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A constructive Lyapunov function for controlling the inverted pendulum

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Abstract— **A constructive control strategy is presented for the stabilization of the inverted pendulum system. Under the assumption that the pendulum is initialized above the upper half plane. To carried out, a suitable procedure is presented in order to find a candidate Lyapunov function, for the whole system. Then the control is proposed directly from the obtained Lyapunov, in such a way, that its time derivative be semi-definite negative with respect to a convenient manifold. Finally, the asymptotically stability of the closed-loop system is concluded by LaSalle theorem.**

Keywords: Underactuated mechanical system, Non-linear System, Lyapunov approach.

I. INTRODUCTION

Controlling the inverted pendulum on a cart (**IPC**) has attracted the attention of many researchers as a challenging benchmark to test advanced control strategies (see [1], [2] and references therein). This mechanical device consists of a free vertical rotating pendulum with a pivot point mounted on a cart. The cart can be moved horizontally by means of a horizontal force, which acts as the input of the system. Since the angular acceleration of the vertical pendulum cannot be directly controlled, the **IPC** is an interesting example of an under-actuated mechanical system. Therefore, many control techniques developed for fully-actuated systems cannot be used directly to stabilize this system. For instance, it is well known that the **IPC** is not input-output linearized by means of static feedback [3]. Besides, the system loses controllability and other geometric properties while the pendulum moves through the horizontal plane [4] and [5]. Nevertheless, the **IPC** is locally controllable around the unstable equilibrium point, hence the stabilization problem can be solved locally by a direct pole placement procedure [8].

Loosely speaking, there are two important problems related to the stabilization of this device. The first problem consist in swinging up the pendulum from the hanging position to the upright vertical position. An energy control strategy is usually applied for this purpose. Once the system is close to the desired upright position with enough low speed, by means of a simple change in the controller, from the non-linear to the linear controller, allows to keep the pendulum at the desired equilibrium (see [7],[6], [9], [10],[12], [13], [11]). The second problem consists in stabilizing the **IPC** around its unstable equilibrium point, assuming that the pendulum is initially above the horizontal plane, or lies inside an open vicinity of zero. This vicinity defines a stability domain for the close-loop system. We mention some of the most important works related to the second problem. In [15] a non-linear controller based on the backstepping procedure is used to solve the stabilization problem in its unstable equilibrium point. Authors build an interconnected system by applying some non-linear transformations, and then, by a recursively backstepping procedure, they derive a stabilizing controller that converges the angle position and the position of the cart to zero. In [14] a non-linear controller forcing the angle of the pendulum to zero is presented, however, this procedure does not guarantee convergence to zero of the cart. This controller was based on nested saturation functions. In [7] the authors propose a stabilization technique using switching and saturation functions, in addition to the Lyapunov method. The resulting closed-loop system possesses a very large region of attraction (for almost all initial conditions). In [16] the authors present a control strategy based on the method of controlled Lagrangians, in conjunction with some symmetry properties satisfied by the **IPC**. This strategy also assures asymptotic stability of the origin for a very large domain of attraction. A similar work with similar tools was presented in [17]. In [18] a semi-global stabilization for the **IPC** by means of a fixed point controller is introduced assuming that the angle is initialized above the upper half plane. This technique consists in finding a cascade form, and then fixed point equations for the control force is obtained by applying the backstepping procedure to develop a control strategy allowing the stabilization of the pendulum around the upper position, starting from any initial position. The proposed scheme is based on the use of saturation functions. The resulting closed-loop system is asymptotically stable without switching to a stabilizing local controller. A similar work, using nested saturation function approach, is presented in [19]. The authors first write an approximate model of the **IPC** model as a chain of integrators, by using of a convenient transformation. And then, they apply a nested saturation control technique to control it. There are several strategies related to the second problem. However, most of these approaches manage the physical model by introducing some non-linear approximations or reducing the order of the system (see [10], [6], [8] and references therein). Finally, we recommend the work presented in [20] to have a better knowledge in this topic.

A constructive nonlinear control law, developed for controlling the **IPC**, is presented in this paper. The control's goal is to make the system to be locally asymptotically stable around the unstable upright position, for a very large attraction domain. It is worth to mention that the main merit of this approach is the construction of a convenient Lyapunov

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function, which allow us to derive the controller and compute the stability domain, in straightforward manner..

The rest of this paper is organized as follows. Section 1 presents the dynamical model of the **IPC**. In section 2 we get a stabilizing nonlinear controller for the **IPC**. The corresponding stability and convergence analysis is carried out in the same section. Section 3 presents some computer simulations. Finally, the conclusions are given in section 4.

II. The inverted pendulum cart system

Consider the inverted pendulum mounted on a cart (**IPS**). The dynamics of this system, is given by [8]

$$\begin{aligned} &mL\cos\theta\ddot{x} + mL^2\ddot{\theta} - gmL\sin\theta = 0;\\ &(M+m)\ddot{x} + Lm\cos\theta\ddot{\theta} - mL\dot{\theta}^2\sin\theta = f, \end{aligned}$$

where x is the cart displacement, θ is the angle that the pendulum forms with the vertical, f is the force applied to the cart, acting as the control input. M and m stands for the cart mass and the pendulum mass concentrated in the bob, L is the pendulum length.

To simplify the algebraic manipulation in the forthcoming developments, we normalize the above equations by introducing the following scaling transformations,

$$q = x/L, \quad u = f/(mg), \quad d\tau = dt\sqrt{g/L}, \quad \delta = M/m.$$

This normalization leads to a simpler system,

$$\begin{aligned} &\cos\theta\ddot{q} + \ddot{\theta} - \sin\theta = 0,\\ &(1+\delta)\ddot{q} + \cos\theta\ddot{\theta} - \dot{\theta}^2\sin\theta = u, \end{aligned}$$

where, with an abuse of notation "." stands for differentiation with respect to the dimensionless time τ. Then, a convenient partial feedback linearization input is given (see [7], [13]),

$$u = \cos\theta\sin\theta - \dot{\theta}^2\sin\theta + v(2 + \sin^2\theta + \delta)$$

which produces the feedback equivalent system:

$$\begin{aligned} &\ddot{\theta} = \sin\theta - \cos\theta v,\\ &\ddot{q} = v. \end{aligned} \tag{1}$$

Notice that for the new input $v = 0$ and $\theta \in [0, 2\pi]$ the last system has two equilibrium points, one being an unstable equilibrium point $(\theta, \dot{\theta}, q, \dot{q}) = (0, 0, 0, 0)$ and the other a stable equilibrium point $(\theta, \dot{\theta}, q, \dot{q}) = (\pi, 0, 0, 0)$. In the sequence , the symbol $\mathbf{x}$ stands for $\mathbf{x} = (\theta, \dot{\theta}, q, \dot{q})$.

III. A stabilization control law

In this section we establish the framework for the stabilization of the **IPS** around its upright position, under the assumption that the pendulum is initialized over the horizontal plane. That is, the pendulum is brought to the unstable top position, with zero displacement of the cart, assuming that the initial angle position of the pendulum satisfies $|\theta(0)| < \pi/2$. For this, first a suitable candidate Lyapunov function is shaped by means of the technique of added integration, as presented in [7]. Secondly, a control law is obtained with its respective domain of attraction. Finally the stability analysis is carried out by means of the LaSalle's invariance theorem.

Construction of the candidate Lyapunov function

Lyapunov function for the two dimensional subsystem:

We begin to stabilize the variables θ and $\dot{\theta}$. For this, we introduce a function E_0, as

$$E_0(\theta, \dot{\theta}) = \frac{1}{2}\left(k_1\cos^2\theta - 1\right)\dot{\theta}^2 + (1 - \cos\theta), \tag{2}$$

where the constant $k_1 > 1$. As we can see, the proposed $E_0(\theta, \dot{\theta})$ is positive definite, for all $|\theta| < \widetilde{\theta}$, where

$$\widetilde{\theta} = \cos^{-1}(\sqrt{1/k_1}). \tag{3}$$

Then the time derivative of E_0 along the trajectories of the system (1), is then given by

$$\dot{E}_0(\theta, \dot{\theta}) = \dot{\theta}\cos\theta\left(v\alpha(\theta) + k_1\beta(\theta, \dot{\theta})\right), \tag{4}$$

where

$$\alpha(\theta) = 1 - k_1\cos^2\theta \quad , \quad \beta(\theta, \dot{\theta}) = (-\dot{\theta}^2 + \cos\theta)\sin\theta. \tag{5}$$

Thus, the following controller, given by

$$v_0 = -\frac{1}{\alpha(\theta)}\left(\dot{\theta}\cos\theta + k_1\beta(\theta, \dot{\theta})\right), \tag{6}$$

makes variables θ and $\dot{\theta}$, to converge asymptotically to zero, because v_0 produces

$$\dot{E}_0(\theta, \dot{\theta}) = -\dot{\theta}^2\cos^2\theta. \tag{7}$$

Of course, the previous positive function E_0 is very suitable just for the asymptotically stabilization of the pendulum's variables. But, this function shows how to shape a convenient Lyapunov function for controlling also the cart velocity $\dot{q}$. For that purpose, one quadratic term is added to the mentioned positive function, in where, the time derivative of the additional term has,somehow , the structure of the relation (4). This idea is inspired by the work presented by Jurdjevic and Quinn [21].

Lyapunov function for the three dimensional subsystem:

Let us propose the unknown variable Z=$z(\theta, \dot{\theta}, \dot{q})$ provided that,

$$\dot{z}(\theta, \dot{\theta}, \dot{q})\dot{\theta}\cos\theta = \dot{E}_0(\theta, \dot{\theta}). \tag{8}$$

So that the variable Z must satisfy

$$\dot{Z} = \dot{\theta}\frac{\partial Z}{\partial\theta} + \sin\theta\frac{\partial Z}{\partial\dot{\theta}} + \left(\frac{\partial Z}{\partial\dot{q}} - \cos\theta\frac{\partial Z}{\partial\dot{\theta}}\right)v. \tag{9}$$

Combining the relations (9), (8) and (4), we obtain after some manipulation, two important relations

$$\dot{\theta}\frac{\partial Z}{\partial\theta} + \sin\theta\frac{\partial Z}{\partial\dot{\theta}} = k_1(\dot{\theta}^2 - \cos\theta)$$

and

$$\frac{\partial Z}{\partial\dot{q}} - \cos\theta\frac{\partial Z}{\partial\dot{\theta}} = (1 - k_1\cos^2\theta).$$

It turns out to be that one solution is given by,

$$Z = \dot{q} + k_1\dot{\theta}\cos\theta. \tag{10}$$

From relations (4), (8) and (10), a convenient positive function may be defined , as

$$E_1(\theta,\dot\theta,\dot q)=\frac{1}{2}(\dot q+k_1\dot\theta\cos\theta)^2+k_dE_0(\dot\theta,\theta), \qquad (11)$$

where k_d is a positive constant. Clearly, the proposed positive function E_1 is also locally positive definite for all $|\theta|<\widetilde{\theta}$. Now, differentiating the above E_1 along of the trajectories of system (1). This yields, after using relation (4), the following

$$\dot E_1(\theta,\dot\theta,\dot q)=\dot w(\theta,\dot\theta,\dot q)\left(v\alpha(\theta)+k_1\beta(\theta,\dot\theta)\right). \qquad (12)$$

where

$$\dot w(\theta,\dot\theta,\dot q)=\dot q+(k_1+k_d)\dot\theta\cos\theta \qquad (13)$$

So that control v_2 can be chosen as

$$v_1=-\frac{1}{\alpha(\theta)}\left(\dot w(\theta,\dot\theta,\dot q)+k_1\beta(\theta,\dot\theta)\right).$$

Since v_1 makes semi-definite negative the time derivative of E_1, then ,

$$\dot E_1(\theta,\dot\theta,\dot q)=-\dot w^2(\theta,\dot\theta,\dot q).$$

It is very easy to show that v_1 is able to stabilize asymptotically the states $\theta,\dot\theta$ and $\dot q$. But, the proof of this fact will be omitted, for it is very similar to the stability analysis, which will be presented below.

Lyapunov function for the whole system:

In the same way , the construction of the candidate Laypunov function for all states, is carried out by adding one quadratic term to the aforementioned positive function E_1. Then, a simple integration of the variable $\dot w$ leads to,

$$w(\theta,q)=q+(k_1+k_d)\sin\theta. \qquad (14)$$

Therefore, a convenient Lyapunov function can be introduced, as

$$E_2(\mathbf{x})=E_1(\theta,\dot\theta,\dot q)+\frac{k_p}{2}w^2(\theta,q). \qquad (15)$$

We can show that the above function $E_2(\mathbf{x})$ is locally definite positive, for all for all $|\theta|<\widetilde{\theta}$. Finally, the time derivative of E_2 along the trajectories of the system (1), is then given by using the relation (12) and the definition of the variables w and $\dot w$ (see (13) and (14)), as

$$\dot E_2(\mathbf{x})=k_pw(\theta,q)\dot w(\theta,\dot\theta,\dot q)+\left(v\alpha(\theta)+k_1\beta(\theta,\dot\theta)\right)\dot w(\theta,\dot\theta,\dot q). \qquad (16)$$

Notice that the time derivative of the above function E_2 has the same structure of the previous time derivative E_1. This is the main trick to construct the candidate Lyapunov function(15), for all the states.

A nonlinear feedback controller for the whole system

Consider the proposed Lyapunov function E_2 with its time derivative $\dot E_2$, given in (15) and (16), respectively. Then, a suitable control law is presented, as

$$v=\frac{-k_i\dot w(\theta,\dot\theta,\dot q)-k_pw(\theta,q)-k_1\beta(\theta,\dot\theta)}{\alpha(\theta)}. \qquad (17)$$

with $k_i>0$. Note that this controller v produces

$$\dot E_2(\mathbf{x})=-k_i\dot w^2(\theta,\dot\theta,\dot q). \qquad (18)$$

Remark : *The proposed controller is well-defined for all* $|\theta|<\widetilde{\theta}$*. To avoid these singularity* $\theta=\pm\widetilde{\theta}$ *(or* $\alpha(\theta)=0$*), it is sufficient that the initial conditions* $x_0=(\theta(0),\dot\theta(0),q(0),\dot q(0))$*; with the assumption that* $|\theta(0)|<\widetilde{\theta}$*, belonging to a neighborhood of the origin, such that*

$$E_2(\mathbf{x}_0)<\widetilde{K}=\frac{k_p}{2}(k_1+k_d)^2\sin^2\widetilde{\theta}+1-\cos\widetilde{\theta} \qquad (19)$$

where $\widetilde{\theta}$ *is given in (3). This is a consequence of the fact that* $E_2(x)$ *is a non-increasing function (18).*

Therefore the inequality (19) defines a stability region for the closed-loop system (see (1) and (17)). That is, for all initial conditions $\mathbf{x}_0$, such that $E_2(\mathbf{x}_0)<\widetilde{K}$; with $|\theta_0|<\widetilde{\theta}$, we guarantee that $E_2(\mathbf{x}(t))<\widetilde{K}$, and also, $|\theta(t)|<\widetilde{\theta}$. This fact allows to define a compact set Ω, as

$$\Omega=\{\mathbf{x}=(\theta,\dot\theta,q,\dot q),|\theta|<\widetilde{\theta}:E_2(\mathbf{x})<\widetilde{K}\}. \qquad (20)$$

The set Ω has the particularity that all the solutions of the closed-loop system, that starts in Ω remain in Ω for ever.

Stability Analysis

Since $E_2(\mathbf{x})$ is positive definite for all $\mathbf{x}\in\Omega$, and $\dot E_2(\mathbf{x})$ is only a semi-definite negative, for all $\mathbf{x}\in R^4$. Obviously , we have stability in the sense of Lyapunov. To complete the proof, we must apply LaSalle's theorem to test the asymptotic stability of the equilibrium point $\mathbf{x}=0$.

Let us first define the set

$$S=\{\mathbf{x}\in\Omega:\left(\dot q+(k_1+k_d)\dot\theta\cos\theta\right)^2=\dot w^2=0\}, \qquad (21)$$

and let M be the largest invariant set in S. LaSalle's theorem guarantee that every solutions starting in a compact set Ω approaches M as $t\to\infty$ (see [**?**]).

Let us then compute the largest invariant set M in S. Clearly, on the set S, it follows that the auxiliary variable w, given by $w=q+(k_1+k_d)\sin\theta$, is a fixed constant on the set S. From the time derivative of E_2(16), it is clear that the controller was selected, so that

$$k_pw(\theta,q)+v\alpha(\theta)+k_1\beta(\theta,\dot\theta)=-k_i\dot w(\theta,q,\dot q).$$

Then, on the set S the above relations leads to

$$k_pw(\theta,q)+v\alpha(\theta)+k_1\beta(\theta,\dot\theta)=0. \qquad (22)$$

Computing the time derivative of the variable $\dot w$ defined in (13), we have

$$k_d\ddot w=v(\alpha(\theta)-1)+(k_1+k_d)\beta(\theta,\dot\theta). \qquad (23)$$

Evidently on the set S, the variable $\ddot w=0$. In such case equation (23) yields the following equality

$$\beta(\theta,\dot\theta)=\frac{v(1-\alpha(\theta))}{(k_1+k_d)}. \qquad (24)$$

Substituting equality (24) in relation (22), we have

$$k_pw(\theta,q)+v\left(\frac{\alpha(\theta)+k_1}{k_1+k_d}\right)=0. \qquad (25)$$

But, on the set S, control v is necessarily a fixed constant, given by the quantity $v = \underline{v}$.[1] Indeed, it follows from this that $w(\theta, q)$ is a fixed constant , on the set S. Besides, $\alpha(\theta)+k_1 > 0$, for all $\theta \in R$ (see the definition in (5)). Therefore, on the set S, system (1) is equivalent to

$$\ddot{\theta} = \sin\theta - \cos\theta \underline{v} \quad ; \quad \ddot{q} = \underline{v} \ . \tag{26}$$

We must analyze the possible cases arising from the above equations:

First, if the constant $\underline{v} \neq 0$ then $\dot{q}\ (t)$ is not bounded on the set S. This fact is a contradiction, for the vector stat $\mathbf{x}(t)$ is bounded, on the set S. Hence, we have $\underline{v} = 0$. This means that $\dot{q}(t)$ being constant, on the set S, , $q(t)$ is not bounded on S. And also we have a contradiction, so that $\dot{q} = 0$, on the set S. In addition , we have

$$q + (k_1 + k_d)\sin\theta = 0. \tag{27}$$

It follows directly from definition (14). Therefore, on the set S, the variables q and θ are constants, defined by $q =\underline{q}$ and $\theta =\underline{\theta}$, respectively. On the other hand, from left-side relation of (26), we have $\sin\underline{\theta}= 0$, on the set S. But, by assuming that $|\theta| < \widetilde{\theta} < \pi/2$, we conclude that $\underline{\theta}= 0$, on the set S. Finally, from (27), it follows that $q = 0$, on the set S. That is to say, the largest invariant set M contained in the set S is the single unstable equilibrium point $\mathbf{x} = 0$. According to La Salle's Theorem all the closed-loop solutions starting in Ω asymptotically converge towards the largest invariant set M, which is the equilibrium point $\mathbf{x} = 0$.

Summarizing the above discussion, we have:

Proposition 1: *Consider the partial linearized system of the **IPS** (1), in the closed-loop with (17), where k_p, k_d, k_i and k_1 are strictly positive constants. Under the assumption that the initial conditions belongs to the compact set Ω (20). Then, the origin of the closed-loop system is locally asymptotically stable and the domain of attraction is the region defined by the inequality (19).*

IV. Numerical simulations

Simulations were carried out to estimate the performance of the proposed nonlinear controller (17), when applied to the normalized system (1). The simulation was performed in Matlab by simple Runge-Kutta algorithm.

To stress the influence of control parameters k_p on the transient behavior, we put together two plots of each state. To do this, we have fixed parameters $k_d = 2.5$ and $k_1 = 4$ whereas parameter k_p was changed from 2 to 0.5. The set of initial conditions was set as $\theta_0 = 1.1$, $\dot{\theta}_0 = 0.1$, $q_0 = 0$ and $\dot{q}_0 = 0$. Figure 1, shows the transient behavior of the variables θ and q, while figure 2, shows the transient behavior of the velocities $\dot{\theta}$ and $\dot{q}$. As we can notice, large values of k_p produce more oscillations and make all the states converge slowly to the desired equilibrium point. Intuitively, the controller injects more potential energy to the system, so that, the system must dissipate all initial potential energy by means of oscillatory movements.

[1]Hereafter, we use the symbol $\underline{X}$ to denote that the variable X is a fix constant, on the set S.

To evaluate the robustness of the closed-loop system respectively to a dissipative force in the not-actuated coordinate θ. A damping term " $-\ 0.1\dot{\theta}$" is added into the right side of the first differential equation of (1). For this simulation, we considered the set of parameters $k_d = 3$, $k_1 = 3$ and $k_p = 1$, and the set of initial conditions as $\theta_0 = 0.8$, $\dot{\theta}_0 = 0$, $q_0 = 1.2$ and $\dot{q}_0 = 0$. Figure 3, shows the closed-loop response of the system when damping is considered. Notice that proposed feedback controller has a good performance, even when damping can destabilize the system.

V. Conclusions

A control strategy for the stabilization of the **IPS** around its unstable equilibrium was examined in this paper. Assuming that the pendulum is initialized above the horizontal plane. The control strategy has been based, on a partial feedback linearization of the **IPC**, following by Lyapunov's approach. First, a constructive procedure for the construction of a convenient candidate Lyapunov function has been presented. Second, a stabilizing control law was obtained directly from the proposed Lyapunov function. Third, the closed-loop stability analysis was carried out by applying LaSalles' invariance Theorem. Besides, the domain of attraction of the closed-loop system can be enlarged, as we desired, for almost the upper half plane.

Finally, the construction of the proposed Lyapunov function, in conjunction with the closed-loop stability analysis, presented in this paper, turned out to be quite simple in contrasted with another works reported in ([14],[17],[19] and [18]). This is because, we only use the simple Lyapunov methodology.

References

[1] Furuta K. & K. Iwase, M, Swing-up time analysis of pendulum, Bulletin of the Polish Academy of Sciences Technical Sciences, **66**(219), pp. 957-984, 2004.

[2] Furuta, K., Control of pendulum: From super mechano-system to human adaptive mechatronics, In Proceedings of 42th IEEE Conference on Decision and Control, pp. 1498-1507, 2003.

[3] Jakubczyk, B. & Respondek, W.,On the linearization of control systems, Bull. Acad. Polon. Sci. Math., **28**, pp. 517-522, 1980.

[4] Fantoni, I. & Lozano, R., Nonlinear control for the underactuacted mecanical system, em Springer-Verlag, London, 2002.

[5] Angeli, D., Almost global stabilisation of the inverted pendulum via continuous state feedback, Automatica, **37**(5), pp. 1072-1088, 2001.

[6] Astrom, K. J. & Furuta, K., Swing up a pendulum by energy control, Automatica, **36**(5), 1072–1088, 2000.

[7] Spong, M. W. & Praly, L., Control of Underactuated mechanical systems using switching and saturation, Procc. of Block Island Workshop on Control Using Logic Based Switching, Springer-Verlag, 1996.

[8] Sira-Ramírez, H. & Agrawal, S. K., Differential flat systems, Marcel Dekker, N.Y., 2004.

[9] Chung, C.C. & Hauser, J., Nonlinear control of a swinging pendulum, Automatica, **36**, pp. 287-295, 2000.

[10] Fradkov, A. L., Swinging control of nonlinear oscillations, International Journal of Control, **31**, pp. 851-862, 1995.

[11] Shiriaev, A.S., Pogromsky, A., Ludvigsen, H. and Egeland, O., On global properties of passivity-based control of an inverted pendulum, International Journal of Robust and Nonlinear Contol, **10**, pp. 283-300, 2000.

[12] Lozano, R., Fantoni, I., & Block, D. J., Stabilization ofthe inverted pendulum around its homoclinic orbit, Systems &Control Letters, **40**, pp. 197-204, 2000.

[13] Spong, M. V., Energy Based Control of a class of Underactuated Mechanical Systems, IFAC World Congress, San Francisco CA., 1996.

[14] Teel, A. R., A nonlinear small gain theorem for the analysis of control system with saturation pendulum, IEEE Trans. on Automatic Control, **41**, pp. 1256-1270, 1996.
[15] Mazenc, F. & Praly, L., Adding integrations, satured controls, and stabilization for feedforward system, IEEE Transactions on Automatic Control, **41**, pp. 1559-1578, 1996.
[16] Chang, D., Leonard, N., & Marsden, J. E., Controlled lagrangians and the stabilization of mechanical systems II: Potential shaping, IEEE Trans. on Automatic Control, **46**, pp. 1556-1571, 2001.
[17] Chang, D., Bloch, A., Leonard, N., Marsden, J.E. & Woolsey, C., Equivalence of Controlled Lagrangian and Controlled Hamiltonian Systems, Control and the Calculus of Variations, **8**, pp. 393-422, 2002.
[18] Olfati-Saber, R., Fixed Point Controllers and Stabilization of the Cart-Pole and Rotating Pendulum, Proc. of the 38th IEEE Conf. on Decision and Control, Phoenix Az. (Dec. 1999), pp. 1174-1181, 1999.
[19] Lozano, R. & Dimogianopoulos, D., Stabilization of a chain of integrators with nonlinear perturbations: Application to the inverted pendulum, Proccedings of the 42nd IEEE Conference on Decision and Control, Maui Hawaii (Dec. 2002), pp. 5191-5196.
[20] Aracil, J. & Gordillo, F., El pendulo invertido : un desafion para el control no lineal, Revista Iberoamericana de Automática e Informática e Industrial, **2**, pp. 8-19, 2005.
[21] Jurdjevic, V., Quinn, J. P., Controllability and Stability, Journal of Differential Equations, **4**, pp. 381-389, 1978.
[22] Khalil, H. K., Nonlinear Systems, Prentice Hall, 2nd. Edition. N.J., 1996.

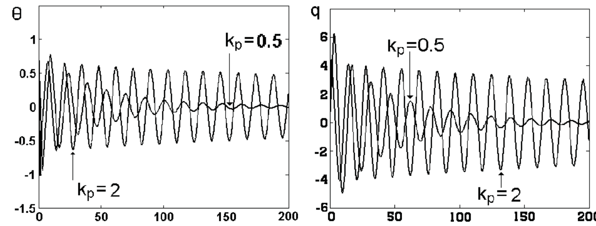

Fig. 1. Closed loop responses of the variables θ and q, for $k_p = 0.5$ and $k_p = 2$.

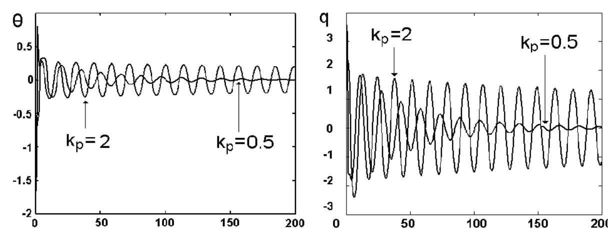

Fig. 2. Closed loop responses of the velocities $\dot{\theta}$ and $\dot{q}$, for $k_p = 0.5$ and $k_p = 2$.

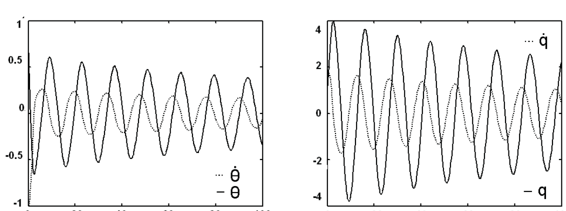

Fig. 3. Feedback controller performance when damping is considering in the direction of θ.