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ISS CONTROL LAWS FOR STABILIZABLE RETARDED SYSTEMS BY MEANS OF THE LIAPUNOV-RAZUMIKHIN METHODOLOGY

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Abstract A disturbance adding to the control law is a typical situation in practice because of actuator errors. In this paper, state feedback control laws which provide input-to-state stability of the closed loop system with respect to a disturbance adding to the control law are investigated for state feedback stabilizable (in the case of disturbance equal to zero) retarded nonlinear systems. The formulas for the input-to-state stabilizing state feedback control law are provided by employing the Liapunov-Razumikhin methodology. An example taken from the past literature is investigated in details, showing the effectiveness and the applicability of the proposed control design.

Keywords: Input-to-State Stabilizability, Retarded Nonlinear Systems, Liapunov-Krasovskii Methodology, Liapunov-Razumikhin Methodology.

INTRODUCTION

In 1989 Sontag showed in the paper [23] that nonlinear systems which are (smooth) feedback stabilizable, are also (smooth) input-to-state stabilizable with respect to disturbances adding to the control input. As well known, those disturbances are very frequent in practice, because of actuator errors. Many contributions concerning the state feedback stabilization and the input-output state feedback linearization of nonlinear retarded systems can be found in the literature (see, for instance, [3,6,8,12,14,15,17,18,26,30]). Liapunov-Razumikhin and Liapunov-Krasovskii methodologies for the input-to-state stability of retarded nonlinear systems have been studied in [25] and in [10,19,28], respectively. As far as the input-to-state stabilizability of stabilizable retarded nonlinear systems is concerned, a contribution is given in [27], where, besides the main results dealing with the relationship between the input-to-state stability and the exponential stability in the unforced case, the input-to-state stabilizability of retarded nonlinear systems which are transformable by a state feedback control law into a linear, delay-free, exponentially stable system is considered, and the formula for the input-to-state stabilizing state feedback control law is provided.

In this paper a general theory for the input-to-state stabilizability of state feedback stabilizable retarded nonlinear systems is provided. It is proved that a state feedback stabilizable (in the case of disturbance equal to zero) retarded nonlinear system admits an input-to-state stabilizable state feedback control law (i.e. admits a state feedback control law such that the closed loop system is input-to-state stable with respect to a disturbance adding to the control law), provided that the control system obtained by closing the loop with the stabilizing (in the case of disturbance equal to zero) state feedback control law admits a Liapunov-Razumikhin function, with a suitable property, by which its global asymptotic stability can be proved. The formula for such input-to-state stabilizing state feedback control law is provided. The delays are arbitrary but it is assumed that they are known.

The effectiveness of the methodology here proposed is shown in details on an example of stabilizable (in the case of disturbance equal to zero) retarded nonlinear system taken from the past literature.

Notations

R denotes the set of real numbers, $R^\star$ denotes the extended real line $[-\infty, +\infty]$, R^+ denotes the set of non negative reals $[0, +\infty)$. The symbol $|\cdot|$ stands for the Euclidean norm of a real vector, or the induced Euclidean norm of a matrix. The essential supremum norm of an essentially bounded function is indicated with the symbol $\|\cdot\|_\infty$. A function $v : R^+ \to R^m$, m positive integer, is said to be *essentially bounded* if $ess\sup_{t\geq 0} |v(t)| < \infty$. For given times $0 \leq T_1 < T_2$, we indicate with $v_{[T_1,T_2)} : R^+ \to R^m$ the function given by $v_{[T_1,T_2)}(t) = v(t)$ for all $t \in [T_1, T_2)$ and $= 0$ elsewhere. An input v is said to be *locally essentially bounded* if, for any $T > 0$, $v_{[0,T)}$ is essentially bounded. For a positive real Δ, $C([-\Delta, 0]; R^n)$ denotes the space of the continuous functions mapping $[-\Delta, 0]$ into R^n, n positive integer. For positive integers m, n, I_m denotes the identity matrix in $R^{m\times m}$, $0_{m,n}$ denotes a matrix of zeros in $R^{m\times n}$. A functional $F : C([-\Delta, 0]; R^n) \to R^{m\times p}$, m, n, p positive integers, is said to be completely continuous if it is continuous and maps closed bounded sets of $C([-\Delta, 0]; R^n)$ into bounded sets of $R^{m\times p}$. Let us here recall that a function $\gamma : R^+ \to R^+$ is: positive definite

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if it is continuous, zero at zero and $\gamma(s) > 0$ for all $s > 0$; of class $\mathcal{G}$ if it is continuous, zero at zero, and nondecreasing; of class $\mathcal{K}$ if it is of class $\mathcal{G}$ and strictly increasing; of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and it is unbounded; of class $\mathcal{L}$ if it monotonically decreases to zero as its argument tends to $+\infty$. A function $\beta : R^+ \times R^+ \to R^+$ is of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each $t \geq 0$ and $\beta(s, \cdot)$ is of class $\mathcal{L}$ for each $s \geq 0$. As usual, ISS stands for both input-to-state stable and input-to-state stability.

PRELIMINARIES

In this section, for the reader's convenience, some previously published results which are fundamental for the understanding of the novel results which will be provided in next sections are briefly reported, with some slight modifications for the purposes of this paper. Let us consider the following retarded nonlinear system

$$\begin{aligned} \dot{x}(t) &= f(x_t) + g(x_t)v(t), \qquad t \geq 0, \qquad a.e., \\ x(\tau) &= \xi_0(\tau), \qquad \tau \in [-\Delta, 0], \end{aligned} \tag{1}$$

where $x(t) \in R^n$, $v(t) \in R^m$ is the input function, measurable and locally essentially bounded, for $t \geq 0$ $x_t : [-\Delta, 0] \to R^n$ is the standard function (see Section 2.1, pp. 38 in [5]) given by $x_t(\tau) = x(t + \tau)$, Δ is the maximum involved delay, f is a locally Lipschitz, completely continuous functional mapping $C([-\Delta, 0]; R^n)$ into R^n, g is a locally Lipschitz, completely continuous functional mapping $C([-\Delta, 0]; R^n)$ into $R^{n\times m}$, $\xi_0 \in C([-\Delta, 0]; R^n)$. It is here supposed that $f(0) = 0$, thus ensuring that $x(t) = 0$ is the trivial solution for the unforced system $\dot{x}(t) = f(x_t)$ with zero initial conditions. Multiple discrete non-commensurate as well as distributed delays can appear in (1). In the following, the continuity of a functional $V : C([-\Delta, 0]; R^n) \to R^+$ is intended with respect to the supremum norm. Given a locally Lipschitz continuous functional $V : C([-\Delta, 0]; R^n) \to R^+$, the upper right-hand derivative D^+V of the functional V is given by (see [1], Definition 4.2.4, pp. 258, see comments before Theorem 2 in [20])

$$D^+V(\phi, v) = \limsup_{h\to 0^+} \frac{1}{h} \left(V(\phi_h) - V(\phi) \right), \tag{2}$$

where $\phi_h \in C([-\Delta, 0]; R^n)$ is given by

$$\phi_h(\theta) = \begin{cases} \phi(\theta + h), & \theta \in [-\Delta, -h), \\ \phi(0) + (f(\phi) + g(\phi)v)(\theta + h), & \theta \in [-h, 0] \end{cases} \tag{3}$$

Definition 1: ([23,19]) *The system (1) is said to be input-to-state stable (ISS) with respect to v, if there exist a $\mathcal{KL}$ function β and a $\mathcal{K}$ function γ such that, for any initial state ξ_0 and any measurable, locally essentially bounded input v, the solution exists for all $t \geq 0$ and furthermore it satisfies*

$$|x(t)| \leq \beta \left(\|\xi_0\|_\infty, t \right) + \gamma \left(\|v_{[0,t)}\|_\infty \right) \tag{4}$$

Theorem 2: ([25]) *If there exist a locally Lipschitz function $\bar{V} : R^n \to R^+$, functions α_1, α_2 of class $\mathcal{K}_\infty$, a function α_3 of class $\mathcal{K}$, functions α_4, ρ of class $\mathcal{G}$, $\alpha_4(s) < s$ for $s > 0$, such that:*

H_1) $\alpha_1(|x|) \leq \bar{V}(x) \leq \alpha_2(|x|)$, $\forall x \in R^n$;

H_2) $D^+V(\phi, v) \leq -\alpha_3(|\phi(0)|)$,
$\forall \phi \in C([-\Delta, 0]; R^n), v \in R^m$:
$\bar{V}(\phi(0)) \geq max\{\alpha_4(\sup_{\tau\in[-\Delta,0]} \bar{V}(\phi(\tau))), \rho(v)\}$,
where $V(\phi) = \bar{V}(\phi(0))$ and D^+V is computed as in (2);

then, the system (1) is input-to-state stable with $\gamma = \alpha_1^{-1} \circ \rho$.

Remark 3: With respect to the published literature: the hypotheses in Theorem 2 are given on the state space domain by using the Definition (2) and the results in [20,21], while in the original version are given on the time domain. •

MAIN RESULTS

Let us consider now the following retarded nonlinear system, corresponding to (1) when the input v is given as the sum of the control input and of the disturbance:

$$\begin{aligned} \dot{x}(t) &= f(x_t) + g(x_t)(u(t) + d(t)), \qquad t \geq 0, \qquad a.e., \\ x(\tau) &= \xi_0(\tau), \qquad \tau \in [-\Delta, 0], \end{aligned} \tag{5}$$

where $u(t) \in R^m$ is the control input, $d(t) \in R^m$ is the disturbance, measurable and locally essentially bounded.

Theorem 4: *Let there exists a locally Lipschitz, completely continuous functional $k : C([-\Delta, 0]; R^n) \to R^m$, such that the closed loop system (5) with $u(t) = k(x_t)$, and no disturbance, described by the equations*

$$\dot{x}(t) = f(x_t) + g(x_t)k(x_t), \tag{6}$$

is globally asymptotically stable. Let, for $\phi \in C([-\Delta, 0]; R^n)$, $h \in (0, \Delta)$,

$$\phi_h^a(\theta) = \begin{cases} \phi(\theta + h), & \theta \in [-\Delta, -h), \\ \phi(0) + (\theta + h) \left(f(\phi) + g(\phi)k(\phi) \right), & \theta \in [-h, 0], \end{cases}$$

$$\delta_h^g(\theta) = \begin{cases} 0_{n\times m}, & \theta \in [-\Delta, -h), \\ (\theta + h)g(\phi), & \theta \in [-h, 0] \end{cases} \tag{7}$$

Let there exist a locally Lipschitz continuous function $\bar{V}_0 : R^n \to R^+$, functions α_1, α_2, α_3 of class $\mathcal{K}_\infty$, functions α_4, ρ of class $\mathcal{G}$, $\alpha_4(s) < s$ for $s > 0$, a locally Lipschitz, completely continuous functional $r : C([-\Delta, 0]; R^n) \to R^m$ such that, defining $V_0(\phi) = \bar{V}_0(\phi(0))$, $\phi \in C([-\Delta, 0]; R^n)$:

i) $\alpha_1(|x|) \leq \bar{V}_0(x) \leq \alpha_2(|x|), \quad \forall x \in R^n$*;*

ii) the functional $D_a^+ V_0 : C([-\Delta, 0]; R^n) \to R^\star$, defined as

$$D_a^+ V_0(\phi) = \limsup_{h \to 0^+} \frac{V_0(\phi_h^a) - V_0(\phi)}{h}, \tag{8}$$

is locally Lipschitz, completely continuous and, moreover,

$$\begin{gathered} D_a^+ V_0(\phi) \leq -\alpha_3(|\phi(0)|), \\ \forall \phi \in C([-\Delta, 0]; R^n) : \\ \bar{V}_0(\phi(0)) \geq \alpha_4 \left(\sup_{\tau \in [-\Delta, 0]} \bar{V}_0(\phi(\tau)) \right); \end{gathered} \tag{9}$$

iii)

$$\begin{gathered} \limsup_{h \to 0^+} \frac{V_0(\phi_h^a + \delta_h^g M) - V_0(\phi_h^a)}{h} = r^T(\phi) M, \\ \forall \phi \in C([-\Delta, 0]; R^n), \ M \in R^m. \end{gathered} \tag{10}$$

Then, the feedback control law

$$u(t) = k(x_t) + p(x_t), \tag{11}$$

with $p : C([-\Delta, 0]; R^n) \to R^m$ defined as

$$p(\phi) = \frac{1}{2} D_a^+ V_0(\phi) r(\phi), \tag{12}$$

is such that the closed loop system (5), (11), described by the following equations

$$\dot{x}(t) = f(x_t) + g(x_t)k(x_t) + g(x_t)p(x_t) + g(x_t)d(t), \tag{13}$$

is input-to-state stable with respect to the measurable and locally essentially bounded disturbance $d(t)$.

Proof. Let us apply Theorem 2 with $\bar{V}_0$ as Razumikhin function. For all $\phi \in C([-\Delta, 0]; R^n)$, $d \in R^m$ such that

$$\begin{gathered} \bar{V}_0(\phi(0)) \geq \\ max \left\{ \alpha_4 \left(\sup_{\tau \in [-\Delta, 0]} \bar{V}_0(\phi(\tau)) \right), \alpha_2 \circ \alpha_3^{-1}(|d|) \right\}, \end{gathered} \tag{14}$$

the following equalities/inequalities hold

$$\begin{aligned} & D^+ V_0(\phi, d) = \limsup_{h \to 0^+} \frac{V_0(\phi_h^a + \delta_h^g (p(\phi) + d)) - V_0(\phi)}{h} = \\ & \limsup_{h \to 0^+} \frac{V_0(\phi_h^a + \delta_h^g (p(\phi) + d)) - V_0(\phi_h^a) + V_0(\phi_h^a) - V_0(\phi)}{h} \\ & \leq \limsup_{h \to 0^+} \frac{V_0(\phi_h^a + \delta_h^g (p(\phi) + d)) - V_0(\phi_h^a)}{h} \\ & + \limsup_{h \to 0^+} \frac{V_0(\phi_h^a) - V_0(\phi)}{h} \leq \\ & - \alpha_3(|\phi(0)|) + r^T(\phi)(p(\phi) + d) = \\ & - \alpha_3(|\phi(0)|) + \frac{1}{2} r^T(\phi) D_a^+ V_0(\phi) r(\phi) + r^T(\phi) d \end{aligned} \tag{15}$$

Taking into account, by condition i), that $\bar{V}_0(\phi(0)) \geq \alpha_2 \circ \alpha_3^{-1}(|d|) \Rightarrow |d| \leq \alpha_3(|\phi(0)|)$, the following inequalities follow

$$\begin{aligned} & D^+ V_0(\phi, d) \leq -\alpha_3(|\phi(0)|) - \frac{1}{2} \alpha_3(|\phi(0)|) r^T(\phi) r(\phi) + \\ & \frac{1}{2} |r(\phi)|^2 \alpha_3(|\phi(0)|) + \frac{1}{2} \alpha_3(|\phi(0)|) \leq -\frac{1}{2} \alpha_3(|\phi(0)|) \end{aligned} \tag{16}$$

Therefore, from Theorem 2, the input-to-state stability of the closed loop system (*13*) with respect to the disturbance $d(t)$ adding to the control law is proved. □

Remark 5: Note that ϕ_h^a is equivalent to ϕ_h in (3) related to the closed loop asymptotically stable system (with no disturbance) described by the equations

$$\dot{x}(t) = f(x_t) + g(x_t)k(x_t) \tag{17}$$

and $\phi_h^a + \delta_h^g M$ is equivalent to ϕ_h in (3) related to the closed loop system with disturbance equal to M, described by the equations

$$\dot{x}(t) = f(x_t) + g(x_t)k(x_t) + g(x_t)M \tag{18}$$

Therefore the term $V(\phi_h^a + \delta_h^g M) - V(\phi_h^a)$ describes a difference between the estimations of the functional V evaluated at the solution of system (17) and at the solution of system (18), respectively. The limit for $h \to 0$ of such difference divided by h turns to be a key point for the stated results (see other papers of the author in this conference concerning the Liapunov-Krasovskii methodology). •

Remark 6: Conditions i), ii) in Theorem 4 are almost standard in the Liapunov-Razumikhin methodology (the main difference is that here α_3 is required to be a class $\mathcal{K}_\infty$ function instead of a positive definite one, see Theorem 1.4, pp. 14 in [4]) for studying the global asymptotic stability of retarded unforced systems (in this case, of the unforced closed loop system (*6*)). The new, key point is given by condition iii) (see (*10*)). The existence of the locally Lipschitz, completely continuous functional $r(\phi)$ is guaranteed for a very large class of Liapunov-Razumikhin functions. For instance, $r(\phi)$ exists for all functions $\bar{V}_0 : R^n \to R^+$ such that, for any given $x, y \in R^n$,

$$\bar{V}_0(x+y) = \bar{V}_0(x) + \bar{V}_1^T(x)y + o(|y|), \tag{19}$$

where $\bar{V}_1 : R^n \to R^n$ is a suitable locally Lipschitz continuous function. In this case

$$r^T(\phi) = \bar{V}_1^T(\phi(0))g(\phi) \tag{20}$$

For, the following equalities hold for (*10*)

$$\begin{aligned}
&\limsup_{h\to 0^+} \frac{V_0(\phi_h^a + \delta_h^g M) - V_0(\phi_h^a)}{h} = \\
&\limsup_{h\to 0^+} \frac{1}{h}\bar{V}_0(\phi(0) + h(f(\phi) + g(\phi)k(\phi)) + hg(\phi)M) - \\
&\frac{1}{h}\bar{V}_0(\phi(0) + h(f(\phi) + g(\phi)k(\phi)) = \\
&\limsup_{h\to 0^+} \frac{1}{h}\bar{V}_0(\phi(0)) + \\
&+ \frac{1}{h}\bar{V}_1^T(\phi(0))(h(f(\phi) + g(\phi)k(\phi)) + hg(\phi)M) + \\
&+ \frac{o\left(|h(f(\phi) + g(\phi)k(\phi)) + hg(\phi)M|\right)}{h} - \\
&- \frac{1}{h}\left(\bar{V}_0(\phi(0)) + \bar{V}_1^T(\phi(0))h(f(\phi) + g(\phi)k(\phi))\right) - \\
&\frac{1}{h}o(|h(f(\phi) + g(\phi)k(\phi))|) \\
&= \bar{V}_1^T(\phi(0))g(\phi)M
\end{aligned} \tag{21}$$

•

ILLUSTRATIVE EXAMPLE

Let us consider the following system, which is a time-invariant case with no uncertainties of the example (34) in [12]

$$\begin{aligned}
&\dot{x}(t) = \\
&\begin{bmatrix} -2x_1(t) + 2x_1(t)x_2^2(t) + 2x_1(t)x_2(t)\sqrt{|x_1(t)x_2(t)|} \\ -2x_2(t) - x_1^2(t)x_2(t) - x_1^2(t)\sqrt{|x_1(t)x_2(t)|} \end{bmatrix} + \\
&\qquad + \begin{bmatrix} x_2(t-\Delta) \\ -x_1^2(t) \end{bmatrix}(4 + |x_1(t-\Delta)|)(u(t) + d(t))
\end{aligned} \tag{22}$$

$d(t)$ is a measurable locally essentially bounded disturbance adding to the control law. Such disturbance is not considered in [12]. The following state feedback control law (a version of (35) in [12] for the time invariant case here considered)

$$\begin{aligned}
&u(t) = k(x_t) = -2\cdot \\
&\frac{4x_1(t)x_2(t-\Delta) - 8x_1^2(t)x_2(t)}{(4 + |x_1(t-\Delta)|)(|4x_1(t)x_2(t-\Delta) - 8x_1^2(t)x_2(t)| + 3)}
\end{aligned} \tag{23}$$

is such that the closed loop system (22), (23) is globally asymptotically stable. This fact is proven in [12] for a larger class of systems which includes system (22). Let us apply Theorem 4 in order to find out an input-to-state stabilizing state feedback control law. The quadratic functional proposed in [12] $\bar{V}_0(x) = x^T \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} x$ can be used here too. The following equality holds ($V_0(\phi) = \bar{V}_0(\phi(0))$), for $\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \in C([-\Delta, 0]; R^2)$,

$$\begin{aligned}
&D_a^+ V_0(\phi) = 2\phi(0)^T \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \cdot \\
&\begin{bmatrix} -2\phi_1(0) + 2\phi_1(0)\phi_2^2(0) + 2\phi_1(0)\phi_2(0)\sqrt{|\phi_1(0)\phi_2(0)|} \\ -2\phi_2(0) - \phi_1^2(0)\phi_2(0) - \phi_1^2(0)\sqrt{|\phi_1(0)\phi_2(0)|} \end{bmatrix} \\
&+ 2\phi^T(0) \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \cdot \\
&\qquad \begin{bmatrix} -2\phi_2(-\Delta)\frac{4\phi_1(0)\phi_2(-\Delta) - 8\phi_1^2(0)\phi_2(0)}{(|4\phi_1(0)\phi_2(-\Delta) - 8\phi_1^2(0)\phi_2(0)| + 3)} \\ 2\phi_1^2(0)\frac{4\phi_1(0)\phi_2(-\Delta) - 8\phi_1^2(0)\phi_2(0)}{(|4\phi_1(0)\phi_2(-\Delta) - 8\phi_1^2(0)\phi_2(0)| + 3)} \end{bmatrix}
\end{aligned} \tag{24}$$

and, from computations in [12],

$$\begin{aligned}
&D_a^+ V_0(\phi) \le -8|\phi(0)|^2 - \\
&- 2\frac{(4\phi_1(0)\phi_2(-\Delta) - 8\phi_1^2(0)\phi_2(0))^2}{(|4\phi_1(0)\phi_2(-\Delta) - 8\phi_1^2(0)\phi_2(0)| + 3)} \le -8|\phi(0)|^2
\end{aligned} \tag{25}$$

Let's now consider the adding term $p(x_t)$ in the new feedback control law proposed in Theorem 4. The term $\limsup_{h\to 0^+} \frac{V_0(\phi_h^a + \delta_h^g M) - V_0(\phi_h^a)}{h}$ has to be computed. The following equality holds, for any given $\phi \in C([-\Delta, 0]; R^n), M \in R$,

$$\begin{aligned}
&\limsup_{h\to 0^+} \frac{V_0(\phi_h^a + \delta_h^g M) - V_0(\phi_h^a)}{h} = \\
&\limsup_{h\to 0^+} \frac{1}{h}\bar{V}_0((\phi^T(0) + h(f(\phi) + g(\phi)k(\phi)) + hg(\phi)M)) - \\
&- \frac{1}{h}\bar{V}_0((\phi^T(0) + h(f(\phi) + g(\phi)k(\phi)))),
\end{aligned} \tag{26}$$

where

$$\begin{aligned}
&f(\phi) = \\
&\begin{bmatrix} -2\phi_1(0) + 2\phi_1(0)\phi_2^2(0) + 2\phi_1(0)\phi_2(0)\sqrt{|\phi_1(0)\phi_2(0)|} \\ -2\phi_2(0) - \phi_1^2(0)\phi_2(0) - \phi_1^2(0)\sqrt{|\phi_1(0)\phi_2(0)|} \end{bmatrix},
\end{aligned} \tag{27}$$

$$g(\phi) = \left[\begin{array}{c} \phi_2(-\Delta)(4+|\phi_1(-\Delta)|) \\ -\phi_1^2(0)(4+|\phi_1(-\Delta)|) \end{array} \right], \tag{28}$$

$$k(\phi) = -2 \cdot \frac{4\phi_1(0)\phi_2(-\Delta) - 8\phi_1^2(0)\phi_2(0)}{(4+|\phi_1(-\Delta)|)(|4\phi_1(0)\phi_2(-\Delta) - 8\phi_1^2(0)\phi_2(0)|+3)}. \tag{29}$$

The following equalities hold

$$\limsup_{h\to 0^+} \frac{V_0(\phi_h^a + \delta_h^g M) - V_0(\phi_h^a)}{h} = \limsup_{h\to 0^+} \frac{1}{h} \cdot$$
$$2(\phi(0) + h(f(\phi)+g(\phi)k(\phi))^T \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} hg(\phi)M) +$$
$$\frac{1}{h}(hg(\phi)M)^T \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} hg(\phi)M) =$$
$$2\phi^T(0) \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} g(\phi)M \tag{30}$$

Therefore, the term $r(\phi)$ in (*10*) is equal to

$$2g^T(\phi) \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \phi(0) \tag{31}$$

The new feedback control law

$$u(t) = k(x_t) + p(x_t) \tag{32}$$

where the functional p is given by

$$p(\phi) = D_a^+ V_0(\phi) g^T(\phi) \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \phi(0) \tag{33}$$

($D_a^+ V_0$ computed in (24)), is such that the closed loop system (22), (32) is (globally) input-to-state stable with respect to the disturbance $d(t)$ adding to the control law. The performed simulations validate the theoretical results. In figs. 1, 2 the behavior of the state variables in the case of feedback control law given by (23) and in the case of feedback control law given by (32) are reported, respectively. The disturbance $d(t)$ is chosen equal to $2 + 2sin(\pi t)$. The initial state variables are chosen constant in $[-\Delta, 0]$ and equal to $[1 \quad -1]^T$. The delay Δ is chosen equal to 0.1. In the first case the state variables diverge to ∞, while in the second case the state variables are kept bounded.

CONCLUSIONS

In this paper, it is proved that retarded systems which are stabilizable by means of a state feedback control law, are also input-to-state stabilizable, by a state feedback control law, with respect to measurable and locally essentially bounded disturbances adding to the control input, provided that a suitable Liapunov-Razumikhin function exists such that the asymptotic stability of the closed loop unforced system can be proved. An example taken from the past literature is investigated in details, showing the effectiveness and the applicability of the methodology here proposed.

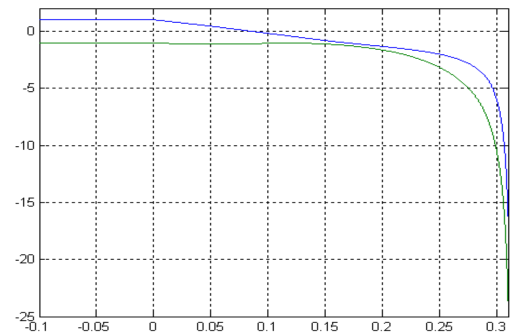

fig. 1: $x_1(t), x_2(t)$ (control law 23)

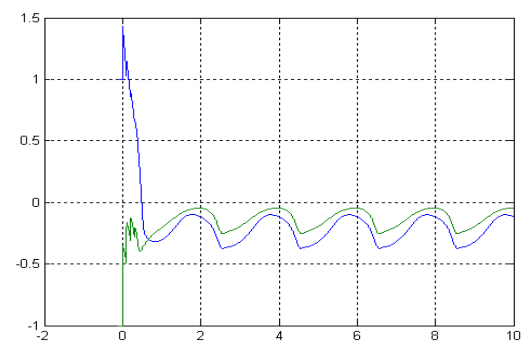

fig. 2: $x_1(t), x_2(t)$ (control law 32)

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