{bmatrix}, \tag{10}$$

with $\epsilon \overset{\odot}{q} = \dot{q}$ and $\epsilon \overset{\odot}{z} = \dot{z}$. We have now transformed our model into that of the standard singular perturbation model [12]. Next, since (10) is linear and time invariant and as we are strictly interested in the input u_{tr}, it suffices to show that the *reduced model* (12) below with $\epsilon = 0$ yields exponential stability. Thus, by setting $\epsilon = 0$ we can define:

$$z = h(q) := -A^{-1}BK_r q. \tag{11}$$

Now since A is Hurwitz in (12), $h(q)$ exists and is unique as needed in the singular perturbation problem. Next by substituting $h(q)$ into $\overset{\odot}{q}$ we obtain the *reduced model*:

$$\begin{aligned} \overset{\odot}{q} &= -DKq + CA^{-1}BK_r q \qquad (12) \\ &= -(D - CA^{-1}B)K_r q \\ &= -(D - CA^{-1}B)(D - CA^{-1}B)^{-1} q \\ &= -q, \end{aligned}$$

clearly exponentially stable. Thus, by the results of singular perturbation there exists an $\epsilon = \epsilon_1$ and a time $t_{O(\epsilon_1)}$ such that

$$q - \overline{q} = O(\epsilon_1) \quad \forall t \geq t_{O(\epsilon_1)}$$

i.e.

$$\eta - \overline{\eta} = O(\epsilon_1) \quad \forall t \geq t_{O(\epsilon_1)},$$

where $\overline{q}$ is the solution of (12) in the t domain, and $\overline{\eta} = \overline{q} + \eta_{ss}$. By (12) it is now easy to deduce that $\overline{q}$ and $\overline{\eta}$ are monotonic; however, this implies that the tuning regulator $u_{tr} = K_r \eta$ is monotonic and since $u_{ss} \in \mathbb{R}^n_+ \setminus \{0\}$, the trajectory $\overline{u} = K_r \overline{\eta}$ will tend toward a nonnegative value monotonically, i.e. $u_{tr} \geq -\delta$ component-wise.

Finally, by setting $\epsilon^* = \epsilon_\delta$ in Theorem 2.2 we have found our control u. Now since δ is arbitrary the result follows. ■

This completes the main results of the paper.

IV. Example

In this section, we present an example which illustrates the contributions of the paper.

Example 4.1: The following plant, which is a stable compartmental system, has been taken from [1] pg.105. This example additionally possesses a random set of constant disturbances and an additional input and output. Consider the reservoir network of Figure 1; note that each reservoir is identified by a number $(1, 2, ..., 6)$ where the water storage level $(x_1, x_2, ..., x_6)$ is a state of the system. Also γ and ϕ are the splitting coefficients of the flows at the branching points. The system is of order 6, as we assume the pump dynamics can be neglected. As pointed out in [1], the dynamics of each reservoir can be captured by a single differential equation: $\dot{x}_i = -\alpha_i x_i + v + e_i\omega, \quad z = \alpha_i x_i$, for all $i = 1, ..., 6$, and where x_i is the water storage (assume in L) and $\alpha > 0$ is the ratio between outflow rate z and storage, with $e_i\omega$ being the disturbance rate into the storage. The input into the reservoir is designated by v and is in (L/s).

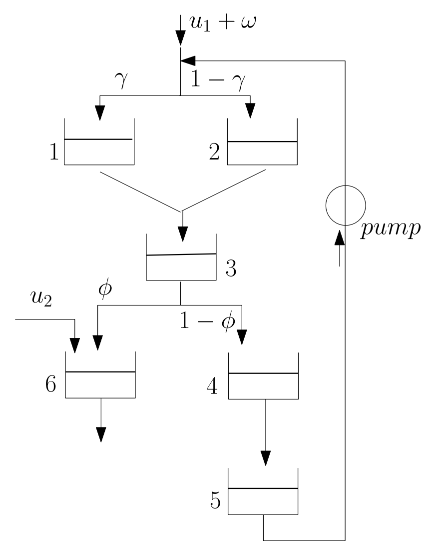

Fig. 1. System set up for Example 4.1.

Consider the case where $\gamma = 0.5$, $\phi = 0.7$, $\alpha_1, ..., \alpha_6 = 0.8,\ 0.7,\ 0.5,\ 1,\ 2,\ 0.8$. Note that all the rates are measured in L/s. This results in the following system:

$$\dot{x} = \begin{bmatrix} -0.8 & 0 & 0 & 0 & 2 & 0 \\ 0 & -0.7 & 0 & 0 & 0 & 0 \\ 0.8 & 0.7 & -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.15 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0.35 & 0 & 0 & -0.8 \end{bmatrix} x$$

$$+\begin{bmatrix} 0.5 & 0 \\ 0.5 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u + \quad [0.5 \;\; 0.5 \;\; 0 \;\; 0 \;\; 0 \;\; 0]^T \omega$$

Also, assume the output y is of the form

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x$$

Next, assume that the initial condition $x_{i0} = 0\ (L)\ \forall i = 1, ..., 6$, i.e. initially there is no water in the tanks, $\omega = 0.5\ (L/s)$, and that the tracking signal is $y_{ref} = [5\ 5]^T\ (L)$.

We now proceed to find the controller $u = u_{ff} + u_{tr}$ (8), which will solve the problem. First we obtain K_r by Algorithm 2.1. By applying $\overline{u}_1 = 1$ and $\overline{u}_2 = 1$ in steady state, we obtain:

$$\begin{aligned} K_r &= \begin{bmatrix} 2.8571 & 0 \\ 2.4107 & 1.2500 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 0.3500 & 0 \\ -0.6750 & 0.8000 \end{bmatrix}. \end{aligned}$$

In similar fashion, we can obtain the gain matrix

$$K_d = \begin{bmatrix} -0.5 \\ 0 \end{bmatrix}.$$

Now, it is easy to see that

$$u_{ss} = K_r y_{ref} + K_d \omega = [1.5\ 0.625]^T > 0,$$

and therefore by Theorem 3.3 we can proceed to use the feedforward and tuning regulator combination to solve Problem 3.4 (or Problem 3.3).

We now let $\epsilon = 0.1$ in (3). Figure 2 illustrates the simulated input response, while Figure 3 shows the output y.

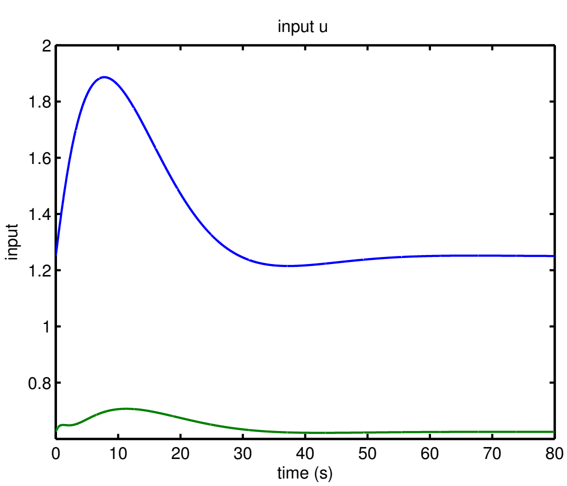

Fig. 2. Input response for Example 4.1.

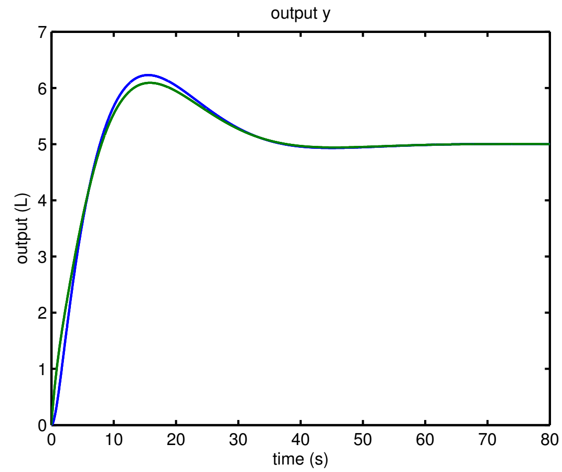

Fig. 3. Output response for Example 4.1.

V. Conclusion

In this paper we have discussed a variation of the servomechanism problem for stable unknown MIMO positive linear systems under constant reference and measurable constant disturbances. In particular, we have shown that the servomechanism problem for positive systems is in general unattainable under nonnegative inputs. However, in some cases the problem is feasible; in these case we have shown how to obtain the largest subset of the reference/disturbance signals and the controller that will solve the problem. We note that in a practical setting it is not always feasible to assume that all inputs and disturbance signals take on positive values, thus the authors are currently working on relaxing such constraints.

References

[1] L. Farina and S. Rinaldi, *Positive Linear Systems: Theory and Applications*, John Wiley & Sons, Inc, New York 2000

[2] B. Roszak and E.J. Davison, "Tuning regulators for tracking SISO positive linear systems", *European Control Conference 2007*, 8 pages, July 3-5, 2007.

[3] T. Kaczorek, *1D and 2D systems*, Springer, New York, 2002.

[4] D. Luenburger, *Introduction to Dynamic Systems: Theory, Models and Applications*, New York:Wiley, 1979.

[5] J.A. Jacquez and C.P. Simon, "Qualitative Theory of Compartmental Systems", *Society for Industrial and Applied Mathematics* , vol. 35(1), pp. 43-79, March 1993.

[6] E.J. Davison, "Multivariable Tuning Regulators: The Feedforward and Robust Control of a General Servomechanism Problem", *IEEE Transactions on Automatic Control*, vol.AC-21, Feb. 1976, pp.35-47.

[7] D.E. Miller and E.J. Davison, "The Self-Tuning Robust Servomechanism Problem", *IEEE Transactions on Automatic Control*, vol.34, May 1989, pp.511-523.

[8] F.G. Shinskey, *Process Control Systems*, New York : McGraw-Hill, 1967, p.101.

[9] E.J. Davison, P.A. Taylor, and J.D. Wright, "On the application of tuning regulators to control a commercial heat exchanger", IEEE Trans on Automatic Control, vol AC-25, no 3, June 1980, pp 361-375.

[10] R.J. Plemmons and R.E. Cline, "The Generalized Inverse of a Nonnegative Matrix", *Proceedings of the American Mathematical Society* vol.31, no.1, Jan. 1972, pp.46-50.

[11] E.J. Davison, S.H. Wang, "Properties of LTI multivariable systems subject to arbitrary output and state feedback", *IEEE Trans on Automatic Control*, vol AC-18, no 1, Feb. 1973, pp 24-32.

[12] H.K. Khalil, *Nonlinear Systems*, Prentice Hall, New Jersey 2002