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Lyapunov-Based Switching Control of Nonlinear Systems Using High-Gain Observers

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***Abstract*— We consider dynamic output feedback practical stabilization of uniformly observable nonlinear systems, based on high-gain observers with saturation. We assume that uncertain parameters and initial conditions belong to known but comparably large compact sets. In this situation, designs based on traditional robust or adaptive techniques, if applicable, would lead to high controller, observer, and adaptation gains. High gains may excite unmodeled dynamics and significantly amplify measurement noise. Moreover, they could be impossible or too costly to implement. In order to reduce the control efforts and improve robustness of a continuous high-gain-observer-based sliding mode control with respect to these non-ideal operational conditions, we have recently proposed a new logic-based switching design strategy. In this paper, we generalize our technique and apply it to a wider class of nonlinear systems and more general Lyapunov-function-based state and output feedback designs. It is important to notice, in particular, that we require neither the sign of the high-frequency gain to be known nor the system to be minimum-phase. The key idea is to split the set of parameters into smaller subsets, design a controller for each of them, and switch the controller if the derivative of the Lyapunov function does not satisfy a certain inequality, after a dwell-time period. We do not order the candidate controllers in advance, as in our earlier work. Instead, we use estimates of the derivatives of the states, provided by an extended order high-gain observer, to calculate instantaneous performance indices. When the controller is falsified, we switch to a new controller that corresponds to the smallest index among the controllers that have not been falsified yet. This modification is important when the number of candidate controllers is high and pre-routed search may lead to an unacceptable transient performance.**

I. INTRODUCTION

Stabilization of nonlinear parametrically uncertain systems is an undoubtedly important problem that have attracted attention of numerous researches and obviously cannot be solved in general. Therefore, typically, it is assumed that the system belongs to a certain class with a fixed structure. Depending on the description of the class, it is often possible to apply one of the classical adaptive or robust control design techniques. In particular, the two approaches, enhanced by a high-gain observer with saturation [3], can be applied to solve the output tracking problem for the class of minimum-phase nonlinear systems in the case when uncertain parameters and initial conditions belong to given compact sets. However, when the set of parameters is large, high controller, observer, and adaptation gains are needed. High gains may excite hidden dynamics and amplify measurement noise. Moreover, they could be unimplementable in practice. In order to avoid unnecessarily high gains we have recently proposed an alternative solution [7], inspired by the logic-based switching control design [15]. The key idea of [7] is to check whether a certain inequality for the derivative of the Lyapunov function is satisfied. The inequality can be verified using the estimates of the states and their derivatives, as soon as the peaking time of the high-gain observer is over. If it is satisfied, then the current controller ensures converges of the trajectories to a set where the tracking error is small. If not, then switching is necessary. It is clear from our analysis in [7] that the sliding-mode design for the candidate controllers is not essential and any Lyapunov function-based design could have been used instead. Moreover, for the problem of state feedback control design, similar switching logic, with a high-gain observer providing derivatives of the states, could be used in order to improve robustness and transient performance of the closed-loop system. We present this generalization below with an important improvement in the switching logic. We avoid following pre-routed search, that may result in an unacceptable transient performance when the number of candidate controllers is high. We use the available on-line information to identify the set to which parameters belong and to choose a candidate controller to be put into the loop when the inequality for the derivative of the Lyapunov function fails. However, the most important extension of our previous result is the class of systems that can be handled with our approach. This class contains now not only minimum-phase nonlinear systems with a known sign of the high-frequency gain of [7] but also uniformly observable nonlinear systems with possibly unstable zero-dynamics and the sign of high-frequency gain depending on the values of the uncertain parameters. The motivation for the class of nonlinear systems, introduced below, is the well-known procedure of dynamic extension [18], [17] for output feedback control. The idea is to augment the system with additional dynamics, typically by adding a chain of integrators at the input. The original system can be parametrically uncertain, non-minimum phase, and even with unknown high-frequency gain. The dynamic extension yields a system with additional outputs and no zero-dynamics. As was noticed by Tornambé [18], implementing a high-gain observer to estimate derivatives of the output, we end-up with a state feedback control problem for the

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extended system. However, in the nonlinear case, designing state feedback law and then implementing it employing the estimates of a high-gain observer could lead to a shrinking region of attraction and unacceptable transient behavior due to the peaking phenomenon. This difficulty can be overcome if the control law is saturated outside of the region of interest as suggested by Esfandiari and Khalil [5]. These ideas, combined, in particular, with backstepping and robust high-gain feedback design, were used by Teel and Praly [17] to develop a powerful tool for semiglobal feedback stabilization of the class of uniformly observable systems. Their results were extended by many researchers, see e.g. [3], [11]. The available techniques apply to parametrically uncertain systems. However, when the parameters of the model belong to a known but comparably large compact set, it is typically assumed that all the systems in the family are minimum-phase and that at least the sign of high-frequency gain is fixed and known. The stability of the zero dynamics (often in the form of input-to-state stability) is essential when high-gain feedback is used to overcome uncertainty. Knowledge of the control direction is important in the problem of smooth stabilization [16] and discontinuous switching control seems to be the most efficient way to avoid it [13]. Recently, logic-based switching feedback design was developed by Morse [15] and others [9], [10], [8], [1], [4] as an alternative to the classical adaptive control. It promises superior and, to some extent, provable guaranteed performance. Moreover, advantage of allowing discontinuity in control is well-known [14] and, in many cases, there is no need to impose either one of the structural assumptions, mentioned above, provided switching in control is allowed. Our goal in what follows is to develop a procedure of achieving stability via switching between several candidate controllers, designed along the lines of [17], [3] to achieve satisfactory performance for a small subset of parameter uncertainty. We continue with a precise description of the class of systems. After that, we discuss the design of extended high-gain observers and the switching strategy. The formulation of the main result conclude the theoretical part of the paper. The last part of the paper presents an illustrative example of a nonlinear systems with an unknown sign of high-frequency gain that can be stabilized using scale-independent hysteresis-based logic [8]. We present some simulation results and compare the performance under our design and the alternative one.

II. CLASS OF SYSTEMS

We consider the regulation problem for a nonlinear system that could be represented in the form

$$\dot{x} = Ax + B\phi(p, x, \zeta, u), \quad \dot{\zeta} = \psi(p, x, \zeta, u), \quad y = Cx, \tag{1}$$

where the triple $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times 1} \times \mathbb{R}^{1\times n}$ represents a chain of n integrators, i.e.

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}^T;$$

$\phi(\cdot)$ and $\psi(\cdot)$ are known continuously differentiable functions; $x = [x_1, \cdots, x_n]^T \in \mathbb{R}^n$ and $\zeta \in \mathbb{R}^s$ are vectors of state variables; $y \in \mathbb{R}$ and ζ are measured outputs; $p \in \mathbb{R}^r$ is a vector of unknown parameters; and $u \in \mathbb{R}$ is the control input.

We restrict ourselves to the single-input case only for simplicity of presentation. Extension to the multi-input case is straightforward and can be done along the lines of [3]. We assume that the differential equation for the x-subsystem is dropped if $n = 0$ and that $n \geq 2$ otherwise (since x could be incorporated into ζ if $n = 1$). Similarly, there is no ζ-subsystem if $s = 0$. We do not assume any special structure of $\psi(\cdot)$ and therefore we consider a class of parametrically uncertain nonlinear systems that contains in particular the following important subclasses.

- Feedback linearizable systems with no zero-dynamics.
- Uniformly observable parametrically uncertain systems. Specifically, the model (1) includes the case of an uncertain system representable by an n-th order input-output model, augmented with a series of integrators at the input side (as in [3], [18], [12]),
- General nonlinear systems with all state variables available for feedback.

In each case we assume that the vector p belongs to a known, relatively large, compact set $\mathcal{P}$. In order to simplify the control design, this set is partitioned into smaller subsets as suggested in [15] (see also [13] and references therein)

$$p \in \mathcal{P} = \bigcup_{i=1}^{N} \mathcal{P}^{(i)}.$$

Our main assumption is as follows. For a given compact set $\Omega \subset \mathbb{R}^{n+s}$ of possible initial conditions and every

$$i \in \mathcal{I} = \{1, \cdots, N\}$$

there exists a family of continuously differentiable, bounded in x (saturated outside of the region of interest), state feedback control laws with all partial derivatives bounded in x as well,

$$u = u^{(i)} \equiv g^{(i)}(x, \zeta) \tag{2}$$

such that for every $p \in \mathcal{P}^{(i)}$ all the trajectories of the closed-loop system (1), (2) initiated inside Ω are bounded and $y(t) \to 0$ as $t \to \infty$. Moreover, we assume that a corresponding family of Lyapunov functions $V^{(i)}(x, \zeta)$ and auxiliary $\mathcal{K}^{\infty}$ functions $\alpha_1^{(i)}(\cdot)$, $\alpha_2^{(i)}(\cdot)$, and $\alpha_3^{(i)}(\cdot)$ are known such that

$$\alpha_1^{(i)}\left(\left\|\begin{bmatrix}x\\ \zeta\end{bmatrix}\right\|\right) \le V^{(i)}(x,\zeta) \le \alpha_2^{(i)}\left(\left\|\begin{bmatrix}x\\ \zeta\end{bmatrix}\right\|\right);$$

$$\frac{\partial V^{(i)}(x,\zeta)}{\partial x}\left[Ax + B\phi\left(\bar{p}, x, \zeta, g^{(i)}(x,\zeta)\right)\right] + \frac{\partial V^{(i)}(x,\zeta)}{\partial \zeta}\,\psi\left(\bar{p}, x, \zeta, g^{(i)}(x,\zeta)\right) \le -\alpha_3^{(i)}\left(\left\|\begin{bmatrix}x\\ \zeta\end{bmatrix}\right\|\right) \tag{3}$$

$\forall \bar{p} \in \mathcal{P}^{(i)}$ and $\forall (x,\zeta) \in U^{(i)}(R)$ with

$$U^{(i)}(R) = \left\{(x,\zeta):\ V^{(i)}(x,\zeta)) \le R\right\},$$

where $R > 0$ is chosen so that Ω is in the interior of $U^{(i)}(R)$.

Our goal is to design an output feedback control law to guarantee practical regulation. The family of the control laws (2) cannot be implemented because

- the vector x is not available for feedback and
- the value $i^* \in \mathcal{I}$, for which $p \in \mathcal{P}^{(i^*)}$, is not known.

The first problem could be resolved with the help of an appropriate high-gain observer (HGO). To deal with the second one we suggest to use 'discontinuous adaptation' in the form of logic-based switching.

III. SWITCHING LOGIC

First of all, we need to obtain a robust estimate $\hat{x}$ of the vector x or, equivalently, of $n-1$ derivatives of the main output y, in order to apply the control law

$$u = \hat{u}^{(i)} = g^{(i)}(\hat{x}, \zeta), \tag{4}$$

which is close to (2), provided $\|x - \hat{x}\|$ is small. However, it is also useful to obtain estimates of $y^{(n)}$ and $\dot{\zeta}$ as well, to be used for parameter identification. Estimating 'extra' derivatives is an alternative to the filter design of classical adaptive control theory [2, Sec. 2.3] aiming to obtain a quasi-static regression model suitable for parameter identification. In what follows we also use these extra derivatives in order to check whether (3), which could be rewritten as

$$\frac{\partial V^{(i)}}{\partial x}(x,\zeta)\,\dot{x} + \frac{\partial V^{(i)}}{\partial \zeta}(x,\zeta)\,\dot{\zeta} \le -\alpha_3^{(i)}\left(\left\|\begin{bmatrix}x\\ \zeta\end{bmatrix}\right\|\right), \tag{5}$$

is satisfied in order to decide whether or not to switch.

Differentiating the equations of the closed-loop system (1), (4) with a fixed i we obtain

$$y^{(n)} = \phi_1^{(i)}(p, x, \zeta, \hat{x}, \dot{\hat{x}}), \qquad \ddot{\zeta} = \psi_1^{(i)}(p, x, \zeta, \hat{x}, \dot{\hat{x}}), \tag{6}$$

where $\phi_1^{(i)}(p, x, \zeta, \hat{x}, \dot{\hat{x}})$ and $\psi_1^{(i)}(p, x, \zeta, \hat{x}, \dot{\hat{x}})$ are certain functions, depending nonlinearly on $\hat{u}^{(i)} \equiv g^{(i)}(\hat{x}, \zeta)$ and linearly on its partial derivatives. It is crucial to notice that, since the functions $g^{(i)}(\hat{x}, \zeta)$, $\dfrac{\partial g^{(i)}(\hat{x},\zeta)}{\partial \hat{x}}$, and $\dfrac{\partial g^{(i)}(\hat{x},\zeta)}{\partial \zeta}$ are globally bounded in $\hat{x}$, the functions $\phi_1^{(i)}(\cdot)$ and $\psi_1^{(i)}(\cdot)$ satisfy a linear growth bound in $\dot{\hat{x}}$.

We are ready now to design a high-gain observer for the extended closed-loop system, following the standard procedure [3].

The variables x and $y^{(n)}$ are estimated by $\hat{x}$ and $\hat{x}_{n+1}$, provided by

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + B\hat{x}_{n+1} + H(\varepsilon_i)(y - C\hat{x}), \\ \dot{\hat{x}}_{n+1} &= \hat{\phi}_1^{(i)}(y, \zeta, \hat{x}, \dot{\hat{x}}) + (\alpha_{n+1}/\varepsilon_i^{n+1})(y - C\hat{x}), \end{aligned} \tag{7}$$

where $\hat{\phi}_1^{(i)}(y, \zeta, \hat{x}, \dot{\hat{x}})$ is a nominal model of $\phi_1^{(i)}(\cdot)$, which can be taken as zero, $H(\varepsilon_i) = \left[\dfrac{\alpha_1}{\varepsilon_i}, \dfrac{\alpha_2}{\varepsilon_i^2}, \cdots, \dfrac{\alpha_n}{\varepsilon_i^n}\right]^T$, $\alpha_1, \alpha_2, \cdots, \alpha_{n+1}$ are chosen such that the polynomial $\xi^{n+1} + \alpha_1\xi^n + \ldots + \alpha_n\xi + \alpha_{n+1}$ is Hurwitz, and $\varepsilon_i > 0$ is a small parameter to be specified.

Similarly, $\dot{\zeta}$ is estimated by $\hat{\zeta}_2$, provided by the following observer (see [6] for some simpler alternatives that are more suitable for special classes mentioned in the introduction):

$$\dot{\hat{\zeta}} = \hat{\zeta}_2 + \frac{2(\zeta - \hat{\zeta})}{\varepsilon_i}, \quad \dot{\hat{\zeta}}_2 = \hat{\psi}_1^{(i)}(y, \zeta, \hat{x}, \dot{\hat{x}}) + \frac{\zeta - \hat{\zeta}}{\varepsilon_i^2}, \tag{8}$$

where $\hat{\psi}_1^{(i)}(y, \zeta, \hat{x}, \dot{\hat{x}})$ is a nominal model for $\psi_1^{(i)}(\cdot)$, which can be taken as zero.

We assume that the initial conditions for (7) and (8) belong to a given (arbitrary) compact set $\hat{\Omega} \in \mathbb{R}^{n+1+2s}$.

Using high-gain observers to estimate additional derivatives brings up a new challenge in the analysis that has not been encountered before in high-gain-observer-based controllers as exemplified by [3]. In such controllers, when the closed-loop system is represented in a singularly perturbed form, the fast subsystem is an $O(\varepsilon)$ perturbation of a Hurwitz linear system and the perturbation term is bounded by a function that is independent of the fast variables. In the current problem, the $O(\varepsilon)$ perturbation term will not be bounded uniformly in the fast variables. However, it will satisfy a linear growth bound in those variables. It is interesting that singular perturbation analysis in the presence of peaking fast variables can handle a linear growth bound on the fast variables. In fact, the singular perturbation result in the very first paper on peaking in high-gain observers [5] does allow the perturbation term on the right-hand side of the fast subsystem to be bounded by a function that grows linearly in the fast variables. While Theorem 1 of [5] cannot be applied directly to our current problem (even if all ε_i are chosen equal to each other), since switching between several control laws results in discontinuous right-hand sides of the slow subsystem, its technique is used in the analysis.

The next issue we are going to address is how to choose the correct candidate control law or, equivalently, the appropriate index $i \in \mathcal{I}$. It is intuitively clear that if $i = i^*$ $\left(p \in \mathcal{P}^{(i^*)}\right)$ and the high-gain observers above are capable of providing good estimates, then the inequality (3) is satisfied up to a small error, the trajectories of the closed-loop system cannot leave the compact set $U^{(i^*)}(R)$, and

$y(t)$ approaches an invariant set where it is small, provided ε_{i^*} is sufficiently small. It is crucial to notice that if (3) is satisfied for a value of $i \neq i^*$, we can still show ultimate boundedness (practical regulation). Obviously, we cannot check the inequality (5) on-line because it depends on the derivatives $\dot{x}$ and $\dot{\zeta}$. However, using the estimates provided by (7) and (8), the following inequality is easy to verify:

$$\frac{\partial V^{(i)}}{\partial x}(\hat{x},\zeta)\,\dot{\hat{x}}+\frac{\partial V^{(i)}}{\partial \zeta}(\hat{x},\zeta)\,\hat{\zeta}_2 \leq a_0-\alpha_3^{(i)}\left(\left\|\begin{bmatrix}\hat{x}\\ \zeta\end{bmatrix}\right\|\right), \quad (9)$$

where $a_0 > 0$ is a small constant introduced to deal with an $O(\varepsilon_i)$ estimation errors induced by the high-gain observers.

After a short peaking period, we can find out whether inequality (9) is satisfied. As long as it is satisfied, we maintain the controller that is currently in the loop. When it is violated, we switch to another controller and exclude the previous one from the list of candidate controllers. How to choose the next controller is a crucial decision that affects the performance of the system. We can switch systematically according to a pre-sorted list, as in our earlier work [7]. The performance, however, might not be acceptable if we have to switch through a long sequence of controllers before we settle at one for which the inequality (9) is satisfied. A more intelligent way is to use on-line information to decide on the next controller. Noting that, after the peaking period, the estimates $\hat{x}_{n+1}$ and $\hat{\zeta}_2$ satisfy the equations

$$\hat{x}_{n+1} - \phi(p,\cdot) = O(\varepsilon_i), \qquad \hat{\zeta}_2 - \psi(p,\cdot) = O(\varepsilon_i),$$

for the true parameter p, we can use

$$J(p) = \left|\hat{x}_{n+1} - \phi\Big(p,\hat{x},\zeta,\hat{u}^{(i)}\Big)\right| + \left\|\hat{\zeta}_2 - \psi\Big(p,\hat{x},\zeta,\hat{u}^{(i)}\Big)\right\|$$

as a performance index to be minimized over all possible values of p. Invoking additional assumptions on how the functions $\phi(\cdot)$ and $\psi(\cdot)$ depend on p, we may use gradient, normalized gradient, least square, or another standard estimation algorithm [2]. Naturally, the algorithm choice will impact the performance, as discussed in [14] for a similar problem. For computational simplicity, we adopt the following approach. For each set $\mathcal{P}^{(j)}$, we choose a nominal parameter $p^{(j)}$. Assuming that the sets $\mathcal{P}^{(j)}$ are small, it is reasonable to expect $\left|J(p) - J\big(p^{(j)}\big)\right|$ to be small for all $p \in \mathcal{P}^{(j)}$. Hence we use $J\big(p^{(j)}\big)$ as an index for the set $\mathcal{P}^{(j)}$. If $\mathcal{I}$ is the set of indices to choose from at some switching time, the next index is taken as

$$i = \arg\min_{j\in\mathcal{I}} \left\{ J\Big(p^{(j)}\Big)\right\}. \quad (10)$$

Finally, we pick a small positive dwell-time constant τ, which is greater then peaking time, and proceed according to the following algorithm.

- Step 1. Define initial time, say $t_0 := 0$, the set of indices $\mathcal{I} := \{1, 2, \cdots, N\}$, and an arbitrary initial value for $i \in \mathcal{I}$, say $i := 1$.
- Step 2. Put the controller $u = \hat{u}^{(i)}$, defined by (4) and (7), into the loop for $t \in [t_0, t_0 + \tau)$.
- Step 3. For $t \geq t_0+\tau$ we continuously check the inequality (9) using current estimates from (7) and (8). We keep the controller $u = \hat{u}^{(i)}$ in the loop until the moment of time $t_i \geq t_0 + \tau$ when the inequality fails.
- Step 4. At $t = t_i$ we redefine $\mathcal{I} := \mathcal{I} \setminus \{i\}$ and choose a new value for i using (10).
- Step 5. Set $t_0 := t_i$ and go back to Step 2.

IV. MAIN RESULT

Theorem. *Consider the closed loop-system (1), (4), (7), (8) under the switching logic described above and with initial states in the compact set* $\Omega \times \hat{\Omega}$*. There exist positive numbers* $\tau_{\min}$ *and* $\tau_{\max}$*, with* $\tau_{\min} < \tau_{\max}$*, and for every* $\tau \in (\tau_{\min}, \tau_{\max})$ *there exists* $\bar{\varepsilon} \in (0, 1)$ *such that if* $\varepsilon_i \in (0, \bar{\varepsilon})$*, for* $i = 1, \cdots, N$*, then the trajectories will be bounded and* $|y(t)|$ *will be ultimately bounded by a bound that can be made arbitrary small, provided* a_0 *in (9) and* ε_i *are chosen sufficiently small.*

Proof is omitted due to space limitations. It is available in [6].

V. EXAMPLE

To demonstrate our design procedure we consider a nonlinear system with an unknown high-frequency gain in the strict-feedback form. Our goal is to compare performance achieved via our design and via a recently proposed logic-based switching strategy [15], [9], [10], [13].

The following system has been investigated in [8, pp. 76–82] (see also [10])

$$\dot{z}_1 = p_1 z_1^3 + p_2 z_2, \qquad \dot{z}_2 = u, \qquad y = z_1 - r, \quad (11)$$

where z_1 and z_2 are the state variables, $p = [p_1, \quad p_2]^T$ is a vector of unknown parameters that belong to the set

$$\mathcal{P} = \bigcup_{i=1}^{42} \left\{p^{(i)}\right\} = \{-1, -0.9, \ldots, 0.9, 1\} \times \{-1, 1\} \subset \mathbb{R}^2,$$

u is the control input, r is a constant reference, and y is the output error.

Our design procedure is applicable for solving both output and state feedback regulation problems.

We start with the case when only y is available. To transform (11) into the form (1) let

$$x_1 = z_1 - r, \qquad x_2 = p_1 z_1^3 + p_2 z_2, \qquad x = \begin{bmatrix}x_1\\ x_2\end{bmatrix},$$

so that

$$\dot{x} = \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}x_2\\ 3p_1(x_1+r)^2 x_2 + p_2 u\end{bmatrix}. \quad (12)$$

We use feedback linearization and pole-placement to derive the control law

$$u = -[\omega^2 x_1 + 2\eta\omega\hat{x}_2 + 3q_1(x_1+r)^2\hat{x}_2]/q_2, \tag{13}$$

where $q = [\,q_1, \quad q_2\,] \in \mathcal{P}$ is the nominal value for p, $\hat{x}_2$ is an estimate for x_2, provided by the high-gain observer

$$\dot{\hat{x}}_1 = \hat{x}_2 + \frac{3(y-\hat{x}_1)}{\varepsilon},\ \dot{\hat{x}}_2 = \hat{x}_3 + \frac{3(y-\hat{x}_1)}{\varepsilon^2},\ \dot{\hat{x}}_3 = \frac{y-\hat{x}_1}{\varepsilon^3}, \tag{14}$$

and $\omega > 0$ and $\eta > 0.25$ are chosen to ensure acceptable transient performance of the closed-loop system (12), (13) with $q = p$, i.e.,

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\omega^2 x_1 - 2\eta\omega x_2. \tag{15}$$

The Lyapunov function candidate can be taken as

$$V(x_1, x_2) = \omega(1+\eta)x_1^2 + x_1x_2 + x_2^2/\omega,$$

so that along the trajectories of (15)

$$\dot{V} = -W(x_1,x_2) = -\omega^2 x_1^2 - (4\eta-1)x_2^2.$$

Therefore, we start with $i = 1$, put the first candidate controller (13) with $q = p^{(i)}$, i.e.

$$u = -\operatorname{Sat}\left(\left[\omega^2 x_1 + \left(2\eta\omega + 3p_1^{(i)}(x_1+r)^2\right)\hat{x}_2\right]/p_2^{(i)}\right),$$

where $\operatorname{Sat}(\cdot)$ is a smooth saturation function, into the loop and switch to another one as soon as the dwell-time period is over and the inequality

$$\frac{\partial V}{\partial x_1}(x_1,\hat{x}_2)\,\hat{x}_2 + \frac{\partial V}{\partial x_2}(x_1,\hat{x}_2)\,\hat{x}_3 + W(x_1,\hat{x}_2) \le a_0,$$

fails, where the output of the high-gain observer (14) is used. The performance index for (10) can be taken as

$$J\left(p^{(j)}\right) = \left|\hat{x}_3 - 3p_1^{(j)}(x_1+r)^2\hat{x}_2 - p_2^{(j)}u\right|.$$

In the state feedback case we let

$$\zeta_1 = z_1 - r, \qquad \zeta_2 = z_2 + \frac{q_1 r^3}{q_2}, \qquad \zeta = \begin{bmatrix}\zeta_1\\ \zeta_2\end{bmatrix},$$

so that

$$\dot{\zeta} = \begin{bmatrix}\dot{\zeta}_1\\ \dot{\zeta}_2\end{bmatrix} = \begin{bmatrix} p_1(\zeta_1+r)^3 + p_2(\zeta_2 - q_1r^3/q_2)\\ u\end{bmatrix}. \tag{16}$$

Following the idea of [8], we use the regulator

$$\begin{aligned} u &= -[\omega^2\zeta_1 + 2\eta\omega\varphi(q,\zeta) + 3q_1(\zeta_1+r)^2\varphi(q,\zeta)]/q_2,\\ \varphi(q,\zeta) &= q_1(\zeta_1+r)^3 + q_2(\zeta_2 - q_1r^3/q_2),\end{aligned} \tag{17}$$

such that the closed-loop system (16), (17) with $q = p$ is equivalent to (15). We use the family of Lyapunov functions

$$V^{(i)}(\zeta_1,\zeta_2) = V\left(\zeta_1, \varphi(p^{(i)},\zeta)\right),$$

that are obtained from the Lyapunov function used for the output feedback, and, correspondingly, the inequalities

$$\frac{\partial V^{(i)}}{\partial \zeta_1}(\zeta_1,\zeta_2)\,\hat{\zeta}_2 + \frac{\partial V^{(i)}}{\partial \zeta_2}(\zeta_1,\zeta_2)\,u + W\left(\zeta_1, \varphi(p^{(i)},\zeta)\right) \le a_0,$$

with the high-gain observer

$$\dot{\hat{\zeta}}_1 = \hat{\zeta}_2 + \frac{2(\zeta_1 - \hat{\zeta}_1)}{\varepsilon_i}, \qquad \dot{\hat{\zeta}}_2 = \frac{\zeta_1 - \hat{\zeta}_1}{\varepsilon_i^2}, \tag{18}$$

and the performance index

$$J\left(p^{(j)}\right) = \left|\hat{\zeta}_2 - \varphi(p^{(j)},\zeta)\right|.$$

When all the state variables are available for feedback, the controller of [8] is an alternative approach. It uses scale-independent hysteresis-based switching logic design. The first step of that approach is to design a multi-estimator for the closed-loop system (16), (17) [8] and the hardest part is to verify that the so-called "ij-injected system" is strongly detectable with a known gain function. The second step, is to use this gain function in order to define "performance indices" $\mu^{(j)}$, $j \in \{1,\cdots,42\}$. Switching is organized as follows. Starting with $i := 1$, we check the inequality

$$\mu^{(i)}(t) \le (1+h)\min_{j\in\{1,\cdots,42\}}\{\mu^{(j)}(t)\},$$

where $h > 0$ is a fixed hysteresis constant. As soon as the inequality fails, we redefine

$$i := \operatorname*{argmin}_{j\in\{1,\cdots,42\}}\{\mu^{(j)}(t)\}$$

and switch to the corresponding candidate controller.

It is shown in [8] that the solutions of the closed-loop hybrid system are well-defined, switching has to stop in finite time (with some value $i = i_0 \in \{1,\cdots,42\}$ and it is possible but not necessary that $i_0 = i^*$), all signals are bounded, and $\lim_{t\to\infty}|y(t)| = 0$.

The simulation results for $r = 1.0$, $\omega = 1.0$ and $\eta = 0.7$ are shown in Fig. 1 and Fig. 2. In each figure, the first and the second rows show results for the Lyapunov-based switching logic for the case of output and state feedback ($\tau = 0.03$, $\varepsilon = 0.001$, and $a_0 = 0.01$) correspondingly, and the third row shows results for the scale-independent hysteresis-based logic ($h = 0.01$ and $\lambda = 0.5$). We show the system's regulated state $x_1(t) = \zeta_1(t)$ (column 1), the generated control input $u(t)$ (column 2), and the index, $i(t)$, of the controller put in the loop (column 3).

In the case when $i^* = 20$, real system parameter is identified perfectly. However, the output feedback regulator outperforms both state feedback regulators. It is worth to notice at this point that the output feedback design in this case is not observer-based and so it is not surprising.

In the case when $i^* = 40$ both regulators based on our logic deliver similar performance though parameters are not identified perfectly for one of them. The regulator based on [8] suffers from the fact that during a short transient period identification is not possible and the first controller is kept in the loop even if it drives the systems states away from the desired equilibrium due to the fact that it is based on a wrong sigh of the high-frequency gain.

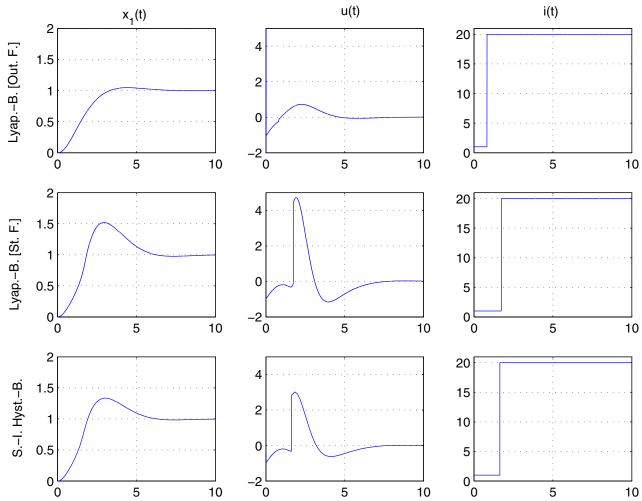

Fig. 1. $(i^* = 20)$

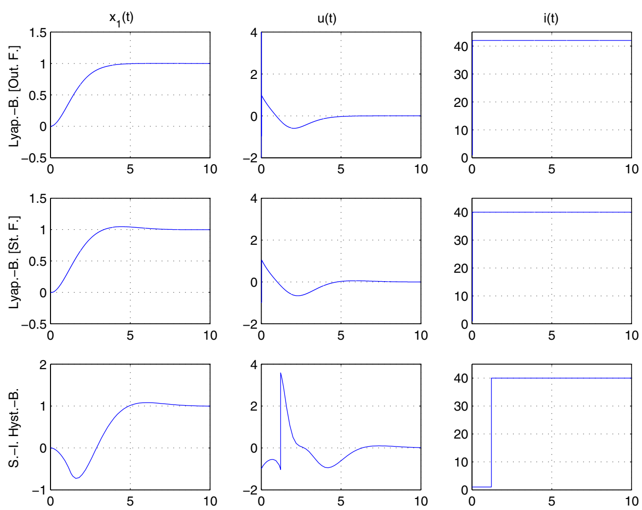

Fig. 2. $(i^* = 40)$

VI. CONCLUSION

Extending our previous result [7] for minimum-phase nonlinear systems, we have proposed new design technique for output and state feedback control to practically stabilize a class of parametrically uncertain uniformly observable nonlinear systems. We do not impose any kind of minimum-phase assumption neither do we assume that the sigh of the high-frequency gain is parameter independent. The following is the summary of our approach:

- split the large set of parameters into a finite number of smaller subsets,
- use any Lyapunov function-based technique to design a smooth partial state feedback candidate controller for each subset,
- design an extended order HGO, providing the estimates for the states, needed to implement the candidate controllers, and for the derivatives of the states,
- obtain the performance indices for the subsets, based on the algebraic equations that must be instantaneously satisfied by the parameters, states and the derivatives of the states,
- use the estimates, provided by the observer, to check whether the inequality for the derivative of the Lyapunov function, corresponding to the current controller, is satisfied and if it fails switch to the controller that corresponds to the smallest performance index.

Application of the developed procedure has been demonstrated on an example of uncertain non-linear system, previously reported in the literature. Main direction of our future research is to investigate robustness of this approach with respect to measurement noise and unmodeled dynamics (see [6] for some preliminary results).

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