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Extensions on a Certainty-Equivalence Feedback Design with a Class of Feedbacks Which Guarantee ISS

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Abstract— **In this paper a new class of nonlinear feedbacks that guarantee input-to-state stability (ISS) w.r.t. the measurement error is identified. In particular, it is shown that (inverse) optimal feedback laws, that are separable into a globally Lipschitz part and a nonlinearity satisfying a certain inequality condition, guarantee ISS. As a consequence, those state feedbacks in conjunction with any globally asymptotically convergent observer lead to a globally asymptotically stable closed-loop. The theoretical results are applied to several control problems, e.g. the nonlinear output feedback design of a single-link robot arm or an active magnetic bearing system.**

I. INTRODUCTION

Nonlinear output feedback design is one of the most challenging problems in nonlinear control. Various examples in literature show that the certainty-equivalence implementation of a globally stabilizing state feedback and a globally converging observer can lead to an unstable closed-loop, see e.g. [7]. Mainly two approaches have been proven to be successful in tackling the nonlinear output feedback design problem. The first approach is based on the use of high-gain observers [6, 10], the second one uses ISS-related concepts [8, 11]. For a more detailed discussion of the nonlinear output feedback design consult [13] and the references therein.

This paper continuous the work on an ISS-related separation principle based on (inverse) optimal feedbacks, introduced in [1]. The closed-loop is considered as a cascade system in which the observer error acts as a measurement disturbance. Thus, the state feedback has to render the system robust against this disturbance, for example a feedback that renders the closed-loop input-to-state stable w.r.t. the measurement error guarantees the necessary robustness. In this paper a new class of nonlinear state feedbacks that guarantee ISS w.r.t. the measurement error is presented which includes the results of [1] as a special case. More precisely, it is shown that (inverse) optimal feedback laws that are separable into a globally Lipschitz part and a nonlinear part, satisfying a certain inequality condition, guarantee the ISS property. A main advantage of this result is that an explicit condition for nonlinearities is given that can be contained in the state feedback. The results are illustrated on various examples, e.g. a single-link robot arm or an active magnetic bearing.

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The remainder of the paper is organized as follows: In Section II the ISS property w.r.t. the measurement error is established for the new class of state feedbacks. In Section III a nonlinear separation principle is stated for this class of state feedbacks and in Section IV the theoretical results are applied to different control problems. Conclusions are given in Section IV.

II. CLASS OF FEEDBACKS THAT GUARANTEE ISS

In this section the main results of the paper are presented, namely, a class of state feedbacks $u = k(x)$ is proposed that render the closed-loop system $\dot{x} = f(x) + G(x)k(x + e)$ input-to-state stable w.r.t the measurement error e. This ISS property is established for a class of nonlinear (inverse) optimal feedbacks. A novelty in this work is that a rather easily verifiable condition for the admissible nonlinearities in the state feedback is established. As discussed at the end of this section, the new result enlarges the class of state feedbacks proposed in [1]. To establish this result, two main assumptions are needed.

Assumption 1: The nonlinear control system is assumed to be of the form

$$\begin{aligned} \dot{x} &= f(x) + G(x)u \\ y &= h(x), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^q$ the control input and $y \in \mathbb{R}^p$ the measurable output. The vector fields $f : \mathbb{R}^n \to \mathbb{R}^n$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times q}$, and $h : \mathbb{R}^n \to \mathbb{R}^p$ are assumed to be sufficiently smooth with $f(0) = 0,\ h(0) = 0$.

Assumption 2: The globally stabilizing state feedback is assumed to be of the form

$$u^* = k(x) = -\frac{1}{2}R^{-1}(x)G^T(x)V_x^T(x), \tag{2}$$

i.e. V has to satisfy the Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{aligned} &V_x(x)f(x) + V_x(x)G(x)k(x) \\ &\quad + q(x) + k^T(x)R(x)k(x) = 0, \end{aligned} \tag{3}$$

where $q(x) \geq c\|x\|^2$ and R is a positive definite matrix with $\lambda_{min}I \leq R(x) \leq \lambda_{max}I, \lambda_{max} > \lambda_{min} > 0$. V is assumed to be a positive definite, radially unbounded C^1 function and V_x denotes the gradient (column vector) of V. Hence, the state feedback (2) minimizes the cost functional

$$V(x(0)) = \int_0^\infty \big(q(x(t)) + u^T(t)R(x(t))u(t)\big)\, dt. \tag{4}$$

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Using Assumptions 1 and 2, the first main result is established in the following theorem.

Theorem 1: Suppose that Assumptions 1 and 2 hold. Then the closed-loop system

$$\dot{x} = f(x) + G(x)k(x+e) \tag{5}$$

is input-to-state stable with respect to $e \in \mathbb{R}^n$, if the state feedback (2) is of the form

$$u^* = k(x) = m(x) + Q(y)p(x_i). \tag{6}$$

In (6) the function $m : \mathbb{R}^n \to \mathbb{R}^q,\ m(0) = 0$, is globally Lipschitz with Lipschitz constant γ, i.e. $\|m(x+e)-m(x)\| \leq \gamma\|e\|$, and the function $p : \mathbb{R}^n \to \mathbb{R}^q$ is of the form

$$p(x_i) = \begin{bmatrix} p_1(x_{i_1}) \\ \vdots \\ p_q(x_{i_q}) \end{bmatrix}, \tag{7}$$

where the components p_j are functions in a single state $x_{i_j}, i_j \in \{1 \dots n\}, j = 1 \dots q$ that satisfy the inequality condition

$$p_j^2(x_{i_j}) > a^2 \left(p_j(x_{i_j} + z) - p_j(x_{i_j})\right)^2 \tag{8}$$

for all nonzero constants $a, z \in \mathbb{R}$ whenever $|x_{i_j}|$ is sufficiently large. Furthermore, the matrix $Q(y)$ in (6) is a $q \times q$ diagonal matrix whose entries are only depending on the measurable output y, i.e. $Q(y)$ is of the form

$$Q(y) = \begin{bmatrix} Q_{11}(y) & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & Q_{qq}(y) \end{bmatrix}. \tag{9}$$

Proof: One has to show that the state feedback (6) guarantees ISS w.r.t. the measurement error. The value function V in (3) can be used as an ISS Lyapunov function candidate. It has to be shown that there exists a class $\mathcal{K}$ function ρ and a class $\mathcal{K}_\infty$ function β such that

$$V_x(x)(f(x) + G(x)k(x+e)) \leq -\beta(\|x\|) \tag{10}$$

holds whenever $\|x\| \geq \rho(\|e\|)$, see e.g. [6]. This is equal to

$$\begin{aligned} V_x(x)f(x) + V_x(x)G(x)(m(x+e) \\ +Q(y)p(x_i+e_i)) < -\beta(\|x\|) \end{aligned} \tag{11}$$

for any given e whenever $\|x\|$ is sufficiently large. The proof is divided into two parts.

Part 1: In the first part an upper bound for the left-hand side of equation (11) is estimated analogous to [1]. This is done by successively overestimating the left-hand side. At first $V_x(x)G(x)(m(x)+Q(y)p(x_i))$ is added and subtracted. This leads to

$$\begin{aligned} &V_x(x)\left(f(x) + G(x)(m(x) + Q(y)p(x_i))\right) \\ &\quad +V_x(x)G(x)\left(m(x+e) - m(x)\right. \\ &\quad \left. + Q(y)p(x_i+e_i) - Q(y)p(x_i)\right) \end{aligned} \tag{12}$$

that can be further transformed, using the HJB equation (3) and the relation (2), into

$$\begin{aligned} &-q(x) - (m(x) + Q(y)p(x_i))^T \\ &\quad \times R(x)(m(x) + Q(y)p(x_i)) \\ &-2(m(x) + Q(y)p(x_i))^T R(x) \\ &\quad \times(\triangle m(x,e) + Q(y)\triangle p(x_i,e_i)) \end{aligned} \tag{13}$$

with $\triangle m(x,e) = m(x+e) - m(x)$ and $\triangle p(x_i,e_i) = p(x_i+e_i) - p(x_i)$. As $\lambda_{min}I < R(x) < \lambda_{max}I$ and $2ab \leq \alpha_1^{-1}\|a\|^2+\alpha_1\|b\|^2$ with $a = R(x)(m(x)+Q(y)p(x_i))$ and $b = (\triangle m(x,e) + Q(y)\triangle p(x_i,e_i))$, the expression (13) is bounded from above by

$$\begin{aligned} &-q(x) - \lambda_{min}\|m(x) + Q(y)p(x_i)\|^2 \\ &\quad +\frac{\lambda_{max}^2}{\alpha_1}\|m(x) + Q(y)p(x_i)\|^2 \\ &+\alpha_1\|\triangle m(x,e) + Q(y)\triangle p(x_i,e_i)\|^2. \end{aligned} \tag{14}$$

Choosing now α_1 such that $\frac{\lambda_{min}}{2} = \frac{\lambda_{max}^2}{\alpha_1}$ and considering that $q(x) \geq c\|x\|^2$, one obtains

$$\begin{aligned} &-c\|x\|^2 - \frac{\lambda_{min}}{2}\left(|m(x)\|^2\right. \\ &\quad \left.+ 2m^T(x)Q(y)p(x_i) + \|Q(y)p(x_i)\|^2\right) \\ &+\frac{2\lambda_{max}^2}{\lambda_{min}}\|\triangle m(x,e) + Q(y)\triangle p(x_i,e_i)\|^2. \end{aligned} \tag{15}$$

Applying Young's inequality $2a^Tb \leq \alpha_2^{-1}\|a\|^2 + \alpha_2\|b\|^2$ to the term $2m^T(x)Q(y)p(x_i)$, with $a = m(x)$ and $b = Q(y)p(x_i)$, (15) is overestimated by

$$\begin{aligned} &-c\|x\|^2 \\ &-\frac{\lambda_{min}}{2}\left((1-\alpha_2^{-1})\|m(x)\|^2 + (1-\alpha_2)\|Q(y)p(x_i)\|^2\right) \\ &+\frac{2\lambda_{max}^2}{\lambda_{min}}\|\triangle m(x,e) + Q(y)\triangle p(x_i,e_i)\|. \end{aligned} \tag{16}$$

It is possible to choose $\alpha_2 < 1$ such that the term

$$-c\|x\|^2 + \frac{\lambda_{min}}{2}(\alpha_2^{-1} - 1)\|m(x)\|^2 \tag{17}$$

is negative definite, i.e. α_2 has to be chosen such that $\frac{\lambda_{min}}{2}(\alpha_2^{-1} - 1)\gamma^2 < c$, where γ is the Lipschitz constant of $m(x)$. With $\tilde{c} = c - \frac{\lambda_{min}}{2}(\alpha_2^{-1} - 1)\gamma^2 > 0$ and $\tilde{d} = \frac{\lambda_{min}}{2}(1-\alpha_2) > 0$ one obtains

$$\begin{aligned} &-\tilde{c}\|x\|^2 - \tilde{d}\|Q(y)p(x_i)\|^2 \\ &+\frac{2\lambda_{max}^2}{\lambda_{min}}\|\triangle m(x,e) + Q(y)\triangle p(x_i,e_i)\|^2. \end{aligned} \tag{18}$$

Finally the triangular inequality and $(\|a\|+\|b\|)^2 \leq 2(\|a\|^2 + \|b\|^2)$ can be used to derive

$$\begin{aligned} &-\tilde{c}\|x\|^2 - \tilde{d}\|Q(y)p(x_i)\|^2 \\ &\quad +\frac{4\lambda_{max}^2}{\lambda_{min}}\|\triangle m(x,e)\|^2 \\ &+\frac{4\lambda_{max}^2}{\lambda_{min}}\|Q(y)\triangle p(x_i,e_i)\|^2, \end{aligned} \tag{19}$$

where one defines $\tilde{\lambda} = \frac{4\lambda_{max}^2}{\lambda_{min}}$ to get, by considering the Lipschitz property $\|\triangle m(x,e)\| \leq \gamma\|e\|$, the upper bound

$$\begin{aligned} -\tilde{c}\|x\|^2 - \tilde{d}\|Q(y)p(x_i)\|^2 \\ +\tilde{\lambda}\gamma^2\|e\|^2 + \tilde{\lambda}\|Q(y)\triangle p(x_i,e_i)\|^2 \end{aligned} \quad (20)$$

for equation (11). Therefore, one arrives at

$$\begin{aligned} V_x(x)(f(x) + G(x)k(x+e)) \\ \leq -\tilde{c}\|x\|^2 + \tilde{\lambda}\gamma^2\|e\|^2 \\ -\tilde{d}\|Q(y)p(x_i)\|^2 + \tilde{\lambda}\|Q(y)\triangle p(x_i,e_i)\|^2. \end{aligned} \quad (21)$$

Part 2: In the second part of the proof it will be shown that (21) satisfies

$$\begin{aligned} -\tilde{c}\|x\|^2 + \tilde{\lambda}\gamma^2\|e\|^2 - \tilde{d}\|Q(y)p(x_i)\|^2 \\ +\tilde{\lambda}\|Q(y)\triangle p(x_i,e_i)\|^2 < -\beta(\|x\|) \end{aligned} \quad (22)$$

for a sufficiently large $\|x\|$ with $\beta(\|x\|) = \frac{\tilde{c}}{2}\|x\|^2$. Thus, it has to be shown that

$$\begin{aligned} -\frac{\tilde{c}}{2}\|x\|^2 + \tilde{\lambda}\gamma^2\|e\|^2 - \tilde{d}\sum_{j=1}^{q} Q_{jj}^2(y)p_j^2(x_{i_j}) \\ +\tilde{\lambda}\sum_{j=1}^{q} Q_{jj}^2(y)\left(p_j(x_{i_j}+e_{i_j}) - p_j(x_{i_j})\right)^2 < 0 \end{aligned} \quad (23)$$

holds for sufficiently large $\|x\|$. Since p_j satisfy (8), i.e.

$$p_j^2(x_{i_j}) > a^2(p_j(x_{i_j}+z) - p_j(x_{i_j}))^2$$

holds for sufficiently large $|x_{i_j}|$ and since $Q_{jj}^2(y) \geq 0$ and $\tilde{d} > 0$, one can see that the expression

$$Q_{jj}^2(y)\tilde{d}\left(-p_j^2(x_{i_j}) + \frac{\tilde{\lambda}}{\tilde{d}}(p_j(x_{i_j}+e_{i_j}) - p_j(x_{i_j}))^2\right) \quad (24)$$

in the sum (23) gets negative (nonpositive) whenever $|x_{i_j}|$ becomes large enough. Consequently (23) holds whenever $|x_{i_j}|$ is sufficiently large. In the case that $\|x\|$ becomes sufficiently large, but not any of the states $\{x_{i_1} \dots x_{i_q}\}$, then still the term $\frac{\tilde{c}}{2}\|x\|^2$ becomes sufficiently large, such that for a given (fixed) e the inequality (23) holds. ■

The next corollary is an extension of Theorem 1. It allows that the components of p in (6) depend on linear combinations of states and not only on a single state.

Corollary 2: Suppose that Assumptions 1 and 2 hold. Then the closed-loop system

$$\dot{x} = f(x) + G(x)k(x+e) \quad (25)$$

is input-to-state stable with respect to $e \in \mathbb{R}^n$, if the state feedback (2) is of the form

$$u^* = k(x) = m(x) + Q(y)p(c_i^T x). \quad (26)$$

In (26) the function $m : \mathbb{R}^n \to \mathbb{R}^q$, $m(0) = 0$, is globally Lipschitz with Lipschitz constant γ, i.e. $\|m(x+e) - m(x)\| \leq \gamma\|e\|$, and the function $p : \mathbb{R}^n \to \mathbb{R}^q$ is of the form

$$p(c_i^T x) = \begin{bmatrix} p_1(c_{i_1}^T x) \\ \vdots \\ p_q(c_{i_q}^T x) \end{bmatrix}, \quad (27)$$

where $c_{i_j} \in \mathbb{R}^n$ and the components p_j are functions that satisfy the inequality condition

$$p_j^2(c_{i_j}^T x) > a^2\left(p_j(c_{i_j}^T x + z) - p_j(c_{i_j}^T x)\right)^2 \quad (28)$$

for all nonzero constants $a, z \in \mathbb{R}$ whenever $|c_{i_j}^T x|$ is sufficiently large. Furthermore, the matrix $Q(y)$ is a $q \times q$ diagonal matrix whose entries are only depending on the measurable output y, i.e. $Q(y)$ is of the form

$$Q(y) = \begin{bmatrix} Q_{11}(y) & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & Q_{qq}(y) \end{bmatrix}. \quad (29)$$

Proof: The proof of this corollary is based on the proof of Corollary 1 in [1]. A linear transformation $\Phi\xi = x$ is used to transform the system into the new coordinates ξ. By defining a new value function for the transformed system, Corollary 2 follows from Theorem 1. The only difference is that $Q(y)$ occurs in the feedback structure. As it can be rewritten as $Q(h(x))$, it gets under the state transformation $Q(h(\Phi\xi)) = Q(\tilde{h}(\xi))$. The system output equation changes as well to $y = \tilde{h}(\xi)$, and consequently $Q(y)$ remains a function of the output. ■

This result allows a variety of nonlinearities in the feedback structure. All polynomials of arbitrarily high degree in a single state or in a linear combination of the states, e.g. x_1^5 or $(x_1 + 2x_2)^3$, are allowed to be contained in the feedback, since all polynomials satisfy (8). Hence the results of [1] are included in Theorem 1. Furthermore, other nonlinearities like e.g. $|x_i|^3$ can now be included in the feedback as well. Notice that $|x_i|^3$ satisfies the inequality condition (8) since $a^2(p(x_i+z) - p(x_i))^2$ satisfies

$$\begin{aligned} a^2((|x_i+z|)^3 - |x_i|^3)^2 \\ < a^2(3|x_i|^2|z| + 3|x_i||z|^2 + |z|^3)^2, \end{aligned} \quad (30)$$

and the right-hand side of (30) is a polynomial in $|x_i|$ of degree 4, when z is considered to be constant. Thus, for sufficiently large $|x_i|$ the function $p^2(x_i) = |x_i|^6$ will always be greater than the right-hand side of (30) as it is a polynomial of a higher order. Therefore the condition (8) is satisfied for all nonzero constants $a, z \in \mathbb{R}$. Another important nonlinearity that can be included in the feedback structure is $\sqrt{|x_i|}$, which is not Lipschitz at zero. Notice that $a^2(\sqrt{|x_i+z|} - \sqrt{|x_i|})^2$ is bounded by a constant for large $|x_i|$ as

$$\begin{aligned} & a^2(\sqrt{|x_i+z|} - \sqrt{|x_i|})^2 \\ & \leq a^2(2|x_i| + |z| - 2\sqrt{|x_i|||x_i| - |z||}) \\ & \leq a^2\left(2|x_i| + |z| - 2|x_i|\sqrt{\left|1 - \frac{|z|}{|x_i|}\right|}\right) \\ & \leq a^2(2|x_i| + |z| - 2c|x_i|). \end{aligned}$$

Whereby the last line follows if $|x_i|$ is sufficiently large and thus c can be chosen arbitrarily close to one, e.g. 0.99. Hence, for any z, a, there exists a $c < 1$ $(2a^2(1-c) < 1)$ such that

$$2a^2(1-c)|x_i| + a^2|z| \leq |x_i| \tag{31}$$

holds for sufficiently large $|x_i|$. Furthermore, the state feedback may also be of the form $u = k(x_1, x_2, x_3) = -x_1 - x_2 - x_3 + x_1^3 x_3^3$, for example if x_1 is measurable, i.e. $y = x_1$. Corollary 2 allows to replace the single state x_i in each nonlinearity p_j by a linear combination of states $c_i^T x$, e.g. $\sqrt{|x_2 + x_3|}$ or $-x_1 - x_2 - x_3 + y^3|x_2 + x_3|^3$ are allowed as well.

III. NONLINEAR SEPARATION PRINCIPLE

In the previous section a new class of state feedbacks has been introduced that render input affine systems ISS w.r.t. the measurement error. In this section their robustness is used to establish a nonlinear separation principle. The following assumption on the observer is needed.

Assumption 3: Consider an observer for the control system (1) with a globally uniform asymptotic observer error dynamics

$$\dot{e} = a(e, x), \tag{32}$$

where $e \in \mathbb{R}^n$ and $e = x - \hat{x}$. More precisely, it is assumed that there exists a Lyapunov function W and a positive definite function α such that

$$W_e(e)a(e, x) < -\alpha(W(e)) \tag{33}$$

holds for all nonzero e, x.

Under the Assumptions 1-3 the question arises, whether the closed-loop, given by the input affine system (1), the convergent observer (32), and the stabilizing feedback (2), in the certainty-equivalence implementation

$$\begin{aligned} \dot{x} &= f(x) + G(x)k(x + e) \\ \dot{e} &= a(e, x) \end{aligned} \tag{34}$$

is globally asymptotically stable. Considering the properties of the closed-loop (34) and the convergence property (33) of the observer, the following statement can be made.

Theorem 3: Suppose that Assumptions 1-3 hold and that the state feedback is of the form (6), i.e. it guarantees input-to-state stability w.r.t. the measurement error. Then the closed-loop (34) is globally asymptotically stable.

Proof: Since the observer satisfies the convergence property (33), the error dynamics converges uniformly in x, i.e. the dynamics becomes $\dot{e} = a(t, e)$. Hence, the closed-loop system (34) has a behavior similar to a cascade system, with the error dynamics $\dot{e} = a(t, e)$ as the driving system and the controlled system $\dot{x} = f(x) + G(x)k(x + e)$ as driven system. It is a well known result of the ISS theory that the origin of such a cascade system is globally asymptotically stable if the driven system is ISS [6, Lemma 4.7]. ■

IV. EXAMPLES

In this section the theoretical results of the previous sections are applied to design nonlinear output feedback laws for various control problems. This demonstrates that the proposed class of (inverse) optimal state feedback is not too restrictive and that it can be applied to practical control problems.

A. Single-Link Robot Arm

A model for a single-link robot arm, shown in Figure 1, is given by [3]

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 - \frac{F_2}{J_2}x_2 - \frac{K}{J_2}x_1 - \frac{mgd}{J_2}(\cos x_1 - 1) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{K^2}{J_1 J_2 N^2}x_1 - \frac{K}{J_2 N}x_3 - \frac{F_1}{J_1}x_4 + u, \end{aligned} \tag{35}$$

where x_1 and x_3 represent the angular at the side of the motor respectively the arm and x_2 and x_4 represent the appropriate angular velocities. The mass m, positioned on the arm, is moved by a electrical motor M. The parameter K is the stiffness of the spring while N, J_1, J_2, F_1, F_2 are constants representing transmission, inertia, and friction. In the following the systems parameters are chosen as $K = 1, J_1 = 0.5, J_2 = 1, m = 0.1, d = 0.15, g = 9.81, N = 1, F_1 = 0.1, F_2 = 0.1$. The control input u is the torque delivered from the motor to the elasticity. The problem is to stabilize the origin of (35) using the measurement of the position, i.e. $y = x_1$. Since the system (35) is Lipschitz nonlinear, a linear feedback $u = k^T x$ in conjunction with a nonlinear observer is applied to stabilize the system. For the feedback design a quadratic positive definite function of the form

$$V(x) = x^T P x \tag{36}$$

is considered as a candidate for the solution of the HJB equation. The matrix P is chosen such that

$$A^T P + PA - PBR^{-1}B^T P = -Q$$

is satisfied with A as the linear part of (35), $B = [0\ 0\ 0\ 1]^T$,

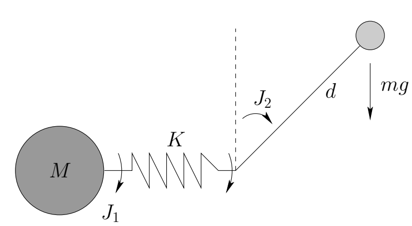

Fig. 1. Single-Link Robot Arm.

$R = 1$, and $Q = \text{diag}([5\ 2\ 2\ 2])$. Hence, the solution of the algebraic Riccati equation is

$$P = \begin{bmatrix} 9.2504 & 3.6285 & -0.8059 & 0.6797 \\ 3.6285 & 12.328 & 7.1056 & 2.606 \\ -0.8059 & 7.1056 & 9.1729 & 3.1486 \\ 0.6797 & 2.606 & 3.1486 & 2.6874 \end{bmatrix}$$

and the corresponding feedback law is given by

$$\begin{aligned} u^* &= -R^{-1}B^T P x \\ &= -0.6797x_1 - 2.606x_2 - 3.1486x_3 - 2.6874x_4. \end{aligned} \tag{37}$$

The feedback (37) is inverse optimal with respect to the cost functional (4) with $R = 1$ and $q(x) = -V_x(x)f(x) + \frac{1}{4}(V_x(x)G(x))^2$, that is positive definite. Since (37) is a linear (inverse) optimal state feedback, i.e. it satisfies condition (8), the controlled system is input-to-state stable with respect to the measurement error and (37) can be combined with any convergent observer, e.g. with the observer of [12]. Hence, all assumptions of Theorem 3 are satisfied, and the origin of the closed-loop system consisting of (35), (37), and a convergent observer is globally asymptotically stable.

B. Active Magnetic Bearing

Active magnetic bearings (AMB) are envisioned to replace conventional ball bearings in numerous future applications. For example, it is assumed that they will play an important role for high-speed flywheel batteries, that will be used for the energy storage in future spaceships. In the following, a simplified one-dimensional AMB system, depicted in Figure 2, is considered. A model for the system is given by [2]

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \epsilon x_3 + x_3|x_3| \\ \dot{x}_3 &= u, \end{aligned} \tag{38}$$

where x_1 represents the position q and x_2 the velocity of the mass. The third state x_3 is proportional to the magnetic flux. The system parameter $\epsilon \geq 0$ is in general smaller than one, and in the following it is $\epsilon = 0.5$. The voltage $V_1 = V$ and $V_2 = -V$ is used as the control input u.

The control objective is to stabilize the origin of the system (38) using the measurements of the rotor position x_1 and the velocity x_2, while the magnetic flux x_3 is not available for feedback. In the first step an (inverse) optimal feedback is designed. Motivated by the fact, that the system (38) consists of a linear system and a nonlinearity that belongs to the sector $[0, \infty]$, i.e. the structure of the system (38) is similar to the structure of a Lure' system [6], the candidate function

$$V(x) = x^T P x + \gamma \int_0^{x_3} s|s|ds \tag{39}$$

for the solution of the HJB equation is considered. V is positive definite since P is a positive definite matrix and the nonlinearity $\eta(s) = s|s|$ belongs to the sector $[0, \infty]$.

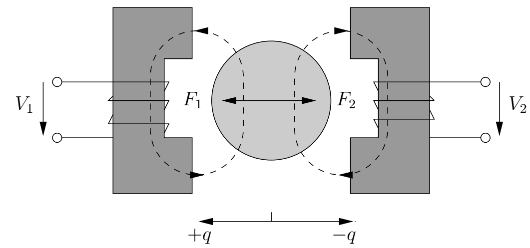

Fig. 2. Active Magnetic Bearing.

The matrix P is chosen as the solution of a Riccati matrix equation

$$A^T P + PA - PBR^{-1}B^T P = -Q,$$

where A is the linear part of (38), $B = [0\ 0\ 1]^T$ and the weightings are $Q = \text{diag}([1\ 9\ 1])$ and $R = 1$. Then, the matrix P is given by

$$P = \begin{bmatrix} 4.2634 & 4.5884 & 1 \\ 4.5884 & 17.5625 & 4.2634 \\ 1 & 4.2634 & 2.2942 \end{bmatrix}. \tag{40}$$

The parameter γ is chosen such that V satisfies the HJB equation, e.g. $\gamma = 9.1769$. A feedback law that minimizes a cost functional of the form (4) with $q(x) > x_1^2 + x_2^2 + x_3^2 + 12.53x_3^2|x_3| + 29.55x_3^4$ and $R = 1$ is given by $u^* = -\frac{1}{2}R^{-1}(V_x(x)G(x))^T = -x_1 - 4.2634x_2 - 2.2942x_3 - 4.5884x_3|x_3|$. Notice that this feedback law is separable into a globally Lipschitz part and the nonlinearity $\eta(x_3) = x_3|x_3|$ that satisfies (8), i.e.

$$\begin{aligned} &\eta^2(x_3) = |x_3|^4 > a^2((x_3 + z)|x_3 + z| - x_3|x_3|)^2 \\ &= a^2(z|x_3| + x_3|z| + z|z|)^2 = a^2(\eta(x_3 + z) - \eta(x_3))^2 \end{aligned}$$

holds for all nonzero constants $a, z \in \mathbb{R}$ whenever $|x_3|$ becomes sufficiently large. Hence all the assumptions of Theorem 1 are satisfied and u^* guarantees ISS w.r.t. the measurement error. A reduced-order observer [2] can be implemented to reconstruct the unmeasurable state x_3. Then, due to Theorem 3, the certainty-equivalence feedback

$$u = -x_1 - 4.2634x_2 - 2.2942\hat{x}_3 - 4.5884\hat{x}_3|\hat{x}_3|, \tag{41}$$

where $\hat{x}_3$ is the estimate of the observer [2], globally asymptotically stabilizes the origin of (38).

C. Nonlinear Control of Linear Systems

A nonlinear state feedback for a linear system, that satisfies for example a nonquadratic cost functionals, can be used to reduce overshoots or to get a better performance for large deviations form the operating point. One approach to control linear systems by nonlinear state feedbacks has been proposed in [4]. In the following, it is shown that for the approach in [4] a certainty-equivalence output feedback design is possible. Consider a linear time-invariant system of the form

$$\dot{x} = Ax + Bu,\ y = Cx \tag{42}$$

with $x \in \mathbb{R}^n$ and $u, y \in \mathbb{R}$. Following [4], the first step is to design a Linear Quadratic Regulator $u^*_{lin} = -B^T P x = -k^T x$ for the system (42) that satisfies a quadratic cost functional with $R = 1$ and $Q > 0$. Applying this feedback to the system (42) leads to the new system matrix $\tilde{A} = A - BB^T P$ with the right and left eigenvectors u_k and v_k, i.e. $\tilde{A}u_k = \lambda_k u_k, \quad \tilde{A}^T v_k^T = \lambda_k v_k^T$, where λ_k are the corresponding eigenvalues. It is assumed that the eigenvalues λ_k are distinct, whereon $\{u_k\}, \{v_k\}$ are linearly independent, and that $\{u_k\}, \{v_k\}$ are normalized such that $u_k v_j = \delta_{kj}$, $(k, j = 1, \ldots, n)$. In order to find a function V that satisfies the HJB equation one defines $\zeta_l(x) = (v_l x)^4$, where v_l is a left eigenvector of $\tilde{A}$, i.e. $\zeta_l(x)$ satisfies

$$x^T \tilde{A}^T \frac{\partial}{\partial x} \zeta_l(x) = \mu_l \zeta_l(x). \tag{43}$$

Introducing $\phi(x) = \frac{1}{4\mu_l}\zeta_l(x)$, the positive definite function

$$V(x) = x^T P x + \phi(x) = x^T P x + \frac{1}{4\mu_l}(v_l x)^4 \tag{44}$$

defines a nonlinear feedback law for (42) by setting

$$u = -B^T \frac{\partial V}{\partial x} = -B^T P x - \frac{1}{\mu_l} B^T v_l^T (v_l x)^3. \tag{45}$$

This feedback is of the desired form $u = m(x) + p(c_i^T x)$. In order to show the optimality of this feedback law, one can use V as a solution candidate for the HJB equation, i.e.

$$\begin{aligned} & V_x(x) f(x) - \frac{1}{4} V_x(x) G(x) (V_x(x) G(x))^T \\ & = x^T (A^T P + PA - PBB^T P) x \\ & + \frac{\partial \phi}{\partial x}(Ax - BB^T P x) - \frac{1}{4} \left(\frac{\partial \phi}{\partial x}\right)^T \left(\frac{\partial \phi}{\partial x}\right). \end{aligned} \tag{46}$$

Using $A - BB^T P = \tilde{A}$ and $\phi(x) = \frac{1}{4\mu}\zeta(x)$, with $\zeta(x)$ satisfying (43), equation (46) turns into

$$-x^T Q x - \frac{1}{4}(v_l x)^4 - \frac{1}{4}\left(\frac{\partial \phi}{\partial x}\right)^T \left(\frac{\partial \phi}{\partial x}\right) < 0. \tag{47}$$

Thus, V solves the HJB equation and the feedback (45) is (inverse) optimal. All assumptions of Corollary 2 are satisfied and the feedback guarantees ISS w.r.t. the measurement error. Hence, a certainty-equivalence implementation of (45) with a convergent observer, e.g. a Luenberger observer, globally asymptotically stabilizes the origin of (42). The design of such a control feedback is illustrated via the system

$$\dot{x}_1 = x_2, \; \dot{x}_2 = -2x_1 - 3x_2 + u, \tag{48}$$

where the first part of the design is already done and $A = \tilde{A}$. In the following, the polynomial feedback u_{nl} is designed. The feedback design is done, using the left-eigenvector $v_l = [2 \;\; 1]^T$, whereon the function $\zeta(x) = (2x_1 + x_2)^4$ is defined. The method proposed above leads to the HJB solution candidate $V(x) = \frac{1}{16}(2x_1 + x_2)^4$ which defines the nonlinear part of the optimal feedback law as $u^*_{nl} = p(v^T x) = -\frac{1}{4}(2x_1 + x_2)^3$. Simulations show that the nonlinear term in (45) makes the system converge faster to the origin for large initial conditions or it avoids a large overshoot of the state x_1.

V. CONCLUSIONS

The contribution of this paper is twofold. The main result of the first part is a generalization of the results in [1]. It has been shown that (inverse) optimal state feedback laws, that are separable into a global Lipschitz part and a nonlinearity which satisfies a certain inequality condition, guarantee ISS w.r.t. the measurement error. This result allows that these feedback laws include polynomial functions as well as other strongly nonlinear (non-Lipschitz) functions like for example $|x_i|^9$ or $y x_i^4$. Furthermore it is shown that the non-Lipschitz term can be scaled with an output-depending nonlinear function. An important consequence of the established results is that the closed-loop consisting of this class of state feedbacks and any uniformly convergent observer is globally asymptotically stable. Hence, new conditions for a globally stabilizing nonlinear output feedback design are presented in this paper.

In the second part, the practical applicability of the established nonlinear separation principle has been illustrated on several examples. In particular, nonlinear output feedback controllers have been designed for a single-link robot arm and an active magnetic bearing.

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