2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

ISS CONTROL LAWS FOR STABILIZABLE RETARDED SYSTEMS BY MEANS OF THE LIAPUNOV-KRASOVSKII METHODOLOGY

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ABSTRACT A disturbance adding to the control law is a typical situation in practice because of actuator errors. In this paper, state feedback control laws which provide input-to-state stability of the closed loop system with respect to a disturbance adding to the control law are investigated for state feedback stabilizable (in the case of disturbance equal to zero) retarded nonlinear systems. The formulas for the input-to-state stabilizing state feedback control law are provided by employing the Liapunov-Krasovskii methodology. An example taken from the past literature is investigated in details, showing the effectiveness and the applicability of the proposed control design.

Keywords: Input-to-State Stabilizability, Retarded Nonlinear Systems, Liapunov-Krasovskii Methodology.

INTRODUCTION

In 1989 Sontag showed in the paper [23] that nonlinear systems which are (smooth) feedback stabilizable, are also (smooth) input-to-state stabilizable with respect to disturbances adding to the control input. As well known, those disturbances are very frequent in practice, because of actuator errors. Many contributions concerning the state feedback stabilization and the input-output state feedback linearization of nonlinear retarded systems can be found in the literature (see, for instance, [3,6,8,12,14,15,17,18,26,30]). Liapunov-Krasovskii methodologies for the input-to-state stability of retarded nonlinear systems have been studied in [25] and in [10,19,28], respectively. As far as the input-to-state stabilizability of stabilizable retarded nonlinear systems is concerned, a contribution is given in [27], where, besides the main results dealing with the relationship between the input-to-state stability and the exponential stability in the unforced case, the input-to-state stabilizability of retarded nonlinear systems which are transformable by a state feedback control law into a linear, delay-free, exponentially stable system is considered, and the formula for the input-to-state stabilizing state feedback control law is provided.

This work is supported by the Italian MIUR project PRIN 2005.

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In this paper a general theory for the input-to-state stabilizability of state feedback stabilizable retarded nonlinear systems is provided. It is proved that a state feedback stabilizable (in the case of disturbance equal to zero) retarded nonlinear system admits an input-to-state stabilizable state feedback control law (i.e. admits a state feedback control law such that the closed loop system is input-to-state stable with respect to a disturbance adding to the control law), provided that the control system obtained by closing the loop with the stabilizing (in the case of disturbance equal to zero) state feedback control law admits a Liapunov-Krasovskii functional, with a suitable property, by which its global asymptotic stability can be proved. The formula for such input-to-state stabilizing state feedback control law is provided. It is assumed that the disturbance is measurable and locally essentially bounded. The essential boundedness of the disturbance in $[0,+\infty)$ is not assumed.

The effectiveness of the methodology here proposed is shown in details on an example of stabilizable (in the case of disturbance equal to zero) retarded nonlinear systems taken from the past literature.

Notations

R denotes the set of real numbers, $R^\star$ denotes the extended real line $[-\infty,+\infty]$, R^+ denotes the set of non negative reals $[0,+\infty)$. The symbol $|\cdot|$ stands for the Euclidean norm of a real vector, or the induced Euclidean norm of a matrix. The essential supremum norm of an essentially bounded function is indicated with the symbol $\|\cdot\|_\infty$. A function $v : R^+ \to R^m$, m positive integer, is said to be *essentially bounded* if $ess\sup_{t\geq 0}|v(t)| < \infty$. For given times $0 \leq T_1 < T_2$, we indicate with $v_{[T_1,T_2)} : R^+ \to R^m$ the function given by $v_{[T_1,T_2)}(t) = v(t)$ for all $t \in [T_1,T_2)$ and $= 0$ elsewhere. An input v is said to be *locally essentially bounded* if, for any $T > 0$, $v_{[0,T)}$ is essentially bounded. For a positive real Δ, $C([-\Delta,0];R^n)$ denotes the space of the continuous functions mapping $[-\Delta,0]$ into R^n, n positive integer. For positive integers m, n, I_m denotes the identity matrix in $R^{m\times m}$, $0_{m,n}$ denotes a matrix of zeros in $R^{m\times n}$. A functional $F : C([-\Delta,0];R^n) \to R^{m\times p}$, m,n,p positive integers, is said to be completely continuous if it is continuous and maps closed bounded sets of $C([-\Delta,0];R^n)$ into bounded sets of $R^{m\times p}$. Let us here recall that a function $\gamma : R^+ \to R^+$ is: positive definite

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if it is continuous, zero at zero and $\gamma(s) > 0$ for all $s > 0$; of class $\mathcal{G}$ if it is continuous, zero at zero, and nondecreasing; of class $\mathcal{K}$ if it is of class $\mathcal{G}$ and strictly increasing; of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and it is unbounded; of class $\mathcal{L}$ if it monotonically decreases to zero as its argument tends to $+\infty$. A function $\beta : R^+ \times R^+ \to R^+$ is of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each $t \geq 0$ and $\beta(s, \cdot)$ is of class $\mathcal{L}$ for each $s \geq 0$. With M_a is indicated any functional mapping $C([-\Delta, 0]; R^n)$ into R^+ such that, for some $\mathcal{K}_\infty$ functions $\gamma_a, \bar{\gamma}_a$, the following inequalities hold

$$\gamma_a(|\phi(0)|) \leq M_a(\phi) \leq \bar{\gamma}_a(\|\phi\|_\infty), \ \forall \phi \in C([-\Delta, 0]; R^n) \tag{1}$$

For example, the $\|\cdot\|_{M_2}$ norm, given by (see [2]) $\|\phi\|_{M_2} = \left(|\phi(0)|^2 + \int_{-\Delta}^{0} |\phi(\tau)|^2 d\tau\right)^{\frac{1}{2}}$, is a M_a functional. As usual, ISS stands for both input-to-state stable and input-to-state stability.

PRELIMINARIES

In this section, for the reader's convenience, some previously published results which are fundamental for the understanding of the novel results which will be provided in next sections are briefly reported, with some slight modifications for the purposes of this paper. Let us consider the following retarded nonlinear system

$$\begin{aligned} &\dot{x}(t) = f(x_t) + g(x_t)v(t), \qquad t \geq 0, \qquad a.e., \\ &x(\tau) = \xi_0(\tau), \qquad \tau \in [-\Delta, 0], \end{aligned} \tag{2}$$

where $x(t) \in R^n$, $v(t) \in R^m$ is the input function, measurable and locally essentially bounded, for $t \geq 0$ $x_t : [-\Delta, 0] \to R^n$ is the standard function (see Section 2.1, pp. 38 in [5]) given by $x_t(\tau) = x(t + \tau)$, Δ is the maximum involved delay, f is a locally Lipschitz, completely continuous functional mapping $C([-\Delta, 0]; R^n)$ into R^n, g is a locally Lipschitz, completely continuous functional mapping $C([-\Delta, 0]; R^n)$ into $R^{n\times m}$, $\xi_0 \in C([-\Delta, 0]; R^n)$. It is here supposed that $f(0) = 0$, thus ensuring that $x(t) = 0$ is the trivial solution for the unforced system $\dot{x}(t) = f(x_t)$ with zero initial conditions. Multiple discrete non-commensurate as well as distributed delays can appear in (2). In the following, the continuity of a functional $V : C([-\Delta, 0]; R^n) \to R^+$ is intended with respect to the supremum norm. Given a locally Lipschitz continuous functional $V : C([-\Delta, 0]; R^n) \to R^+$, the upper right-hand derivative D^+V of the functional V is given by (see [1], Definition 4.2.4, pp. 258)

$$D^+V(\phi, v) = \limsup_{h\to 0^+} \frac{1}{h} \left(V(\phi_h) - V(\phi)\right), \tag{3}$$

where $\phi_h \in C([-\Delta, 0]; R^n)$ is given by

$$\phi_h(\theta) = \begin{cases} \phi(\theta + h), & s \in [-\Delta, -h), \\ \phi(0) + (f(\phi) + g(\phi)v)(\theta + h), & \theta \in [-h, 0] \end{cases} \tag{4}$$

Definition 1: ([23,19]) *The system (2) is said to be input-to-state stable (ISS) if there exist a $\mathcal{KL}$ function β and a $\mathcal{K}$ function γ such that, for any initial state ξ_0 and any measurable, locally essentially bounded input v, the solution exists for all $t \geq 0$ and furthermore it satisfies*

$$|x(t)| \leq \beta\left(\|\xi_0\|_\infty, t\right) + \gamma\left(\|v_{[0,t)}\|_\infty\right) \tag{5}$$

Theorem 2: ([19]) *If there exist a locally Lipschitz functional $V : C([-\Delta, 0]; R^n) \to R^+$, functions α_1, α_2 of class $\mathcal{K}_\infty$, and functions α_3, ρ of class $\mathcal{K}$ such that:*

H_1) $\alpha_1(|\phi(0)|) \leq V(\phi) \leq \alpha_2(M_a(\phi))$, $\forall \phi \in C([-\Delta, 0]; R^n)$;

H_2) $D^+V(\phi, v) \leq -\alpha_3(M_a(\phi))$,
$\forall \phi \in C([-\Delta, 0]; R^n), \ v \in R^m : \ M_a(\phi) \geq \rho(|v|)$;

then, the system (2) is input-to-state stable with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.

Remark 3: With respect to the published literature: Theorem 2 makes use of the M_a functional instead of the $\|\cdot\|_a$ norm used in [19], thus weakening the hypotheses. •

MAIN RESULTS

Let us consider now the following retarded nonlinear system, corresponding to (2) when the input v is given as the sum of the control input and of the disturbance:

$$\begin{aligned} &\dot{x}(t) = f(x_t) + g(x_t)(u(t) + d(t)), \qquad t \geq 0, \qquad a.e., \\ &x(\tau) = \xi_0(\tau), \qquad \tau \in [-\Delta, 0], \end{aligned} \tag{6}$$

where $u(t) \in R^m$ is the control input, $d(t) \in R^m$ is the disturbance, measurable and locally essentially bounded.

Theorem 4: *Let there exists a locally Lipschitz, completely continuous functional $k : C([-\Delta, 0]; R^n) \to R^m$, such that the closed loop system (6) with $u(t) = k(x_t)$, and no disturbance, described by the equations*

$$\dot{x}(t) = f(x_t) + g(x_t)k(x_t), \tag{7}$$

is globally asymptotically stable. Let, for $\phi \in C([-\Delta, 0]; R^n)$, $h \in (0, \Delta)$,

$$\begin{aligned} &\phi_h^a(\theta) = \\ &\begin{cases} \phi(\theta + h), & \theta \in [-\Delta, -h), \\ \phi(0) + (\theta + h)\left(f(\phi) + g(\phi)k(\phi)\right), & \theta \in [-h, 0], \end{cases} \\ &\delta_h^g(\theta) = \begin{cases} 0_{n\times m}, & \theta \in [-\Delta, -h), \\ (\theta + h)g(\phi), & \theta \in [-h, 0] \end{cases} \end{aligned} \tag{8}$$

Let there exist a locally Lipschitz continuous functional $V_0 : C([-\Delta, 0]; R^n) \to R^+$, functions α_1, α_2 and α_3 of class $\mathcal{K}_\infty$, a locally Lipschitz, completely continuous functional $r : C([-\Delta, 0]; R^n) \to R^m$ such that, for any $\phi \in C([-\Delta, 0]; R^n)$ and any $M \in R^m$:

i) $\alpha_1(|\phi(0)|) \le V_0(\phi) \le \alpha_2(M_a(\phi))$;

ii) the functional $D_a^+ V_0 : C([-\Delta, 0]; R^n) \to R^\star$, defined as

$$D_a^+ V_0(\phi) = \limsup_{h \to 0^+} \frac{V_0(\phi_h^a) - V_0(\phi)}{h} \tag{9}$$

is locally Lipschitz, completely continuous and, moreover, $D_a^+ V_0(\phi) \le -\alpha_3(M_a(\phi))$;

iii)

$$\limsup_{h \to 0^+} \frac{V_0(\phi_h^a + \delta_h^g M) - V_0(\phi_h^a)}{h} = r^T(\phi) M. \tag{10}$$

Then, the feedback control law

$$u(t) = k(x_t) + p(x_t), \tag{11}$$

with $p : C([-\Delta, 0]; R^n) \to R^m$ defined as

$$p(\phi) = \frac{1}{2} D_a^+ V_0(\phi) r(\phi), \tag{12}$$

is such that the closed loop system (6), (11), described by the following equations

$$\dot{x}(t) = f(x_t) + g(x_t)k(x_t) + g(x_t)p(x_t) + g(x_t)d(t), \tag{13}$$

is input-to-state stable with respect to the measurable and locally essentially bounded disturbance $d(t)$.

Proof.

In order to prove the input-to-state stability of the closed loop system (*13*) with respect to the disturbance $d(t)$, let us apply Theorem 2 by using the locally Lipschitz continuous functional $V = V_0$. The functional V satisfies the hypothesis H_1 in Theorem 2. As far as the hypothesis H_2 in Theorem 2 is concerned, the derivative $D^+V(\phi, d)$ of the functional V with respect to the closed loop system (*13*), according to (3), is as follows

$$D^+V(\phi, d) = \limsup_{h \to 0^+} \frac{V(\phi_h^a + \delta_h^g(p(\phi) + d)) - V(\phi)}{h} \tag{14}$$

The following equalities/inequalities hold

$$\begin{aligned} D^+V(\phi, d) = \limsup_{h \to 0^+} \frac{V(\phi_h^a + \delta_h^g(p(\phi) + d)) - V(\phi_h^a)}{h} + \\ \frac{V(\phi_h^a) - V(\phi)}{h} \le -\alpha_3(M_a(\phi)) + r^T(\phi)(p(\phi) + d) \le \\ -\alpha_3(M_a(\phi)) - \frac{1}{2}\alpha_3(M_a(\phi)) r^T(\phi) r(\phi) + r^T(\phi) d \end{aligned} \tag{15}$$

If $\alpha_3(M_a(\phi)) \ge |d|$ (this is not an hypothesis about suitable essential boundedness of the disturbance, but just concerns the hypothesis H_2 in Theorem 2), then the following inequalities hold

$$\begin{aligned} D^+V(\phi, d) \le -\alpha_3(M_a(\phi)) - \frac{1}{2}\alpha_3(M_a(\phi)) r^T(\phi) r(\phi) + \\ \frac{1}{2}\alpha_3(M_a(\phi)) r^T(\phi) r(\phi) + \frac{1}{2}\alpha_3(M_a(\phi)), \end{aligned} \tag{16}$$

and, therefore, the inequality follows

$$D^+V(\phi, d) \le -\frac{1}{2}\alpha_3(M_a(\phi)) \tag{17}$$

So, by theorem 2, the proof of the input-to-state stability of the closed loop system (*13*) is accomplished. ☐

ILLUSTRATIVE EXAMPLE

Let us consider the retarded nonlinear system studied in [18] in the case of no disturbance adding to the control law, described by the following nonlinear equations

$$\begin{aligned} \dot{x}_1(t) &= -4x_1(t) - x_1(t - \Delta) - x_3^2(t - \Delta) \\ \dot{x}_2(t) &= x_1(t - \Delta) - x_2(t - \Delta) + (1 + x_1^2(t))(u(t) + d(t)) \\ \dot{x}_3(t) &= x_2(t) + x_1(t) x_3(t - \Delta) \end{aligned} \tag{18}$$

In [18] the following change of coordinates is introduced

$$\begin{bmatrix} z_1(t) \\ z_2(t) \\ \xi(t) \end{bmatrix} = \begin{bmatrix} x_3(t) \\ x_2(t) + x_1(t) x_3^2(t - \Delta) \\ x_1(t) \end{bmatrix} \tag{19}$$

by which the system (18) can be rewritten as

$$\begin{aligned} \begin{bmatrix} \dot{z}(t) \\ \dot{\xi}(t) \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} T(z_t, z_{t-\Delta}) \\ -4\xi(t) - \xi(t - \Delta) - z_1^2(t - \Delta) \end{bmatrix} + \\ \begin{bmatrix} 0 \\ 1 + \xi^2(t) \\ 0 \end{bmatrix} (u(t) + d(t)), \end{aligned} \tag{20}$$

where $z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$, $T : C([-\Delta, 0]; R^2) \times C([-\Delta, 0]; R^2) \to R$ is a suitable continuous functional (see [18]). By the feedback control law

$$u(t) = k(z_t, z_{t-\Delta}) = \frac{-T(z_t, z_{t-\Delta}) + G^T z(t)}{1 + \xi^2(t)}, \tag{21}$$

where $G \in R^2$ is such that the matrix $H = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} G^T$ is Hurwitz (see (45) and the expression of $v(t)$

in [18]), the closed loop system, in the new coordinates, is described by the following nonlinear equations

$$\begin{bmatrix} \dot{z}(t) \\ \dot{\xi}(t) \end{bmatrix} = \begin{bmatrix} Hz(t) + \begin{bmatrix} 0 \\ (1+\xi_1^2(t)) \end{bmatrix} d(t) \\ -4\xi(t) - \xi(t-\Delta) - z_1^2(t-\Delta) \end{bmatrix} \tag{22}$$

Without loss of generality, let us suppose that the vector G is chosen such that the matrix H has real negative distinct eigenvalues $-\lambda_1$, $-\lambda_2$. In [18], as said before, the disturbance adding to the control law is not considered, that is $d(t) = 0$, and it is proved that the unforced closed loop system (20),(21) is (locally) asymptotically stable. Here we apply Theorem 4 in order to find out a feedback control law

$$u(t) = k(z_t, z_{t-\Delta}) + p(z_t) \tag{23}$$

such that the forced closed loop system (20), (23) is (globally) input-to-state stable with respect to the disturbance $d(t)$. By using a new partial change of coordinates $\hat{z}(t) = Rz(t)$, by which $RHR^{-1} = \begin{bmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix}$, it results that the global asymptotic stability of the closed loop system (22) (with disturbance equal to zero) is proved by means of the following Liapunov-Krasovskii functional $V_0 : C([-\Delta, 0]; R^3) \to R^+$, given, for $\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} \in C([-\Delta, 0]; R^3)$, by

$$\begin{aligned} V_0(\phi) &= a\,\big(k_1(R_{1,1}\phi_1(0) + R_{1,2}\phi_2(0))^4 + \\ & k_2(R_{2,1}\phi_1(0) + R_{2,2}\phi_2(0))^4 + k_3\phi_3^2(0) + \\ & \int_{-\Delta}^{0} \alpha(\tau)((R_{1,1}\phi_1(\tau) + R_{1,2}\phi_2(\tau))^4 + \\ & (R_{2,1}\phi_1(\tau) + R_{2,2}\phi_2(\tau))^4)d\tau + \\ & \int_{-\Delta}^{0} \beta(\tau)\phi_3^2(\tau)d\tau\Big), \end{aligned} \tag{24}$$

where a is any positive real (it may be a useful further tuning parameter at the aim of keeping the new feedback control law suitably bounded), $\alpha(\tau) = -a_1\frac{\tau}{\Delta} + a_2\frac{\tau+\Delta}{\Delta}$, $\beta(\tau) = -b_1\frac{\tau}{\Delta} + b_2\frac{\tau+\Delta}{\Delta}$, $a_1, a_2, b_1, b_2, k_1, k_2, k_3$ are positive reals which are chosen (it is always possible) such to satisfy

$$\begin{aligned} & 4k_3 max\{(R_{1,1}^{-1})^4, (R_{1,2}^{-1})^4\} < a_1 < a_2 < \\ & 4min\{k_1\lambda_1, k_2\lambda_2\}, \\ & k_3 < b_1 < b_2 < 5k_3, \end{aligned} \tag{25}$$

$R_{i,j}$, and $R_{i,j}^{-1}$, $i, j = 1, 2$, are the elements in the i^{th} row and j^{th} column of R and R^{-1}, respectively. For the functional V_0, $D_a^+ V_0$ is given by

$$\begin{aligned} D_a^+ V_0(\phi) &= a\,\big((a_2 - 4k_1\lambda_1)(R_{1,1}\phi_1(0) + R_{1,2}\phi_2(0))^4 + \\ & (a_2 - 4k_2\lambda_2)(R_{2,1}\phi_1(0) + R_{2,2}\phi_2(0))^4 + \\ & (b_2 - 8k_3)\phi_3^2(0) - 2k_3\phi_3(0)\phi_3(-\Delta) \\ & - 2k_3\phi_3(0)\phi_1^2(-\Delta) - a_1(R_{1,1}\phi_1(-\Delta) + R_{1,2}\phi_2(-\Delta))^4 \\ & - a_1(R_{2,1}\phi_1(-\Delta) + R_{2,2}\phi_2(-\Delta))^4 - b_1\phi_3^2(-\Delta) \\ & - \frac{a_2 - a_1}{\Delta}\int_{-\Delta}^{0}((R_{1,1}\phi_1(\tau) + R_{1,2}\phi_2(\tau))^4 d\tau - \\ & - \frac{a_2 - a_1}{\Delta}\int_{-\Delta}^{0}(R_{2,1}\phi_1(\tau) + R_{2,2}\phi_2(\tau))^4)d\tau \\ & - \frac{b_2 - b_1}{\Delta}\int_{-\Delta}^{0}\phi_3^2(\tau)d\tau\Big) \end{aligned} \tag{26}$$

As far as the $\limsup_{h\to 0^+} \frac{V_0(\phi_h^a + \delta_h^g M) - V_0(\phi_h^a)}{h}$, where $M \in R$, the following equality holds

$$\begin{aligned} & \limsup_{h\to 0^+} \frac{V_0(\phi_h^a + \delta_h^g M) - V_0(\phi_h^a)}{h} = \\ & a\,\big(4k_1(R_{1,1}\phi_1(0) + R_{1,2}\phi_2(0))^3(1 + \phi_3(0)^2)M + \\ & +4k_2(R_{2,1}(\phi_1(0) + R_{2,2}\phi_2(0))^3)(1 + \phi_3(0)^2)M\big), \end{aligned} \tag{27}$$

by which the functional $r(\phi)$ in iii) of Theorem 4 is defined as

$$\begin{aligned} r(\phi) &= 4ak_1(R_{1,1}\phi_1(0) + R_{1,2}\phi_2(0))^3(1 + \phi_3(0)^2) + \\ & 4ak_2(R_{2,1}\phi_1(0) + R_{2,2}\phi_2(0))^3(1 + \phi_3(0)^2) \end{aligned} \tag{28}$$

Therefore the new feedback control law is equal to

$$u(t) = k(z_t, z_{t-\Delta}) + p(z_t), \tag{29}$$

where $p : C([-\Delta, 0]; R^3) \to R$ is defined as

$$\begin{aligned} p(\phi) &= D_a^+ V_0(\phi)(2ak_1(R_{1,1}\phi_1(0) + \\ & R_{1,2}\phi_2(0))^3(1 + \phi_3(0)^2) + \\ & 2ak_2(R_{1,2}\phi_1(0) + R_{2,2}\phi_2(0))^3(1 + \phi_3(0)^2)), \end{aligned} \tag{30}$$

$D_a^+ V_0(\phi)$ given by (26). By Theorem 4, the closed loop system (20), (29) is (globally) input-to-state stable with respect to the disturbance $d(t)$. As far as the original system (18) is concerned, taking into account the change of coordinates (19), the (global) input-to-state stability of the closed loop system (20), (29) implies the (global) input-to-state stability of the closed loop system (18), (19), (29) (with respect to the disturbance $d(t)$).

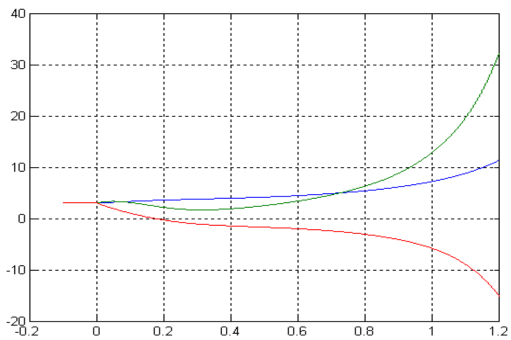

fig. 1: $z(t)$ and $\xi(t)$ (control law 21)

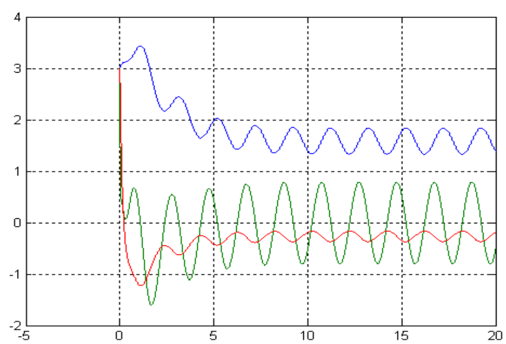

fig. 2: $z(t)$ and $\xi(t)$ (control law 29)

The numerical simulations, performed by Matlab, validate the theoretical results. In the case reported here, the matrix $H = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ (chosen eigenvalues $-1, -2$), the disturbance is equal to $d(t) = 3(1+sin(\pi t))$ and the initial state variables are chosen constant in $[-\Delta, 0]$ and equal to $[3 \quad 3 \quad 3]^T$. The delay Δ is chosen equal to 0.1. The parameter a is chosen equal to 10^{-4}. The other parameters are chosen as $k_1 = 2$, $k_2 = 2$, $k_3 = 1$, $a_1 = 5$, $a_2 = 6$, $b_1 = 2$, $b_2 = 3$. When the feedback control law (21) is used, the variables $z(t)$ and $\xi(t)$ of the closed loop system (20), (21), with disturbance, diverge to ∞ (see fig. 1). When the feedback control law (29) is used, the variables $z(t)$ and $\xi(t)$ of the closed loop system (20), (29), with disturbance, are kept bounded (fig. 2). In this case the feedback control law proposed in [18] for the unforced case ($d(t) = 0$) cannot work in the forced case, while the feedback control law proposed in this paper yields very good performance also for the case with disturbance.

CONCLUSIONS

In this paper, it is proved that retarded systems which are stabilizable by means of a state feedback control law, are also input-to-state stabilizable, by a state feedback control law, with respect to measurable and locally essentially bounded disturbances adding to the control input, provided that a suitable Liapunov-Krasovskii functional exists such that the asymptotic stability of the closed loop unforced system can be proved. Such functional must have some smooth property with respect to a suitable variable, which is a very weak hypothesis. An example taken from the past literature is investigated in details, showing the effectiveness and the applicability of the methodology here proposed.

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