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Stabilization of the strongly damping inertia wheel pendulum by a nested saturation functions

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Abstract— **The stabilization of the strongly damping inertia wheel pendulum around its unstable equilibrium point is presented in this paper. The fact that this system can be rewritten approximately as a chain of integrators with and nonlinear perturbation suggests the use of a nested saturation based controller for making all state variables converge to zero. The proposed control strategy makes the closed-loop system globally asymptotically and locally exponentially stable around the unstable inverted vertical position, even when the physical damping is presented in the model.**

I. INTRODUCTION

Control of the under-actuated inertia wheel pendulum (**IWP**) has attracted the attention of several researchers as a test bed for the effectiveness of control design techniques proposed by control theory [1], [2], [3], [4]. This device is constituted by a physical pendulum with a rotating wheel at the end, that freely spins about an axis parallel to the pendulum axis of rotation. The disk is actuated by a DC-motor, while the pendulum is un-actuated. The coupling torque generated by the disk angular acceleration is used as the control of the system. Since the torque of the pendulum cannot be directly driven, this device is one example of an under-actuated mechanical system. That is, it has only one controller and two degrees of freedom. There are mainly two control maneuvers related with this system; the first is swinging the pendulum up from the hanging position to the upright vertical position; the second consists of stabilizing the **IWP** around its unstable equilibrium point, with the two angular positions of the system at the origin. According to this issue, we mention some of the most remarkable works related to this topic. In [1] a control energy approach based on a collocated partial feedback linearization and passivity of the resulting zero dynamics is used to solve the swinging and balance problem of the **IWP**; also, it is shown that this system is feedback linearizable with respect to some suitable output, under the asumptions that the pendulum angle lies in the upperhalf plane and the physical damping force is ignored. In [2], [3], the authors transform the dynamics of the original system into a cascade nonlinear system in a strict feedback form, by using some global transformations. Based on this, a globally asymptotically stabilization around its unstable top position is presented, by means of the standard backstepping procedure. In [4] two nonlinear swinging-up control strategies for solving the swinging and balance of the pendulum about its unstable inverted position are used. These approaches are based on the total energy stored in the system and guarantee convergence of the pendulum to a homoclinic orbit. In [5] the interconnection and damping assignment passivity based control is used for the asymptotic stabilization of the **IWP** around its top position. The obtained closed-loop system guarantees the asymptotically convergence of all the states, for all initial conditions, except for a set of zero measure. To do this, two necessary matching conditions have to be satisfied in order to obtain a stabilizing controller. In none of the mentioned works the undesirable effect of the damping force was considered.

In this paper we deal with the asymptotic stabilization of the under-actuated and strongly damping inertia wheel pendulum (**IWP**) around its unstable top position. Our main contribution is to present a suitable set of transformations, that allows us to accomplish a nested saturation based controller to bring the system to the unstable top position. That is, the obtained closed-loop system makes the strongly damping **IWP** globally asymptotically and locally exponentially stable at the origin, which coincides with the upright equilibrium point. As far as we know, the stabilization of the strongly damping **IWP** has not been thoroughly studied in the literature. In most cases, the problem has been solved designing a simple control law, made possible by ignoring the physical damping, in the hope that this force cannot affect the closed-loop stability. However, this is not always true, because, if the physical damping is presented, it tends to destabilize the closed-loop solution, especially in the top position (see [6] and [7]. This fact can be shown by a simple linearization around the origin. On the other hand, the construction of a candidate Lyapunov function for solving the stabilization of the strongly damping **IWP** turns out to be a very difficult task, since it is necessary to solve a set of partial differential equations. In general, the stabilization of the strongly damping **IWP** cannot be solved by means of control Lagrange or control Hamiltonian approaches, as was pointed out by [6], [8]. This is because the physical damping destroys the original structural properties of the Euler-Lagrange or the Hamiltonian systems. That is, it is not possible to find an additional term that compensates the

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undesirable damping effect by using the energy approach.

This paper is organized as follows. In Section 2 we present the dynamical model of the strongly damping **IWP** and the transformation of the original system in such a way that the obtained system looks like an integrator chain with an additional nonlinear perturbation. In Section 3 we develop the control strategy based on saturation functions. In Section 4 we present some computer simulations. Finally, we devote Section 5 to the conclusions.

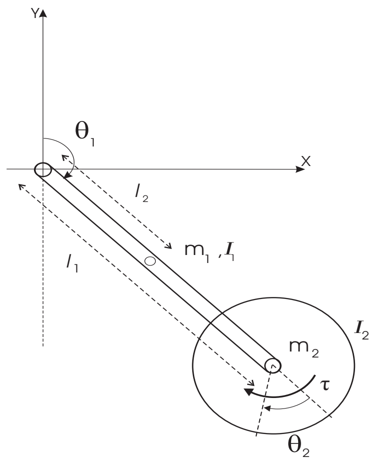

Fig. 1. The under-actuated inertia wheel pendulum (**IWP**)

II. THE INERTIA WHEEL PENDULUM

The **IWP**, depicted in Figure 1, is a planar inverted pendulum with a revolving wheel at the end. The wheel pendulum is actuated while the the pendulum join at the base is unactuated. The model of this system is described by [4] as

$$\begin{array}{l}\left(I_1+I_2+m_1l_2^2+m_2l_1^2\right)\ddot{\theta}_1+I_2\ddot{\theta}_2-\eta g\sin(\theta_1)+\delta_1\dot{\theta}_1=0\\ I_2\ddot{\theta}_1+I_2\ddot{\theta}_2+\delta_2\dot{\theta}_2=\tau_{1,2}\end{array} \tag{1}$$

where θ_1 is the pendulum angle, θ_2 is the disk angle and τ is the torque input applied on the disk. The remaining parameters are described in the following table:

$m_{1,2}$: pendulum and wheel masses.
$l_{1,2}$: pendulum length and distance to the center of the pendulum mass.
$I_{1,2}$: moments of pendulum and wheel inertia.
$\delta_{1,2}$: damping coefficient of the unactuated and the actuated coordinates.
and $\eta = m_1l_2 + m_2l_1$.

As can be seen, θ_1 and θ_2 are the non-actuated and the actuated system coordinates, respectively. That is because τ acts directly on the disk position. Now, to simplify the algebraic manipulation in the forthcoming developments, we rewrite system (1) as

$$\begin{array}{l}(1+\kappa_1)\ddot{\theta}_1+\ddot{\theta}_2-\kappa_2\sin(\theta_1)+\delta\dot{\theta}_1=0\\ \ddot{\theta}_1+\ddot{\theta}_2=v\end{array} \tag{2}$$

where

$$\begin{array}{c}\kappa_1=(I_1+m_1l_2^2+m_2l_1^2)/I_2; \quad \kappa_2=\eta g/I_2;\\ \tau=vI_2+\delta_2\dot{\theta}_2.\end{array} \tag{3}$$

The control objective is to find a continuous feedback v to bring the pendulum to the upright position with the disk position at the origin, even if the linear dissipation force is presented in the non-actuated coordinate.

Comment: When $\delta = 0$, it is well known how to solve the asymptotic stabilization of this system around its unstable top position using an energy based approach or a standard backstepping procedure (see [3], [4], [5]). However, if the physical damping is presented in the model, then the passivity and flatness properties are lost. That is, the closed-loop system may became unstable in the top position or the closed-loop solution may converge to other equilibrium point [7], [9]. This fact can be shown by simple linearization of the closed-loop position around the top position. On the other hand, it is not possible to directly accomplish a model matching approach to solve the asymptotic stabilization of this system [6], [8]. That is, matching controlled Lagrange and matching controlled Hamilton approaches are not suitable for solving the asymptotic stabilization when the physical damping is presented in the **IWP**. This occurs due to the fact that the damping force breaks the symmetric property of the original Euler-Lagrange or Hamilton systems. So in order to avoid this obstacle, we introduce a global transformation that allows us to express system (2) as a chain of integrators with an additional nonlinear perturbation. Thus, a nested-saturation controller can be used for rendering asymptotically stable the origin of the latter model.

A. Transforming the original structure of the system:

Let us introduce the following global change of coordinates:

$$\begin{array}{ll}z_1=(1+\kappa_1)\,\theta_1+\theta_2; & \dot{z}_1=p_1;\\ z_2=\theta_1; & \dot{z}_2=p_2,\end{array} \tag{4}$$

which leads to the following nonlinear system

$$\dot{x} = A_0x + \Delta(x) + b_0u \tag{5}$$

where

$$x=\begin{bmatrix} z_1\\ p_1\\ z_2\\ p_2\end{bmatrix}; \qquad A_o=\begin{bmatrix}0&1&0&0\\0&0&\kappa_2&-\delta\\0&0&0&1\\0&0&0&0\end{bmatrix};$$

$$\Delta(x)=\begin{bmatrix}0\\ \kappa_2\phi(z_2)\\ 0\\ 0\end{bmatrix}; \qquad b_0=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}.$$

The perturbation ϕ and the new controller u are defined as

$$\begin{array}{c}\phi(z_2)=\sin(z_2)-z_2;\\ u=\frac{1}{\kappa_1}(-v-\delta\dot{z}_2+\kappa_2\sin(z_2)).\end{array} \tag{6}$$

Note that the structure of the above system has a similar form to the four cascade integrators with an additional nonlinear perturbation. On the other hand, the new controller u directly acts on the non-actuated coordinate θ_1, which is the pendulum position. Contrarily, in system (2) the torque τ directly drives on the disk position. That is, we slightly change the structure of the original strongly damping **IWP**.

III. CONTROL STRATEGY

In this section we establish the framework of our control strategy. The idea behind it consists of bringing all the states very close to the origin, where the nonlinear perturbation can be bounded by the square of the pendulum angle position. Afterwards, the stability analysis can be carried out by using a robust linear system stability. In other words, we force the system states (5) to behave as an exponentially linear system with a very small perturbation. For this purpose we use a nested saturation based controller. This technique, introduced in [10], [11], has been used for the stabilization of linear cascade integrators and for controlling a wide class of under-actuated system [12], [13], [14], [15], [16].

So, we proceed as follows: first, a linear transformation is used to directly propose a stabilizing controller. Secondly, it is shown that the proposed controller guarantees the boundedness of all states. Finally, we show that the closed-loop system is locally exponentially asymptotically stable after some finite time.

Before developing the control strategy, we introduce some convenient definition:
We say that function $\sigma_m[s] : R \to R$ is a linear saturation function, if it satisfies

$$\sigma_m[s] = \begin{cases} s & if \ |s| \le m \\ m sign(s) & if \ |s| > m \end{cases}. \quad (7)$$

A nested based controller: Inspired in the previous work of [11], we propose a convenient linear transformation that allows us to propose, in a direct way, the necessary stabilizing controller u for the nonlinear system (5).

Let us first introduce a global linear transformation $q = Sx$, which is selected such that

$$SA_oS^{-1} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad and \quad Sb_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

After some simple algebraic manipulations, we can propose T, as

$$S = \begin{bmatrix} \frac{1}{\kappa_2} & \frac{\delta+3\kappa_2}{\kappa_2^2} & 3+\frac{\delta^2}{\kappa_2^2}+\frac{3\delta}{\kappa_2} & 1 \\ 0 & \frac{1}{k_2} & 2+\frac{\delta}{\kappa_2} & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (8)$$

So that system (5) can be rewritten as

$$\begin{aligned} \dot{q}_1 &= u + q_2 + q_3 + q_4 + \left(\frac{\delta+3\kappa_2}{\kappa_2}\right)\phi(q_3 - q_4) \\ \dot{q}_2 &= u + q_3 + q_4 + \phi(q_3 - q_4) \\ \dot{q}_3 &= u + q_4 \\ \dot{q}_4 &= u \end{aligned} \quad (9)$$

To stabilize the above system, we propose the following nested based controller u, as:

$$u = -q_4 - k\sigma_\alpha \left[\frac{1}{k}\left(q_3 + \sigma_\beta\left[q_2 + \sigma_\gamma\left[q_1\right]\right]\right)\right], \quad (10)$$

where k is a scaling positive constant.
Note that the closed-loop system, defined by equations (9) and (10), is globally Lipschitz. Consequently, all the states $\{q_i\}$[1] cannot have a finite time scape [17].

A. Boundedness of all states:

Now, we show in four simple steps that the closed-loop solution of the proposed closed-loop system, (9) and (10), ensures that all the states are bounded. Moreover the bound of each state directly depends on the designed parameters of the controller (10).

Step 1: To show that the state q_4 is bounded, we introduce an auxiliary function V_1, as:

$$V_1 = \frac{1}{2}q_4^2 \quad (11)$$

Differentiating (11) and using the fourth differential equation of (9), we have:

$$\dot{V}_1 = -q_4^2 - q_4 k\sigma_\alpha\left[q_3/k + \sigma_\beta\left[q_2 + \sigma_\gamma\left[q_1\right]\right]/k\right]$$

If $|q_4| > k\alpha$ then, from the above, we have that $\dot{V}_1 \le 0$. Therefore, there is a finite time T_1 after which, we have:

$$|q_4(t)| < k\alpha; \forall t > T_1.$$

That is, q_4 is bounded after some finite time T_1.

Step 2: We proceed to analyze the behavior of the state q_3. To do this, we introduce an auxiliary positive function V_2, as:

$$V_2 = \frac{1}{2}q_3^2. \quad (12)$$

Substituting the proposed controller (10) into the third differential equation of (9), we have:

$$\dot{q}_3 = -k\sigma_\alpha\left[\frac{1}{k}\left(q_3 + \sigma_\beta\left[q_2 + \sigma_\gamma\left[q_1\right]\right]\right)\right]. \quad (13)$$

Differentiating (12) and using (13), we obtain:

$$\dot{V}_2 = -q_3 k\sigma_\alpha\left[\frac{1}{k}\left(q_3 + \sigma_\beta\left[q_2 + \sigma_\gamma\left[q_1\right]\right]\right)\right],$$

where the control parameters, α and β, have to be selected such that $\alpha > 2\beta/k$. If $|q_3| > \beta$ then $\dot{V}_2 \le 0$. Therefore, there is a finite time $T_2 > T_1$, after which, we have:

$$|q_3(t)| < \beta; \forall t > T_2.$$

Consequently, q_3 is also bounded after some finite time T_2. On the other hand, defining the auxiliary variable

$$w = q_3 + \sigma_\beta\left[q_2 + \sigma_\gamma\left[q_1\right]\right],$$

[1] Here after, we use $\{x_i\}$ to denote $x = [x_1, x_2, x_3, x_4]^T$.

we have that $|w(t)| \leq |q_3(t)| + \beta$, for all $t > 0$, and, evidently, $|w(t)| < 2\beta$ after $t > T_2$. Since $\alpha > 2\beta/k$ clearly then

$$k\sigma_\alpha\left[\frac{1}{k}w\right] = w; t > T_2.$$

From the above, we have that control u turns out to be

$$u = -q_4 - q_3 - \sigma_\beta\left[q_2 + \sigma_\gamma\left[q_1\right]\right]; t > T_2. \quad (14)$$

Remark 1: *After $t > T_2$, we have that*

$$|q_3 - q_4| < \beta + k\alpha < \frac{\alpha k}{2} + \alpha k = \mu_k. \quad (15)$$

Because control parameter k can be selected as we desired, we can fix it as $\mu_k < 1$. Consequently, $|q_3(t) - q_4(t)| < \mu_k < 1$, for all $t > T_2$. Then, applying the following inequality

$$|\sin(x) - x| \leq |\sin(1) - 1|\, x^2 = \overline{\theta}x^2; \forall\, |x| < 1, \quad (16)$$

into the definition of function ϕ, we clearly have

$$|\phi(q_3 - q_4)| \leq \overline{\theta}\,|q_3 - q_4|^2 < \overline{\theta}\mu_k^2; \forall t > T_2. \quad (17)$$

Step 3: Substituting (14) into the second differential equation of (9), we obtain:

$$\dot{q}_2 = -\sigma_\beta\left[q_2 + \sigma_\gamma\left[q_1\right]\right] + \phi(q_3 - q_4); t > T_2, \quad (18)$$

where β and γ must satisfy $\beta > 2\gamma + \overline{\theta}\mu_k^2$. In order to show that q_2 is bounded, we need to introduce the auxiliary function V_3, as:

$$V_3 = \frac{1}{2}q_2^2. \quad (19)$$

Differentiating (19) and using (18), it produces:

$$\dot{V}_3 = -q_2\left(\sigma_\beta\left[q_2 + \sigma_\gamma\left[q_1\right]\right] + \phi(q_3 - q_4)\right).$$

Obviously, if $|q_2| > \gamma + \overline{\theta}\mu_k^2$ then $\dot{V}_3 \leq 0$ and there is a finite time $T_3 > T_2$, after which, we have:

$$|q_2(t)| < \gamma + \overline{\theta}\mu_k^2; \forall t > T_3.$$

Consequently, q_2 is bounded and control u turns out to be

$$u = -q_4 - q_3 - q_2 - \sigma_\gamma\left[q_1\right]; \forall t > T_3. \quad (20)$$

Step 4: Substituting equation (20) into the first differential equation of (9), we have:

$$\dot{q}_1 = -\sigma_\gamma\left[q_1\right] + \left(\frac{\delta}{\kappa_2} + 3\right)\phi(q_3 - q_4); \forall t > T_3. \quad (21)$$

To show that q_1 is bounded, we define the auxiliary positive function V_4, as:

$$V_4 = \frac{1}{2}q_1^2. \quad (22)$$

Differentiating (22) and using (21), we have:

$$\dot{V}_4 = -q_1\left(\sigma_\gamma\left[q_1\right] + \left(\frac{\delta}{\kappa_2} + 3\right)\phi(q_3 - q_4)\right). \quad (23)$$

Where γ must be selected such that $\gamma > (\delta/\kappa_2 + 3)\,\overline{\theta}\mu_k^2$. If $|q_1| > (\delta/\kappa_2 + 3)\overline{\theta}\mu_k^2$ then $V_4 \leq 0$ and, there is $T_4 > T_3$ such that

$$|q_1(t)| < \left(\frac{\delta}{\kappa_2} + 3\right)\overline{\theta}\mu_k^2; \forall t > T_4.$$

That is, all the states $\{q_i\}$ are bounded after $t > T_4$.

We summarize this section with the following Lemma that allows to compute the set of control parameters $\{\alpha, \beta, \gamma, \mu_k\}$, needed to guarantee the boundedness of all states.

Lemma 1: *Given the positive constants δ and κ_2 and fixing $\mu_k \in (0, 1)$[2], the following inequalities*

$$\alpha > 2\beta; \quad \beta > 2\gamma + \overline{\theta}\mu_k^2; \quad \gamma > \left(\tfrac{\delta}{\kappa_2} + 3\right)\overline{\theta}\mu_k^2, \quad (24)$$

are fulfilled, provided that parameters γ, β and α are selected as:

$$\begin{array}{l}\gamma = \lambda\overline{\theta}\mu_k^2\left(\tfrac{\delta}{\kappa_2} + 3\right); \quad \beta = \lambda\overline{\theta}\mu_k^2(7 + \tfrac{2\delta}{\kappa_2});\\ \alpha = 2\lambda\overline{\theta}\mu_k^2(7 + \tfrac{2\delta}{\kappa_2}),\end{array} \quad (25)$$

where $\lambda > 1$.

B. Convergence of all states to zero

We will prove that the closed-loop system given by (9) and (14) is asymptotically stable and locally exponentially stable, under the assumption of the Lemma 1. That is, if the control parameters k, γ and β are selected according to Lemma 1, then the vector state q converges to zero.

We must note that after $t > T_4$, the control law is no longer saturated, that is,

$$u = -q_1 - q_2 - q_3 - q_4,$$

and the closed-loop system turns out to be

$$\begin{array}{l}\dot{q}_1 = -q_1 + \left(\tfrac{\delta}{\kappa_2} + 3\right)\phi(q_3 - q_4),\\ \dot{q}_2 = -q_1 - q_2 + \phi(q_3 - q_4),\\ \dot{q}_3 = -q_1 - q_2 - q_3,\\ \dot{q}_4 = -q_1 - q_2 - q_3 - q_4,\end{array} \quad (26)$$

Now, in order to demonstrate the convergence of all the states to zero, we use the following Lyapunov function

$$V = \frac{1}{2}q^T q, \quad (27)$$

Differentiating (26) along the trajectories of (26), we obtain

$$\dot{V} = -q^T M q + \left(q_2 + (\frac{\delta}{\kappa_2} + 3)q_1\right)\phi(q_3 - q_4) \quad (28)$$

where M is given by

$$M = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}.$$

[2]Recalling that $k = 2\mu_k/3\alpha$.

Note that $\lambda_{\min}\{M\} = 1/2$ and therefore $M > 0$. Recalling that after $t > T_4$, the states q_1, q_2 and function ϕ satisfy the following inequalities

$$|q_1| < \overline{\theta}\mu_k^2 \left(\tfrac{\delta}{\kappa_2} + 3\right); \qquad |q_2| < \overline{\theta}\mu_k^2 \left(\tfrac{\delta}{\kappa_2} + 4\right);$$
$$|\phi(q_3 - q_4)| < \overline{\theta}(q_3 - q_4)^2.$$

Substituting the above inequalities into the second term of relation (28), we have after using the triangle inequality that

$$|(\delta/\kappa_2 + 3)q_1 + q_2|\,|\phi(q_3 - q_4)| < \overline{K}(q_3 - q_4)^2 \le 2\overline{K}(q_3^2 + q_4^2); \quad (29)$$

where

$$\overline{K} = \overline{\theta}\mu_k^2 \left(\delta/\kappa_2 + 3\right)^2 + \overline{\theta}\mu_k^2 \left(\delta/\kappa_2 + 4\right). \quad (30)$$

Note that $\overline{K}$ can be as small as needed, because $\mu_k \in (0, 1)$ is selected as desired.

Therefore applying the inequality (29) into the time derivative of V (28), we evidently have

$$\dot{V} < -\frac{1}{2}\left[q_1^2 + q_2^2 + q_3^2 + q_4^2\right] + 2\overline{K}(q_3^2 + q_4^2).$$

If we force the positive constant $\overline{K} < 1/4$, then $\dot{V} < 0$, for all $q \neq 0$. That is, if $\overline{K}$ is selected such that $\overline{K} < 1/4$, then vector state q locally exponentially converges to zero

From the above discussion, we have:

Proposition 1: *Consider the strongly damping IWP system, as described in (2), in closed-loop with*

$$v = \kappa_1 q_4 + \kappa_1 k \sigma_\alpha \left[\tfrac{1}{k}\left(q_3 + \sigma_\beta\left[q_2 + \sigma_\gamma\left[q_1\right]\right]\right)\right]$$
$$-\delta\dot{\theta}_1 + \kappa_2 \sin(\theta_1),$$

where q is obtained via $\{q_i\} = S\{x_i\}$, where matrix S is given in (8), *and the set of x_i are defined, as*

$$x_1 = (1 + \kappa_1)\,\theta_1 + \theta_2, \quad x_2 = \dot{x_1}, \quad x_3 = \theta_1, \quad x_4 = \dot{\theta}_1.$$

Under the assumption that the control parameters $\{\alpha$, β, γ, $k\}$ are selected according to Lemma 1. Then, the closed-loop system is globally asymptotically and locally exponentially stable, provided that $\overline{K} < 1/4$ where the estimated $\overline{K}$ is given in (30).

Comment:The torque τ which is related with the input control is partiality bounded. Due to the fact the proposed control included linear terms of the angular velocities as we can see in (3) and the **Proposition 1** .

We omit to present a digital simulation because we do not have enough space.

IV. SIMULATIONS RESULTS

In order to test the performance of the obtained control law we carried out two numerical simulations using the MATLAB^{TM} system. The **IWP** physical parameters were set as $m_1 = 0.01$kg, $m_2 = 0.1$kg, $l_1 = 0.5$m, $l_2 = 0.35$m, $I_1 = 3.5 \times 10^{-3}$kgm^2 and $I_2 = 1.4 \times 10^{-2}$kgm^2 for both simulations. However, in the first experiment the additional linear damping term was set as $\delta_1 = 0.5$, while in the second it was set as $\delta_1 = 0.05$.

In the experiment, we transferred the pendulum position from the lower stable equilibrium point to the upright unstable equilibrium point. That is, we fixed the initial conditions as $\theta_1(0) = \pi$[rad], $\theta_2(0) = 0$, $\dot{\theta}_1(0) = 0$ and $\dot{\theta}_2(0) = 0$. The structural parameters, defined in (3), are given by $\kappa_1 = 0.80357$, $\kappa_2 = 36.75$ and $\delta = 35.71$. The control parameters, designed according to **Lemma 1** and **Proposition 1**, are fixed as $\alpha = 0.23$, $\beta = 0.1171$, $\gamma = 0.05195$ and $\mu_k = 0.33$. Figure 2 shows the close-loop system response. As we can see from this figure, the state θ_1 converges to zero faster than the state θ_2. This means that, while the wheel angular position is decreased, the pendulum angular position moves to within a very small vicinity of the origin. Once the pendulum is very close to the origin, the control action starts to regulate the wheel dynamics. In other words, firstly the control action brings the pendulum into a small vicinity of zero, while the wheel angular position decreases until it reaches its minimum; secondly the control, little by little, brings the wheel angular position to the origin. Note that this particular control maneuver cannot be carried out if we use energy based control methods, because the rest lower point is not inside of the stability domain of these kinds of control strategies (see for example, [4] and [5]). In the second experiment we took the same initial conditions and the same physical parameter as in the first experiment, except for the structural parameter related with the damping which was set as $\delta = 3.5$ and all the control parameters were selected as before, except μ_k that was set as $\mu_k = 0.\dot{4}$. Figure 3 shows the closed.loop response of all states. As can be seen from this figure, θ_1 and $\dot{\theta}_1$ have a similar behaviour as in the first experiment, but the time response was considerably improved. However, the numerical values of states θ_2 and $\dot{\theta}_2$ are smaller than the ones in the first experiment. This was expected, because the needed compensation for the linear damping is directly proportional to the number of spins and the angular velocity of the wheel. In other words, if the undesirable damping effect is incremented, then the wheel control action has to be stronger to accomplish the control maneuver. Obviously, the opposite is also true.

V. CONCLUSIONS

A nested saturation based controller allows us to solve a number of interesting non-linear control stabilization problems. This powerful technique allows us to propose the necessary stabilizing controller without the necessity of having a candidate Lyapunov function for the whole system. In this case, we have applied this technique for the stabilization of the strongly damping **IWP** around its upright equilibrium point. Intuitively, the proposed controller consist of two stages. Firstly, we bring the pendulum close enough to the vertical unstable equilibrium point; secondly, we start to regulate the wheel angle position, until all the system states are confined inside a very small vicinity of zero, which can be estimated and contracted as desired. Afterwards, the closed-loop system behaves as an exponential linear system with a small perturbation, where it can be bounded by the

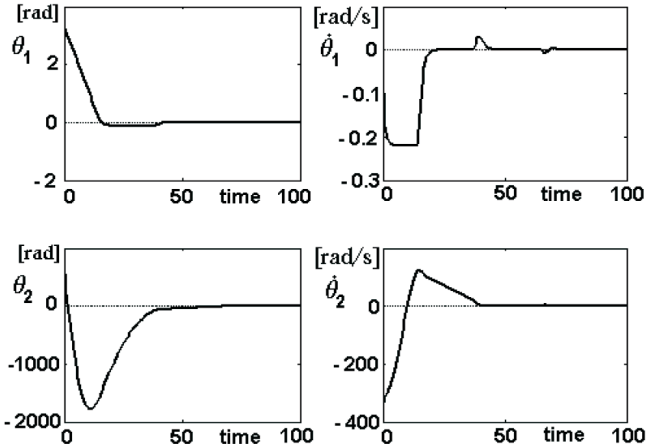

Fig. 2. Closed-loop response of all states, when $\delta = 35.71$.

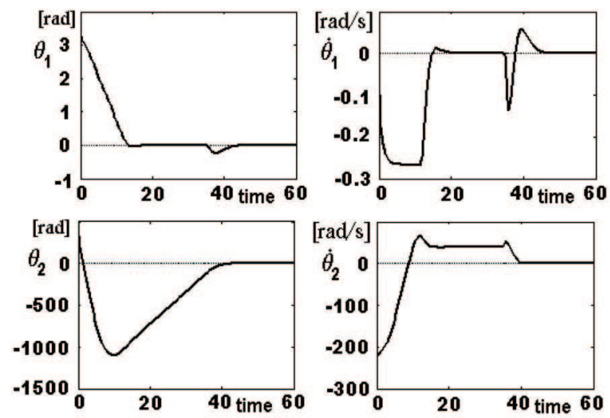

Fig. 3. Closed-loop response of all states, when $\delta = 3.5$.

square of the pendulum angular position. The latter closed-loop system, which is almost a linear system, turns out to be asymptotically stable at the origin. Convergence to zero of the closed-loop system is assured by using a simple Lyapunov method. We emphasize that the stabilization of the strongly **IWP** cannot be assured if we use the energy shaping and passivity-based controllers, as was pointed out by [6], [8].

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