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Semi-global Robust Output Regulation for a Class of Nonlinear Systems Using Output Feedback

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Abstract— **This paper studies the semi-global robust output regulation problem for the class of nonlinear affine systems in normal form. The same problem was studied before by Khalil under the assumption that the solution of the regulator equations is polynomial. By using the nonlinear internal model approach, we have relaxed the polynomial assumption on the solution of the regulator equations.**

I. Introduction

The output regulation problem aims to design a feedback control law for a given plant such that the output of the plant can asymptotically track a class of reference inputs and/or reject a class of disturbances while maintaining the closed-loop stability. Here both the class of reference inputs and the class of the disturbances are generated by autonomous differential equations called exosystem. The problem has been extensively studied for linear systems since 1970s [2], [3], [4], and for nonlinear systems since 1990s [9], [10], [12]. The robust version of this problem is addressed in [1], [5], [7], [8] where only small parameter perturbations and/or local convergence are considered. Also, the solvability of the various versions of the regional and/or semi-global robust output regulation problem has been studied in [11], [13], [15], [16], [18]. Nevertheless, the results of all these papers rely on a key assumption that the regulator equations of the system admit a polynomial solution. This assumption essentially requires that the nonlinear systems contain only polynomial nonlinearities. Recently, a general framework for studying the robust output regulation problem was proposed in [6]. Under this framework, the robust output regulation problem for a given plant can be systematically converted into a robust stabilization problem for an appropriately defined augmented system. Moreover, this general framework admits a class of nonlinear internal models such that the polynomial condition [11], [16] can be relaxed by some milder assumption. This general framework has been successfully applied to solve the global robust output regulation problem for lower triangular nonlinear systems [6]. In this paper, we will further apply this framework to study a semi-global robust output regulation problem for a class of nonlinear systems in normal form. As in [6], our approach consists of two steps. First, convert the semi-global robust output regulation problem of the given system into a semi-global robust stabilization problem of an appropriately defined augmented system. Second, solve the semi-global robust stabilization problem of the augmented system using the approach adapted from the work of [17] and [18]. It should be noted that while the first step follows straightforwardly from the general framework developed in [6], the accomplishment of the second step is non-trivial. This is because the zero dynamics of the augmented system consist of two parts. The first part is a variation of the zero dynamics of the given plant, and the second part is the dynamics that govern the nonlinear internal model. Appropriate assumptions have to be made on the plant and the internal model so that the robust stabilization problem of the augmented system is tractable.

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II. Problem Formulation

Consider a nonlinear system described as follows,

$$\begin{array}{rcl} \dot{z} &=& f_0(z,x,v,w) \\ \dot{x}_i &=& x_{i+1}, \;\; i=1,\cdots,i-1 \\ \dot{x}_r &=& f_1(z,x,v,w)+u \\ \dot{v} &=& A_1 v \\ e &=& x_1 - q(v,w) \end{array} \quad (1)$$

where $z \in \Re^m$, $x = \text{col}(x_1,\cdots,x_r)$ with $x_i \in \Re$, $i=1,\cdots,r$, are the plant states, $u(t) \in \Re$ is the control input, $e(t) \in \Re$ is the tracking error, $v(t) \in \Re^q$ is the exogenous signal representing the disturbance and/or the reference input, and $w \in \Re^N$ is the uncertain parameter. All the functions are sufficiently smooth with $f_0(0,0,0,w)=0$, $f_1(0,0,0,w)=0$, and $q(0,w)=0$. The exosystem is neutrally stable, i.e., all the eigenvalues of A_1 are simple and have zero real parts.

The class of output feedback control laws considered here are described by

$$\begin{array}{rcl} u &=& k(\zeta,e) \\ \dot{\zeta} &=& f_z(\zeta,e) \end{array} \quad (2)$$

where ζ is the compensator state vector of dimension n_c to be specified later.

Semi-global robust output regulation problem. Given compact sets $Z \subset \Re^m$, $X \subset \Re^r$, $Z_c \subset \Re^{n_c}$, $V_0 \subset \Re^q$ and $W \subset \Re^N$, which contain the origins of the respective Euclidean spaces, a controller of the form (2) is said to solve the *robust output regulation problem* for the system (1) with respect to $W \times V_0 \times Z \times X \times Z_c$ if the closed-loop system (1)-(2) with its state being denoted by $x_c = \text{col}(z,x,\zeta)$ has the following two properties:

Property 1. For any $w \in W$, $v(0) \in V_0$, and $x_c(0) \in X_c$ where $X_c = Z \times X \times Z_c$, the trajectories of the closed-loop system starting from $x_c(0)$ and $v(0)$ exist and are bounded for all $t \geq 0$.

Property 2. The tracking error $e(t)$ of the trajectories described in Property 1 approaches zero asymptotically, i.e., $\lim_{t\to\infty} e(t) = 0$.

If for any given triple $W \times V_0 \times X_c$, there exists a controller of the form (2) such that the closed-loop system (1)-(2) has Properties 1 and 2, then we say that the *semi-global robust output regulation problem* of the system (1) is solvable. ▪

Remark 2.1: Since the exosystem is neutrally stable, for any $v(0) \in V_0$, there exists a compact set $V \subset \Re^q$ containing the origin of $\Re^q$ such that $v(t) \in V$ for all $t \geq 0$. ▪

Under the general framework for the robust output regulation problem developed in [6], the semi-global robust output regulation problem can be converted into a semi-global robust stabilization problem for an appropriately defined augmented system. To fulfill this conversion, we need the following standard assumptions.

A1: Let $\mathbf{x}(v,w) = \mathrm{col}(\mathbf{x}_1(v,w), \cdots, \mathbf{x}_r(v,w))$ where

$$\begin{aligned}\mathbf{x}_1(v,w) &= q(v,w)\\ \mathbf{x}_i(v,w) &= \frac{\partial \mathbf{x}_{i-1}(v,w)}{\partial v}A_1 v, \quad i = 2,\cdots,r.\end{aligned}$$

There exists a sufficiently smooth function $\mathbf{z}(v,w)$ with $\mathbf{z}(0,0) = 0$ such that, for all $v \in \Re^q$, $w \in \Re^N$,

$$\frac{\partial \mathbf{z}(v,w)}{\partial v}A_1 v = f_0(\mathbf{z}(v,w), \mathbf{x}(v,w), v, w).$$

Remark 2.2: Under assumption A1, the regulator equations associated with the system (1) are solvable by $(\mathbf{z}(v,w), \mathbf{x}(v,w), \mathbf{u}(v,w))$ where

$$\mathbf{u}(v,w) = \frac{\partial \mathbf{x}_r(v,w)}{\partial v}A_1 v - f_1(\mathbf{z}(v,w), \mathbf{x}(v,w), v, w).$$

However, the solvability of the regulator equations is insufficient for solving the robust output regulation problem, some additional conditions have to be imposed on the solution of the regulator equations [1], [5], [8], [11], [16]. Despite the different appearances of these conditions, they amount to requiring the system admit a linear internal model, which in turn essentially requires the system only contain polynomial nonlinearities. Recently, a much less restrictive condition was given in [6] and will be stated below as assumption A2. Let us first note that, if $\pi(v(t),w)$ is a polynomial function of v or a trigonometric polynomial function of t along the trajectory of the exosystem, then there exist an integer r and real numbers $a_1, \cdots, a_r$ such that $\pi(v(t),w)$ satisfies a differential equation of the form [1] and [5],

$$\frac{d^r\pi(v(t),w)}{dt^r} - a_1\pi(v(t),w) - a_2\frac{d\pi(v(t),w)}{dt} - \cdots - a_r\frac{d^{(r-1)}\pi(v(t),w)}{dt^{(r-1)}} = 0 \qquad (3)$$

for all trajectories $v(t)$ of the exosystem and all $w \in \Re^N$. We will call the monic polynomial $P(\lambda) = \lambda^r - a_r\lambda^{r-1} - \cdots - a_2\lambda - a_1$ a zeroing polynomial of $\pi(v,w)$ if $\pi(v,w)$ satisfies (3), and call $P(\lambda)$ a minimal zeroing polynomial of $\pi(v,w)$ if $P(\lambda)$ is a zeroing polynomial of $\pi(v,w)$ of least degree. Let $\pi_i(v,w)$, $i = 1,\cdots,I$, for some positive integer I, be I polynomials in v. They are called pairwise coprime if their minimal zeroing polynomials $P_1(\lambda),\cdots,P_I(\lambda)$ are pairwise coprime. ▪

A2: There exist pairwise coprime polynomials $\pi_1(v,w)$, $\cdots$, $\pi_I(v,w)$ with r_1, $\cdots$, r_I being the degrees of their minimal zeroing polynomials $P_1(s)$, $\cdots$, $P_I(s)$, and sufficiently smooth function $\Gamma : \Re^{r_1+\cdots+r_I} \to \Re$ vanishing at the origin such that, for all trajectories $v(t)$ of the exosystem, and $w \in \Re^N$,

$$\begin{aligned}&\mathbf{u}(v,w)\\ &= \Gamma\Big(\pi_1(v,w), \dot{\pi}_1(v,w), \cdots, \frac{d^{(r_1-1)}\pi_1(v,w)}{dt^{(r_1-1)}},\\ &\cdots, \pi_I(v,w), \dot{\pi}_I(v,w), \cdots, \frac{d^{(r_I-1)}\pi_1(v,w)}{dt^{(r_I-1)}}\Big)\end{aligned}$$

and

the pair (Ψ_i, Φ_i) is observable, $i = 1,\cdots,I$

where $\Psi = [\Psi_1,\cdots,\Psi_I]$ is the Jacobian matrix of Γ at the origin, and Φ_i is the companion matrix of $P_i(s)$.

Remark 2.3: It is noted that assumption A2 includes the polynomial condition in [11], [16] as a special case. ▪

Remark 2.4: Under assumptions A1-A2, let

$$\theta(v,w) = T\mathrm{col}(\theta_1(v,w),\cdots,\theta_I(v,w))$$

where

$$\theta_i(v,w) = \mathrm{col}(\pi_i(v,w), \dot{\pi}_i(v,w), \cdots, \frac{d^{(r_i-1)}\pi_i(v,w)}{dt^{(r_i-1)}}),$$

$i = 1,\cdots,I$, and T is any nonsingular matrix with an appropriate dimension. Then, it is easy to verify that

$$\begin{aligned}\dot{\theta}(v,w) &= \alpha(\theta(v,w))\\ \mathbf{u}(v,w) &= \beta(\theta(v,w))\end{aligned} \qquad (4)$$

where $\alpha(\theta) = T\Phi T^{-1}\theta$ with $\Phi = \mathrm{diag}(\Phi_1,\cdots,\Phi_I)$, and $\beta(\theta) = \Gamma(T^{-1}\theta)$. The triple $\{\theta,\alpha,\beta\}$ is called the steady state generator of system (1) with output u [6]. Since the pair (Ψ,Φ) is observable, for any controllable pair (M,N) with M a Hurwitz matrix of dimension $r_\eta = r_1+\cdots+r_I$ and N a column vector, there exists a nonsingular matrix T satisfying the Sylvester equation $T\Phi - MT = N\Psi$. Define

$$\dot{\eta} = \gamma(\eta,u) = M\eta + N(u - \beta(\eta) + \Psi T^{-1}\eta) \qquad (5)$$

where $\eta \in \Re^{r_\eta}$. System (5) is also introduced in [6] and is called an internal model of (1) with output u. It is noted that when the function $\beta(\cdot)$ is linear, (5) reduces to a linear internal model proposed in [19]. ▪

The combination of the system (1) and the internal model (5) is called the augmented system. Under the coordinate and input transformation,

$$\begin{array}{ll} \bar{\eta} = \eta - \theta(v,w), & \bar{x}_i = x_i - \mathbf{x}_i(v,w), \quad i = 1,\cdots,r \\ \bar{z} = z - \mathbf{z}(v,w), & \bar{u} = u - \beta(\eta) \end{array} \tag{6}$$

the augmented system takes the following form

$$\begin{array}{rcl} \dot{\bar{\eta}} &=& (M + N\Psi T^{-1})\bar{\eta} + N\bar{u} \\ \dot{\bar{z}} &=& \bar{f}_0(\bar{z}, \bar{x}_1, \cdots, \bar{x}_r, v, w) \\ \dot{\bar{x}}_i &=& \bar{x}_{i+1}, \ \ i = 1, \cdots, i-1 \\ \dot{\bar{x}}_r &=& \bar{f}_1(\bar{\eta}, \bar{z}, \bar{x}_1, \cdots, \bar{x}_r, v, w) + \bar{u} \\ \dot{v} &=& A_1 v \\ e &=& \bar{x}_1 \end{array} \tag{7}$$

where

$$\begin{array}{r} \bar{f}_0(\bar{z}, \bar{x}_1, \cdots, \bar{x}_r, v, w) = \\ f_0(\bar{z} + \mathbf{z}(v,w), \bar{x}_1 + \mathbf{x}_1(v,w), \cdots, \bar{x}_r + \mathbf{x}_r(v,w), v, w) \\ -f_0(\mathbf{z}(v,w), \mathbf{x}_1(v,w), \cdots, \mathbf{x}_r(v,w), v, w) \end{array}$$

and

$$\begin{array}{r} \bar{f}_1(\bar{\eta}, \bar{z}, \bar{x}_1, \cdots, \bar{x}_r, v, w) = \\ f_1(\bar{z} + \mathbf{z}(v,w), \bar{x}_1 + \mathbf{x}_1(v,w), \cdots, \bar{x}_r + \mathbf{x}_r(v,w), v, w) \\ -f_1(\mathbf{z}(v,w), \mathbf{x}_1(v,w), \cdots, \mathbf{x}_r(v,w), v, w) \\ +\beta(\bar{\eta} + \theta(v,w)) - \beta(\theta(v,w)). \end{array}$$

It is clear that $\bar{f}_i(0, \cdots, 0, v, w) = 0$, $i = 1, 2$, that is, the origin $\text{col}(\bar{\eta}, \bar{z}, \bar{x}) = 0$ is the equilibrium point of the unforced augmented system (7) for all trajectories $v(t)$ of the exosystem and any $w \in \Re^N$.

Let $d(t) = \text{col}(v(t), w) \in D = V \times W$, then system (7) is a special case of the following more general nonlinear systems

$$\begin{array}{rcl} \dot{x} &=& f(x, u, d(t)) \\ e &=& h(x, d(t)) \end{array} \tag{8}$$

where x is the n-dimensional system state, u the m-dimensional control input, e the p-dimensional output, $d(t)$ the n_d-dimensional uncertain parameters, and $f(0, 0, d(t)) = 0$ and $h(0, d(t)) = 0$ for all $d(t) \in D$.

Semi-global robust stabilization problem. Consider the uncertain nonlinear control system (8) and a dynamic output feedback controller of the form

$$\begin{array}{rcl} u &=& k(\xi, e) \\ \dot{\xi} &=& \varphi(\xi, e) \end{array} \tag{9}$$

where $\xi \in \Re^{n_\xi}$ for some integer n_ξ, and k and φ are globally defined sufficiently smooth functions satisfying $k(0,0) = 0$ and $\varphi(0,0) = 0$. Let $D \subset \Re^{n_d}$, $X \subset \Re^n$, and $\Xi \subset \Re^{n_\xi}$ be some given compact sets containing the origins of the respective Euclidian spaces. If there exists a dynamic output feedback controller of the form (9) such that, for any $x(0) \in X$, $\xi(0) \in \Xi$ and $d(t) \in D$, the solution of the closed-loop system (8) and (9) exists and is bounded for all $t \geq 0$ and all the states converge to the origin asymptotically, then we say that the controller (9) solves the *robust stabilization problem* of the system (8) with respect to $D \times X \times \Xi$. If for every triple (D, X, Ξ), there exists a dynamic output feedback controller of the form (9) that solves the robust stabilization problem of the system (8) with respect to $D \times X \times \Xi$, then we say that the *semi-global stabilization problem* of the system (8) is solvable by output feedback control. ∎

As for the global robust output regulation case studied in [6], the following proposition shows that the solvability of the semi-global robust stabilization problem of the augmented system (7) leads to that of the semi-global robust output regulation problem of the original system (1).

Proposition 2.1: Consider the augmented system (7) with the uncertainty $d(t) = \text{col}(v(t), w) \in D = V \times W$. If a dynamic output feedback controller

$$\begin{array}{rcl} \bar{u} &=& k(\xi, e) \\ \dot{\xi} &=& \varphi(\xi, e) \end{array} \tag{10}$$

solves the robust stabilization problem of (7) with respect to $D \times \bar{Z}_\eta \times \bar{Z} \times \bar{X} \times \Xi$ where $\bar{Z}_\eta \times \bar{Z} \times \bar{X} \times \Xi \subset \Re^{r_\eta} \times \Re^m \times \Re^r \times \Re^{n_\xi}$ are compact sets containing the origin, then the dynamic output feedback controller of the form

$$\begin{array}{rcl} u &=& \beta(\eta) + k(\xi, e) \\ \dot{\xi} &=& \varphi(\xi, e) \\ \dot{\eta} &=& M\eta + N(k(\xi, e) + \Psi T^{-1}\eta). \end{array} \tag{11}$$

solves the robust output regulation problem of the original system (1) with respect to $D_0 \times Z_\eta \times Z \times X \times \Xi$ where $D_0 = V_0 \times W$ and

$$\begin{array}{rcl} X &=& \{x = \bar{x} + \mathbf{x}(v,w), \ \ \bar{x} \in \bar{X}, \ \ \text{col}(v,w) \in D\} \\ Z &=& \{z = \bar{z} + \mathbf{z}(v,w), \ \ \bar{z} \in \bar{Z}, \ \ \text{col}(v,w) \in D\} \\ Z_\eta &=& \{\eta = \bar{\eta} + \theta(v,w), \ \ \bar{\eta} \in \bar{Z}_\eta, \ \ \text{col}(v,w) \in D\}. \end{array}$$

∎

III. Robust Stabilization via Output Feedback

Due to Proposition 2.1, we only need to solve the semi-global robust stabilization problem of the augmented system (7) in order to solve the semi-global robust regulation problem of the original system (1). System (7) is in the standard normal form. The robust stabilization problem of the system of the form (7) has been well studied in literatures [17] and [18]. In particular, Khalil et al gave solvability conditions for the semi-global robust stabilization of the system of the form (7) when d is constant [17]. To make use of the result of [17], let us introduce another transformation $\tilde{\eta} = \bar{\eta} - N\bar{x}_r$. Then, the system (7) is transformed to

$$\begin{array}{rcl} \dot{\tilde{\eta}} &=& Q(\tilde{\eta}, \bar{z}, \bar{x}_1, \cdots, \bar{x}_r, v, w) \\ \dot{\bar{z}} &=& \bar{f}_0(\bar{z}, \bar{x}_1, \cdots, \bar{x}_r, v, w) \\ \dot{\bar{x}}_i &=& \bar{x}_{i+1}, \ \ i = 1, \cdots, i-1 \\ \dot{\bar{x}}_r &=& \bar{f}_1(\tilde{\eta} + N\bar{x}_r, \bar{z}, \bar{x}_1, \cdots, \bar{x}_r, v, w) + \bar{u} \\ e &=& \bar{x}_1 \end{array} \tag{12}$$

where

$$Q(\tilde{\eta}, \bar{z}, \bar{x}_1, \cdots, \bar{x}_r, v, w) =$$
$$M\tilde{\eta} + MN\bar{x}_r - N(\beta^{[2]}(\tilde{\eta} + N\bar{x}_r + \theta) - \beta^{[2]}(\theta)) -$$
$$N(f_1(\bar{z} + \mathbf{z}(v,w), \bar{x}_1 + \mathbf{x}_1(v,w), \cdots, \bar{x}_r + \mathbf{x}_r(v,w), v, w)$$
$$-f_1(\mathbf{z}(v,w), \mathbf{x}_1(v,w), \mathbf{x}_1(v,w), \cdots, \mathbf{x}_r(v,w), v, w))$$

where $\beta^{[2]}(x) = \beta(x) - \Psi T^{-1}x$. It is clear that $Q(0, \cdots, 0, v, w) = 0$. Denote $d(t) = \text{col}(v(t),\ w)$, $\tilde{z} = \text{col}(\tilde{\eta}, \bar{z})$ and $\bar{x} = \text{col}(\bar{x}_1, \cdots, \bar{x}_r)$, then system (12) can be rewritten as

$$\begin{aligned} \dot{\tilde{z}} &= \phi_0(\tilde{z}, \bar{x}, d) \\ \dot{\bar{x}} &= A\bar{x} + B(\phi_1(\tilde{z}, \bar{x}, d) + \bar{u}) \\ e &= \bar{x}_1 \end{aligned} \tag{13}$$

where $A = \begin{bmatrix} 0 & I_{r-1} \\ 0 & 0 \end{bmatrix}_{r\times r}$, $B = \begin{bmatrix} 0_{(r-1)\times 1} \\ 1 \end{bmatrix}_{r\times 1}$ and

$$\begin{aligned} \phi_0(\tilde{z}, \bar{x}, d) &= \begin{bmatrix} Q(\tilde{\eta}, \bar{z}, \bar{x}_1, \cdots, \bar{x}_r, v, w) \\ \bar{f}_0(\bar{z}, \bar{x}_1, \cdots, \bar{x}_r, v, w) \end{bmatrix} \\ \phi_1(\tilde{z}, \bar{x}, d) &= \bar{f}_1(\tilde{\eta} + N\bar{x}_r, \bar{z}, \bar{x}_1, \cdots, \bar{x}_r, v, w). \end{aligned}$$

It is noted that $\phi_0(0, 0, d) = 0$ for all d.

It can be deduced from Corollary 1 of [17] that system (13) is semi-globally stabilizable if the system satisfies Conditions 1 to 3 listed below.

Condition 1: There exist a C^1 function $W_1 : \Re^{m+r_\eta} \to \Re^+$, and class $\mathcal{K}_\infty$ functions α_i, i = 1, 2, 3, and γ_1 such that, for all $(\tilde{z}, \bar{x}) \in \Re^{m+r_\eta} \times \Re^r$, and all $d \in D$,

$$\alpha_1(\|\tilde{z}\|) \le W_1(\tilde{z}) \le \alpha_2(\|\tilde{z}\|)$$
$$\frac{\partial W_1(\tilde{z})}{\partial \tilde{z}} \phi_0(\tilde{z}, \bar{x}, d) \le -\alpha_3(\|\tilde{z}\|), \forall \|\tilde{z}\| \ge \gamma_1(\|\bar{x}\|).$$

Before introducing Condition 2, the following notations are defined. Let K be such that $A + BK$ is Hurwitz, and the symmetric positive definite matrix P be the solution of the Lyapunov equation

$$P(A + BK) + (A + BK)^T P = -I.$$

Also let

$$\begin{aligned} \Omega^1_{c_1} &= \{\ \tilde{z} \in \Re^{m+r_\eta} : W_1(\tilde{z}) \le c_1\} \\ \Omega^2_{c_2} &= \{\bar{x} \in \Re^r : W_2(\bar{x}) \le c_2\} \end{aligned}$$

where $W_2(\bar{x}) = \bar{x}^T P \bar{x}$.

Condition 2: For each pair of $c_1, c_2 > 0$ satisfying

$$\alpha_4(\sqrt{c_2/\lambda_{\min}(P)}) \le c_1$$

where $\lambda_{\min}(P)$ is the minimal eigenvalue of the matrix P, and $\alpha_4 = \alpha_2 \circ \gamma_1$, there exists a scalar nonnegative locally Lipschitz function $\rho(\bar{x})$ such that

$$|\phi_1(\tilde{z}, \bar{x}, d) - K\bar{x}| \le \rho(\bar{x})$$

for all $(\tilde{z}, \bar{x}) \in \Omega^1_{c_1} \times \Omega^2_{c_2}$, all $d \in D$.

Condition 3: In some neighborhood of $(\tilde{z}, \bar{x}) = 0$, there exist a C^1 positive definite function $\tilde{W}_1 : \Re^{m+r_\eta} \to \Re^+$, such that, for all $d \in D$,

$$\frac{\partial \tilde{W}_1(\tilde{z})}{\partial \tilde{z}} \phi_0(\tilde{z}, 0, d) \le -\lambda_1 \|\tilde{z}\|^2$$
$$|\phi_1(\tilde{z}, 0, d)| \le \lambda_2 \|\tilde{z}\|$$
$$\frac{\partial \tilde{W}_1(\tilde{z})}{\partial \tilde{z}} [\phi_0(\tilde{z}, \bar{x}, d) - \phi_0(\tilde{z}, 0, d)] \le \lambda_3 \|\tilde{z}\| \|\bar{x}\|$$

where $\lambda_i > 0$, $i = 1, 2, 3$.

Remark 3.1: Note that, in [17], the uncertainty d is assumed to be constant. It is not difficult to conclude that the results of [17] also apply to the case where d is time-varying as long as d belongs to a compact set. ∎

It is noted that the zero dynamics of the augmented system (7) consist of the zero dynamics of the original plant and the dynamics governing the internal model. Therefore, even though the zero dynamics of the plant satisfy Conditions 1 to 3, there is no guarantee that the zero dynamics of the augmented system also satisfy Conditions 1 to 3. Therefore, it is interesting to further study the problem of imposing conditions on the plant and the internal model (5) so that the zero dynamics of the augmented system can satisfy Conditions 1 to 3.

A3: There exist a C^1 function $V_1(\bar{z})$ and class $\mathcal{K}_\infty$ functions $\bar{\alpha}_i$, $i = 1, 2, 3$ and σ, independent on $d(t)$ such that, for all $(\bar{z}, \bar{x}) \in \Re^m \times \Re^r$, and all $d \in D$,

$$\bar{\alpha}_1(\|\bar{z}\|) \le V_1(\bar{z}) \le \bar{\alpha}_2(\|\bar{z}\|)$$
$$\frac{\partial V_1(\bar{z})}{\partial \bar{z}} \bar{f}_0(\bar{z}, \bar{x}, d) \le -\bar{\alpha}_3(\|\bar{z}\|) + \sigma(\|\bar{x}\|),$$

and in some neighborhood of $\bar{z} = 0$,

$$\left\| \frac{\partial V_1(\bar{z})}{\partial \bar{z}} \right\| \le \bar{\lambda}_1 \|\bar{z}\|,\ \bar{\lambda}_1 > 0.$$

Moreover, the function $\bar{\alpha}_3(\cdot)$ satisfies $\lim_{s\to 0^+} \sup \frac{\bar{\alpha}_3^{-1}(s^2)}{s} < +\infty$.

A4: Let the symmetric positive definite matrix P_M be the solution of the Lyapunov equation

$$P_M M + M^T P_M = -I.$$

Then there exists a positive number R, such that, for all $\tilde{\eta}$ and δ,

$$-2\tilde{\eta}^T P_M N(\beta^{[2]}(\tilde{\eta} + \delta) - \beta^{[2]}(\delta)) \le (1 - R)\tilde{\eta}^T \tilde{\eta}.$$

Theorem 3.1: Suppose the system (13) satisfies assumptions A3 and A4, then there exists some class $\mathcal{K}_\infty$ function $W_1(\tilde{z})$ such that $\alpha_1(\|\tilde{z}\|) \le W_1(\tilde{z}) \le \alpha_2(\|\tilde{z}\|)$ for some class $\mathcal{K}_\infty$ functions α_i, $i = 1, 2$, and

$$\frac{\partial W_1(\tilde{z})}{\partial \tilde{z}} \phi_0(\tilde{z}, \bar{x}, d(t)) \le -\alpha_3(\|\tilde{z}\|),\ \forall \|\tilde{z}\| \ge \gamma_1(\|\bar{x}\|)$$

for some class $\mathcal{K}_\infty$ functions γ_1 and α_3. Further, suppose the system (13) satisfies the additional assumption

A5: For each pair of $c_1, c_2 > 0$ satisfying

$$\alpha_4(\sqrt{c_2/\lambda_{\min}(P)}) \le c_1,$$

where $\alpha_4 = \alpha_2 \circ \gamma_1$, there exists a scalar nonnegative locally Lipschitz function $\rho(\bar{x})$ such that

$$|\phi_1(\tilde{z}, \bar{x}, d) - K\bar{x}| \leq \rho(\bar{x})$$

for all $(\tilde{z}, \bar{x}) \in \Omega^3_{c_1} \times \Omega^2_{c_2}$, and all $d \in D$ where $\Omega^3_{c_1} = \{ \tilde{z} \in \Re^{m+r_\eta} : V_1(\bar{z}) \leq c_1, V_2(\tilde{\eta}) \leq c_1\}$ and $V_2(\tilde{\eta}) = \tilde{\eta}^T P_M \tilde{\eta}$.

Then, the semi-global robust stabilization problem of the system (13) is solvable by a dynamic output feedback controller of the form (9). ∎

Proof. It suffices to prove that system (13) satisfies Condition 1 under assumptions A3 and A4, and Conditions 2 and 3 under assumptions A3-A5.

Verification of Condition 1: Consider the $\tilde{z}$-subsystem of (13), $\dot{\tilde{z}} = \phi_0(\tilde{z}, \bar{x}, d)$, that is,

$$\dot{\tilde{\eta}} = Q(\tilde{\eta}, \bar{z}, \bar{x}, d(t)) \quad (14)$$
$$\dot{\bar{z}} = \bar{f}_0(\bar{z}, \bar{x}, d(t)) \quad (15)$$

where $d(t) = \text{col}(v(t), w)$. It is clear that $\phi_0(0, 0, d) = 0$.

On one hand, system (14) can be rewritten as

$$\begin{aligned} \dot{\tilde{\eta}} &= Q(\tilde{\eta}, \bar{z}, \bar{x}, d(t)) \\ &= M\tilde{\eta} - N(\beta^{[2]}(\tilde{\eta} + \delta) - \beta^{[2]}(\delta)) + \psi(\bar{z}, \bar{x}, d(t)) \end{aligned}$$

where $\delta = N\bar{x}_r + \theta$, and

$$\begin{aligned} &\psi(\bar{z}, \bar{x}, d(t)) = MN\bar{x}_r - N(\beta^{[2]}(N\bar{x}_r + \theta) - \beta^{[2]}(\theta)) \\ &-Nf_1(\bar{z} + \mathbf{z}(v, w), \bar{x}_1 + \mathbf{x}_1(v, w), \cdots, \bar{x}_r + \mathbf{x}_r(v, w), v, w) \\ &+Nf_1(\mathbf{z}(v, w), \mathbf{x}_1(v, w), \mathbf{x}_1(v, w), \cdots, \mathbf{x}_r(v, w), v, w). \end{aligned}$$

Under assumption A4, the derivative of $V_2(\tilde{\eta})$ along system (14) is

$$\begin{aligned} \frac{dV_2(\tilde{\eta})}{dt} &= -\tilde{\eta}^T\tilde{\eta} - 2\tilde{\eta}^T P_M N(\beta^{[2]}(\tilde{\eta} + \delta) - \beta^{[2]}(\delta)) \\ &\quad +2\tilde{\eta} P_M \psi(\bar{z}, \bar{x}, d(t)) \\ &\leq -R\tilde{\eta}^T\tilde{\eta} + 2\tilde{\eta} P_M \psi(\bar{z}, \bar{x}, d(t)) \\ &\leq -\frac{R}{2}\|\tilde{\eta}\|^2 + \frac{2}{R}\|P_M\psi(\bar{z}, \bar{x}, d(t))\|^2. \end{aligned}$$

Noting that $\psi(0, 0, d(t)) = 0$, it is not difficult to find C^1 class $\mathcal{K}_\infty$ functions $\sigma_{\bar{z}}$ and $\sigma_{\bar{x}}$, independent of $d(t)$, such that $\sigma_{\bar{z}}(\|\bar{z}\|) \leq \|\bar{z}\|^2\hat{\sigma}_{\bar{z}}(\|\bar{z}\|)$ for some smooth positive function $\hat{\sigma}_{\bar{z}}$, and

$$\frac{dV_2(\tilde{\eta})}{dt} \leq -\frac{R}{2}\|\tilde{\eta}\|^2 + \sigma_{\bar{z}}(\|\bar{z}\|) + \sigma_{\bar{x}}(\|\bar{x}\|).$$

On the other hand, under A3, by using the changing supply function technique [20], for any positive function $\Delta(\bar{z})$, there exist a smooth function

$$V_1'(\bar{z}) = \int_0^{V_1(\bar{z})} S(s)ds,$$

where $S(s) \geq 1$ is a C^1 nondecreasing function, and class $\mathcal{K}_\infty$ functions $\bar{\alpha}_i'$, $i = 1, 2$, and σ', independent of $d(t)$, such that

$$\bar{\alpha}_1'(\|\bar{z}\|) \leq V_1'(\bar{z}) \leq \bar{\alpha}_2'(\|\bar{z}\|)$$
$$\frac{\partial V_1'(\bar{z})}{\partial \bar{z}}\bar{f}_0(\bar{z}, \bar{x}, d(t)) \leq -\Delta(\bar{z})\bar{\alpha}_3(\|\bar{z}\|) + \sigma'(\|\bar{x}\|).$$

Since the function $\bar{\alpha}_3(\cdot)$ satisfies $\lim_{s\to 0^+}\sup \frac{\bar{\alpha}_3^{-1}(s^2)}{s} < +\infty$, we can choose Δ such that

$$\Delta(\bar{z})\bar{\alpha}_3(\|\bar{z}\|) \geq \|\bar{z}\|^2 \left[\hat{\sigma}_{\bar{z}}(\|\bar{z}\|) + 1\right]. \quad (16)$$

Hence, we have $\Delta(\bar{z})\bar{\alpha}_3(\|\bar{z}\|) \geq \sigma_{\bar{z}}(\|\bar{z}\|) + \|\bar{z}\|^2$. To show (16), note that for any smooth function $\bar{\Delta}(\bar{z})$, there exists smooth function $\Delta(\bar{z})$ such that

$$\Delta(\bar{z})\bar{\alpha}_3(\|\bar{z}\|) \geq \|\bar{z}\|^2\bar{\Delta}(\bar{z}). \quad (17)$$

In fact, since $\bar{\alpha}_3(\cdot)$ satisfies $\lim_{s\to 0^+}\sup \frac{\bar{\alpha}_3^{-1}(s^2)}{s} < +\infty$, there exits some constant $L_1 \geq 1$ such that $\frac{\|\bar{z}\|^2}{L_1^2} \leq \bar{\alpha}_3(\|\bar{z}\|)$ for $\|\bar{z}\| \leq 1$, and since $\bar{\alpha}_3(\cdot)$ is of class $\mathcal{K}_\infty$, for some constant $L_2 > 0$, $\bar{\alpha}_3(\|\bar{z}\|) \geq L_2$ when $\|\bar{z}\| \geq 1$. As a result, (17) holds for

$$\Delta(\bar{z}) \geq L_1^2\bar{\Delta}(\bar{z}) + \frac{1}{L_2}\|\bar{z}\|^2\bar{\Delta}(\bar{z}).$$

Then, (16) holds from (17) by taking $\bar{\Delta}(\bar{z}) \geq \hat{\sigma}_{\bar{z}}(\|\bar{z}\|) + 1$.

Now, let $W_1(\tilde{z}) = V_1'(\bar{z}) + V_2(\tilde{\eta})$, then clearly, $\alpha_1(\|\tilde{z}\|) \leq W_1(\tilde{z}) \leq \alpha_2(\|\tilde{z}\|)$ for some class $\mathcal{K}_\infty$ functions α_i, $i = 1, 2$, and

$$\begin{aligned} &\frac{\partial W_1(\tilde{z})}{\partial \tilde{z}}\phi_0(\tilde{z}, \bar{x}, d(t)) \\ \leq\ & -\frac{R}{2}\|\tilde{\eta}\|^2 + \sigma_{\bar{z}}(\|\bar{z}\|) + \sigma_{\bar{x}}(\|\bar{x}\|) \\ & -\Delta(\bar{z})\bar{\alpha}_3(\|\bar{z}\|) + \sigma'(\|\bar{x}\|) \\ \leq\ & -\frac{R}{2}\|\tilde{\eta}\|^2 - \|\bar{z}\|^2 + \sigma_{\bar{x}}(\|\bar{x}\|) + \sigma'(\|\bar{x}\|). \quad (18) \end{aligned}$$

Clearly, there exist class $\mathcal{K}_\infty$ functions γ_1 and α_3 such that

$$\frac{\partial W_1(\tilde{z})}{\partial \tilde{z}}\phi_0(\tilde{z}, \bar{x}, d(t)) \leq -\alpha_3(\|\tilde{z}\|),\ \forall \|\tilde{z}\| \geq \gamma_1(\|\bar{x}\|).$$

Verification of Condition 2: Under assumption A5, it suffices to prove that $(\tilde{z}, \bar{x}) \in \Omega^1_{c_1} \times \Omega^2_{c_2}$ implies $(\tilde{z}, \bar{x}) \in \Omega^3_{c_1} \times \Omega^2_{c_2}$. In fact, when $\tilde{z} \in \Omega^1_{c_1}$, we have

$$W_1(\tilde{z}) = V_1'(\bar{z}) + V_2(\tilde{\eta}) = \int_0^{V_1(\bar{z})} S(s)ds + V_2(\tilde{\eta}) \leq c_1,$$

which gives

$$\int_0^{V_1(\bar{z})} S(s)ds \leq c_1 \text{ and } V_2(\tilde{\eta}) \leq c_1.$$

Since $S(s) \geq 1$, we further get that $V_1(\bar{z}) \leq c_1$. As a result, $\tilde{z} \in \Omega^3_{c_1}$.

Verification of Condition 3: From inequality (18), it can be seen that α_3 can be chosen such that locally, $\alpha_3(\|\tilde{z}\|) \geq \lambda_1\|\tilde{z}\|^2$ for some $\lambda_1 > 0$. Hence,

$$\frac{\partial W_1(\tilde{z})}{\partial \tilde{z}}\phi_0(\tilde{z}, 0, d(t)) \leq \alpha_3(\|\tilde{z}\|) \leq -\lambda_1\|\tilde{z}\|^2.$$

Second, since $\phi_1(\tilde{z}, 0, d)$ is C^1 and $d \in D$, we have, locally, $|\phi_1(\tilde{z}, 0, d)| \leq \lambda_2\|\tilde{z}\|$ for some $\lambda_2 > 0$.

Third, we have

$$\frac{\partial W_1(\tilde{z})}{\partial \tilde{z}}[\phi_0(\tilde{z},\bar{x},d(t)) - \phi_0(\tilde{z},0,d(t))]$$
$$\leq \left[S(V_1(\bar{z}))\frac{\partial V_1(\bar{z})}{\partial \tilde{z}} + 2P_M\tilde{\eta}\right] \times[\phi_0(\tilde{z},\bar{x},d(t)) - \phi_0(\tilde{z},0,d(t))]$$

Since $\phi_0(\tilde{z},\bar{x},d(t))$ is C^1, then under A3, we have, locally,

$$\frac{\partial W_1(\tilde{z})}{\partial \tilde{z}}[\phi_0(\tilde{z},\bar{x},d(t)) - \phi_0(\tilde{z},0,d(t))] \leq \lambda_3\|\tilde{z}\|\|\bar{x}\|$$

for some $\lambda_3 > 0$. From above, Condition 3 holds with $\tilde{W}_1(\tilde{z}) = W_1(\tilde{z})$. ∎

Remark 3.2: Roughly, assumption A3 implies that the subsystem $\dot{\bar{z}} = \bar{f}_0(\bar{z},\bar{x},d)$ is robust input-to-state stable with $\bar{z}$ as input, $\bar{x}$ as state and d as uncertainty, and it also implies that the equilibrium point of $\dot{\bar{z}} = \bar{f}_0(\bar{z},0,d)$ at $\bar{z} = 0$ is locally exponentially stable if the functions $\bar{\alpha}_1$ and $\bar{\alpha}_2$ take quadratic form in some neighborhood of the origin. A similar assumption can be found in [14] which handles the global robust regulation problem of lower triangular systems. This assumption together with assumption A4 guarantees that the zero dynamics of the augmented system satisfies Conditions 1 and 3. Assumption A5 is made to guarantee that the zero dynamics of the augmented system satisfies Conditions 2. ∎

Remark 3.3: As pointed out in [6], assumption A4 is satisfied when $\left|\beta^{[2]}(\tilde{\eta}+\delta) - \beta^{[2]}(\delta)\right| \leq \frac{(1-R)}{2\|P_M N\|}\|\tilde{\eta}\|$. In particular, assumption A4 is satisfied when $\beta(x)$ is a linear function of x. ∎

By Proposition 2.1 and Theorem 3.1, we have the following conclusion.

Theorem 3.2: Under assumptions A1-A5, the semi-global robust output regulation problem of the system (1) is solvable by a dynamic output feedback controller. ∎

Remark 3.4: The methodology of this paper is different from the one in [16]. In [16], the dynamics of the exosystem is treated as part of the augmented system. In this paper, the augmented system only consists of the given plant and the internal model while the exogenous signal v is treated as a time-varying disturbance, which can be handled in the same way as the unknown parameter w. Therefore, the augmented system is still in the standard normal form. As a result, the semi-global robust stabilization technique in [17] can be directly applied to handle the robust stabilization problem of the augmented system. ∎

IV. Conclusions

This paper addresses the semi-global robust output regulation problem for a class of minimum phase nonlinear systems by using dynamic output feedback. Under the framework of [6], the problem is solved in two steps. In the first step, the semi-global robust output regulation problem is converted into a semi-global robust stabilization problem, and then in the second step, the semi-global robust output regulation problem is solved by solving the corresponding semi-global robust stabilization problem with min-max control proposed in [17]. Moreover, by introducing a nonlinear internal model, the polynomial condition on the solution of the regulator equations is relaxed.

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