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A Cutting Plane Algorithm for Frequency Domain Specification with Application to Bending Modes Attenuation

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***Abstract*— Bending modes attenuation is a very challenging problem, especially if the modes are close to the bandwidth, in which case applying weighting filters is useless. This work aims to present an idea based on the Cutting Plane Algorithm (CPA) to overcome such problems. The interesting point of using CPA instead of optimization under LMI (Linear Matrix Inequality) constraints is that no additional variables are introduced, which has a significant effect on the computation time. The application of Youla parameterization is considered in order to restore the linearity in the compensator parameters, thus the convexity is guaranteed and the CPA is efficiently used.**

Keywords: Youla parameterization, Cutting Plane Algorithm, KYP lemma.

I. INTRODUCTION

The problem of bending modes attenuation can be addressed by different ways according to the method used to design the controller. In the normalized coprime factors synthesis for instance [1], a roll-off filter is applied, which has the classical form of a low-pass filter with unity gain in the bandwidth $[0, \omega_0]$; the frequency range where the bending modes have to be attenuated are beyond ω_0 and the order n of the filter enables to adjust the roll-off :

$$W(s) = \frac{1}{\left(\frac{s}{\omega_0} + 1\right)^n} \tag{1}$$

with s being the Laplace operator.

This filter can be used only if the bending modes are located outside the bandwidth. As a major drawback, it causes undesirable effects on the stability margins if ω_0 is close to the crossover frequency [2]. A second technique is the standard $\mathcal{H}_\infty$ synthesis where a frequency-dependent weighting filter is applied to one of the closed-loop transfer of the plant. Such a filter allows to limit the resonance of the bending modes but it still influences the bandwidth.

As another way, Iwasaki and Hara [3] introduce frequency domain constraints in $\mathcal{H}_\infty$ problems under static gain feedback, without weighting filter. The inconvenience is the number of variables which have to be added. To avoid the problem of added variables in the LMI constraints, Kao [4] presents an alternative using a Hamiltonian matrix and applies a Cutting Plane Algorithm (CPA) instead of Semidefinite Programming.

The purpose of this paper is to show that by using the Youla parameterization to restore the linearity of the constraints with respect to the decision variables, the CPA can be efficiently applied to produce closed-loop gain attenuation in a specified frequency range. It is organized as follows: section 2 contains a brief presentation of Youla parameterization; section 3 concerns the CPA. The main contribution appears in section 4, where an interesting result leads to use a Hamiltonian matrix to handle a frequency domain specification. An illustrative example is finally presented in section 5.

II. YOULA PARAMETERIZATION

Since the work of [5], the Youla parameterization has often been used in multiobjective control problems [6], [7]. For two main reasons: firstly, all compensators stabilizing the plant can be expressed using the Youla parameterization on a convex domain; secondly, most constraints are linear in the Youla parameter if a certain representation is used. For instance in discrete time one can use a FIR structure for the Youla parameter:

$$Q(z) = q_0 + q_1\frac{1}{z} + \cdots + q_{n_q}\frac{1}{z^{n_q}} \tag{2}$$

where n_q is the order of approximation a priori chosen.

In the literature Q is represented either by the transfer function (2) or by a state space realization, according to the constraints formulation. In this work, the state space representation is taken, and the decision variables are located only in C_q and D_q, respectively the output and the direct transmission matrices of the Youla parameter. As for Q, the plant G is taken with state space realization:

$$G: \quad \begin{matrix} \\ z \\ y \end{matrix} \left(\begin{array}{c|cc} & w & u \\ A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right) \tag{3}$$

where z is the output to be controlled despite disturbance w, using control input u and measurement y. Using an initial controller, all stabilizing controllers are defined by the Redheffer product $K = J * Q$ (see the interconnection structure of Figure 1), where J is such that the transfer between u_q and y_q is identically equal to zero; thus G_{zw} (the closed-loop transfer between input w and output z) is affine in Q:

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$$G_{zw} = H - UQV \tag{4}$$

where H, U and V are stable transfer functions, resulting from the interconnection $G * J$.

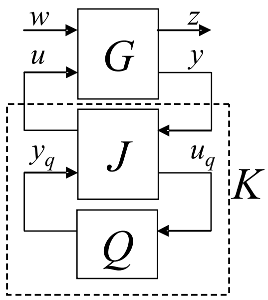

Fig. 1. Closed-loop structure using Youla parameterization

For the case where the plant has only one control input, and if it is possible to consider only one controlled output (in the case of more than one output, a weighted combination of them can be considered), U is scalar and G_{zw} can be rewritten:

$$G_{zw} = H - QUV \tag{5}$$

From state space realizations of H, U, V and Q, a non minimal realization of G_{zw} is as follows:

$$G_{zw} = \left(\begin{array}{c|c} A_{zw} & B_{zw} \\ \hline C_{zw} & D_{zw} \end{array}\right) = \left(\begin{array}{cccc|c} A_h & 0 & 0 & 0 & B_h \\ 0 & A_q & B_qC_v & B_qD_vC_u & B_qD_vD_u \\ 0 & 0 & A_v & B_vC_u & B_vD_u \\ 0 & 0 & 0 & A_u & B_u \\ \hline C_h & -C_q & -D_qC_v & -D_qD_vC_u & D_h - D_qD_vD_u \end{array}\right) \tag{6}$$

According to (6), matrices C_q and D_q are only entering in C_{zw} and D_{zw}, which guarantees the convexity of the algorithm derived in section 4.

Remark 1: if there is more than one control input or more than one controlled output, the formulation (5) can be replaced by a Kronecker product [6], so that C_q and D_q appear again in C_{zw} and D_{zw} only.

III. Cutting Plane Algorithm

This section present the CPA applied to a feasibility problem. As presented in [4], the problem under consideration is:

$$\text{Find } x \quad \text{subj to} \quad \mathcal{S}_x > 0 \tag{7}$$

where x is the vector of decision variables, and $\mathcal{S}_x$ is a set expressed on matrix constraint form. The problem (7) is convex if $\mathcal{S}_x$ is affine on x.

The problem (7) can be rewritten on an equivalent form as an eigenvalue maximization problem :

$$\sup_{x,y} \; y \quad \text{subj to} \quad \begin{cases} \mathcal{S}_x - yI > 0 \\ y < 1 \end{cases} \tag{8}$$

The problem is feasible if $y > 0$. From (8) a concave function is defined:

$$q(x) = \sup \{y : \mathcal{S}_x - yI > 0, y < 1\} \tag{9}$$

Optimizing problem (8) is equivalent to the optimization problem without constraint:

$$y_{opt} = \sup_x q(x) \tag{10}$$

For solving the problem (10), the method of Kelly [8] is commonly used. This method needs the calculation of the sub-gradient of $q(x)$. Since no analytic form is available for $q(x)$, the calculation of the sub-gradient is a difficult problem. In [4] an alternative is proposed, by solving a Linear Programming Problem (LPP) $p_k(x)$ (index k is used because several LPPs will be iteratively defined), after having at best delimited the function $q(x)$ by hyperplanes. The figure bellow gives an interpretation of this bound in the scalar case.

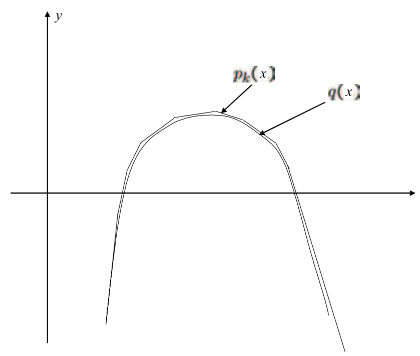

Fig. 2. Bounding $q(x)$ by hyperplanes

The function $q(x)$ is bounded iteratively by a set of hyperplanes leading to a piecewise linear function $p_k(x)$:

$$q(x) \leq p_k(x) := \min_{1 \leq i \leq k} \{a_i x - b_i\} \tag{11}$$

At iteration k the LPP to be solved is:

$$\max_{x_{\min} \leq x \leq x_{\max}} p_k(x) \tag{12}$$

with $x_{\min}$ and $x_{\max}$ defining some numerical limits of the components of vector x.

Before presenting the CPA, it is assumed that there is a mechanism which checks the inequalities and generates the hyperplanes: it will be exposed in the next section.

The algorithm begins with an initial value y_l belonging to the feasible set: this value is simply founded from physical considerations for example. The initialization of the hyperplanes can be done by considering the diagonal of constraints $\mathcal{S}_x - yI > 0$ for example, which lead to an initial value of $y^{(0)}$ and $x^{(0)}$ solution of the LPP (12) for $p_0(x)$.

At each iteration a linear interpolation involving a parameter $\alpha \in [0,1]$ derives $\hat{y}^{(k)}$:

$$\hat{y}^{(k)} = \alpha y^{(k)} + (1-\alpha) y_l \tag{13}$$

The mechanism juges if the set of constraints $\mathcal{S}_x - \hat{y}^{(k)} I > 0$ is verified: if yes, the value of y_l is replaced by $\hat{y}^{(k)}$ else, the mechanism adds new hyperplanes and the $(k+1)^{th}$ LPP is solved. The principle of the CPA is simple, but the main task is to verify the constraints and to generate the hyperplanes.

IV. Frequency domain specification

In this section, continuous-time representations are assumed. If the plant is considered in discrete-time, the Tustin transform can be applied to get an equivalent continuous-time plant:

$$\begin{array}{rcl} A_{zw_c} & = & -\frac{2}{T}I + \frac{4}{T}A_{zw}(I+A_{zw})^{-1} \\ B_{zw_c} & = & \frac{4}{T}(I+A_{zw})^{-1}B_{zw} \\ C_{zw_c}(x) & = & C_{zw}(x)(I+A_{zw})^{-1} \\ D_{zw_c}(x) & = & D_{zw}(x) - C_{zw}(x)(I+A_{zw})^{-1}B_{zw} \end{array} \tag{14}$$

where vector x contains the decision variables in (C_q, D_q) and T is the sample time. For simplicity reasons the index $_{zw}$ or $_{zw_c}$ is omitted in the rest of the paper.

Consider the problem of frequency domain attenuation under gain value γ, where only the output matrices depend on the decision vector x; this problem can be formulated as:

$$\begin{array}{l} \text{find } x \quad \text{subj to :} \\ H(x,\gamma,\omega) = \\ \begin{pmatrix} \hat{G}(j\omega) \\ I \end{pmatrix}^* \begin{pmatrix} Q(x) & F(x) \\ F(x)^T & R(x,\gamma) \end{pmatrix} \begin{pmatrix} \hat{G}(j\omega) \\ I \end{pmatrix} > 0, \\ \forall \omega \in [\omega_1, \omega_2] \end{array} \tag{15}$$

where:

$$\begin{array}{l} \hat{G}(j\omega) = (j\omega I - A)^{-1} B \\ Q(x) = -C(x)^T C(x) \\ F(x) = -C(x)^T D(x) \\ R(x,\gamma) = -D(x)^T D(x) + \gamma^2 I \end{array}$$

The Hamiltonian matrix associated with $H(x,\gamma,\omega)$ is defined as:

$$\mathcal{H}(x,\gamma) = \begin{pmatrix} A - BR(x,\gamma)^{-1}F(x)^T & BR(x,\gamma)^{-1}B^T \\ Q(x) - F(x)R(x,\gamma)^{-1}F(x)^T & -A^T + F(x)R(x,\gamma)^{-1}B^T \end{pmatrix} \tag{16}$$

If the whole frequency range $\omega \in [0,\infty]$ is considered, then the Kalman-Yakubovich-Popov (KYP) lemma [9] gives several equivalent conditions to the positive definiteness of $H(x,\gamma,w)$; only one of them is considered below:

Lemma 1: Assume the pair of matrices $[A,B]$ is stabilizable. The following statements are equivalent:

1) $H(x,\gamma,\omega) > 0$, $\forall \omega \in [0,\infty]$.
2) $R(x,\gamma) > 0$, and the Hamiltonian $\mathcal{H}(x,\gamma)$ has no eigenvalues on the imaginary axis.

A specification on a particular frequency range $[\omega_1,\omega_2]$ is considered by slightly modifying the KYP lemma:

Lemma 2: $(KYP\ modified)$ Assume the pair of matrices $[A,B]$ is stabilizable. The following statements are equivalent:

1) $H(x,\gamma,\omega) > 0$, $\forall \omega \in [\omega_1,\omega_2]$.
2) $R(x,\gamma) \neq 0$, $H(x,\gamma,\omega_2) > 0$, and the Hamiltonian $\mathcal{H}(x,\gamma)$ has no eigenvalues on the imaginary axis belonging in $[j\omega_1, j\omega_2]$.

Proof. The equivalence induced in lemma 2 is mainly due to the fact that if ω is a solution of $H(x,\gamma,\omega)=0$, then the Hamiltonian $\mathcal{H}(x,\gamma)$ has an eigenvalue on the imaginary axis at $j\omega$.

To establish this fact, a state space representation of $H(x,\gamma,\omega)$ can be derived by writing $H(x,\gamma,\omega)$ successively as:

$$H: \left(\begin{pmatrix}\left(\begin{array}{c|c} A & B \\ \hline I & 0\end{array}\right) \\ I\end{pmatrix}\right)^* \begin{pmatrix} Q(x) & F(x) \\ F(x)^T & R(x,\gamma)\end{pmatrix} \left(\begin{pmatrix}\left(\begin{array}{c|c} A & B \\ \hline I & 0\end{array}\right) \\ I\end{pmatrix}\right)$$

$$H: \left(\begin{array}{c|c} A & B \\ \hline I & 0 \\ 0 & I\end{array}\right)^* \begin{pmatrix} Q(x) & F(x) \\ F(x)^T & R(x,\gamma)\end{pmatrix} \left(\begin{array}{c|c} A & B \\ \hline I & 0 \\ 0 & I\end{array}\right)$$

$$H: \left(\begin{array}{c|cc} -A^T & Q(x) & F(x) \\ \hline -B^T & F(x)^T & R(x,\gamma)\end{array}\right) \left(\begin{array}{c|c} A & B \\ \hline I & 0 \\ 0 & I\end{array}\right)$$

$$H: \left(\begin{array}{cc|c} A & 0 & B \\ Q(x) & -A^T & F(x) \\ \hline F(x)^T & -B^T & R(x,\gamma)\end{array}\right)$$

The transmission zeros of H are the poles of H^{-1} for $R(x,\gamma) \neq 0$, which are the eigenvalues of:

$$\begin{array}{l} \begin{pmatrix} A & 0 \\ Q(x) & -A^T \end{pmatrix} - \\ \qquad \begin{pmatrix} B \\ F(x) \end{pmatrix} R(x,\gamma)^{-1} \begin{pmatrix} F(x)^T & -B^T \end{pmatrix} = \\ \begin{pmatrix} A - BR(x,\gamma)^{-1}F(x)^T & BR(x,\gamma)^{-1}B^T \\ Q(x) - F(x)R(x,\gamma)^{-1}F(x)^T & -A^T + F(x)R(x,\gamma)^{-1}B^T \end{pmatrix} \end{array}$$

which is exactly $\mathcal{H}(x,\gamma)$.

If $\omega_2 = \infty$, according to the precedent development, there is no need to add $R(x,\gamma) \neq 0$ as a constraint in lemma 2, because $H(x,\gamma,\infty) = R(x,\gamma)$. The role of $H(x,\gamma,\omega_2) > 0$ constraint is to assure that the modulus of G is less than γ on the whole interval of frequency, even if there is no intersection between the modulus of G and the straight line corresponding to γ in this interval: as showed in Figure 3, for γ_1 as gain constraint there is an intersection, which allows to detect the non-feasibility of the contraint; on the other hand, for γ_2 there is no intersection, and only the constraint $H(x, \gamma, \omega_2) > 0$ enables to detect the non-feasibility.

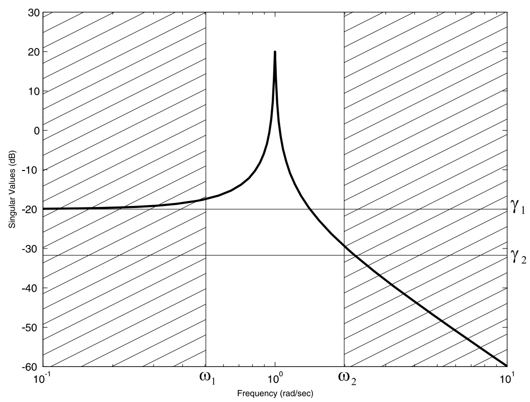

Fig. 3. Example showing the utility of $H(x, \gamma, \omega_2) > 0$ constraint

The lemma 2 is the part of the mechanism allowing to verify the set of constraints $\mathcal{S}_x - \hat{y}^{(k)} I > 0$. It remains to find out the second function of the mechanism, which is the creation of the hyperplanes if the set of constraints are not verified.

First, from constraint (15) which is not affine in x, a linearity on x is restored using the Schur lemma:

$$\hat{H}(x, \gamma, \omega) = \begin{pmatrix} \gamma^2 & \begin{pmatrix} C(x) & D(x) \end{pmatrix} \begin{pmatrix} G \\ I \end{pmatrix} \\ \begin{pmatrix} G^H & I \end{pmatrix} \begin{pmatrix} C(x)^T \\ D(x)^T \end{pmatrix} & I \end{pmatrix} > 0 \tag{17}$$

Assume the Hamiltonian has an eigenvalue $j\hat{\omega}$ belonging in $[j\omega_1, j\omega_2]$. Injecting the values $j\hat{\omega}$ on expression (17) leads to the hyperplane contraints:

$$v_i^T \left(\hat{H}(x, \gamma, \hat{\omega}) - \hat{y}^{(k)} I \right) v_i < 0 \tag{18}$$

where the v_i are the eigenvectors associated to the negative eigenvalues of the matrix in (18).

The Schur lemma is also applied on $R(x, \gamma)$ to restore the affinity in x:

$$\hat{R}(x, \gamma) = \begin{pmatrix} \gamma^2 & D(x) \\ D(x)^T & I \end{pmatrix} > 0 \tag{19}$$

As for (18) hyperplanes are derived from (19).

V. Numerical example

The digital control of a hard-disk read/write head is considered. It is taken from a Matlab demo [10]. The head-disk assembly (HDA) and actuators are modeled by a SISO system where the input is the current i_c driving the voice coil motor and the output is the position error signal $\epsilon_\theta = \theta_{ref} - \theta$. The order of the state space representation is 10 including two rigid-body modes and the first four resonances. The model also includes a small delay $T_r = 10^{-5}$ sec.

Only the rigid modes are considered for the compensator design, although the bending modes are to be attenuated while rejecting an input disturbance with zero static error. The rigid model is first discretized using a zero-order hold with sample time $T = 7.10^{-5}$ sec. To handle disturbance rejection, a $\mathcal{H}_2$ synthesis is applied to the rigid model, which produces an instability when the resulting controller is applied to the complete model (see Figure 4). This instability is due to the resonance peak on the closed-loop transfer function (Figure 5, solid line).

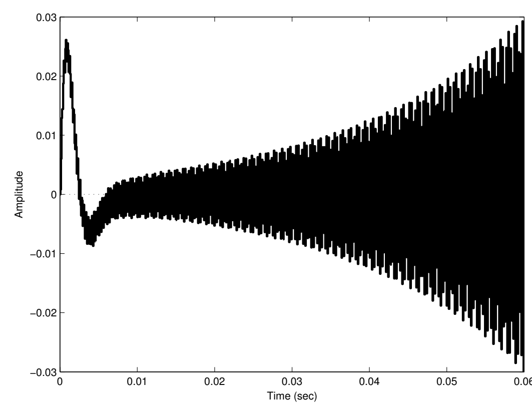

Fig. 4. Instability introduced by the $\mathcal{H}_2$ compensator on the flexible model

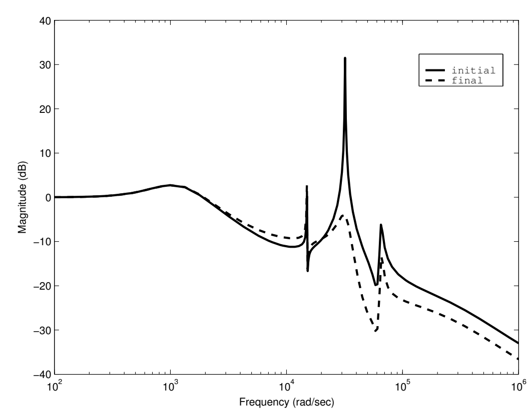

Fig. 5. Reduction of the resonance peak

To reduce the resonance peak while preserving the time-domain performance, the range of frequencies where γ is minimized is chosen as $[2.9.10^4\ , 3.5.10^4]$, while maintaining the $\mathcal{H}_2$ constraint. Since only a problem of feasibility is presented on this work, the value of γ is decreased at each test of feasibility for a given order n_q of the Youla parameter. The value $n_q = 1$ is not sufficient to reduce significantly the magnitude. For $n_q = 2$ the value of $\gamma = 0.1$ on the rigid model is sufficient to reduce the peak resonance (Figure 5, dashed line). The time-domain performance is preserved, as shown in Figure 6.

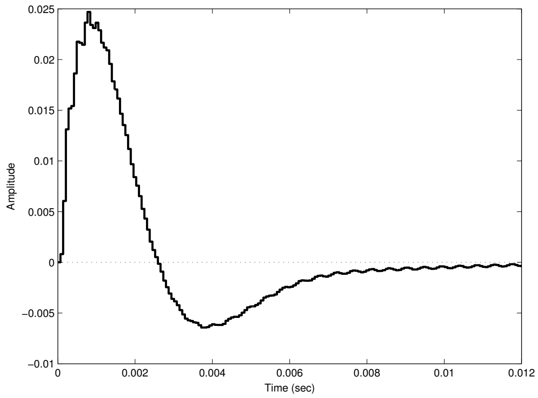

Fig. 6. Impulse response for the CPA compensator

VI. Conclusion

Designing a compensator, by considering gain attenuation in a particular frequency domain (for instance to obtain bending modes attenuation), can be done through the Youla parametrization without weighting filters, thus avoiding to increase the order of the compensator or to deteriorate the stability margins. By considering the cutting plane algorithm, no additional variables have to be introduced, as for instance when using optimization under LMI constraints. The numerical efficiency of the proposed developments has been showed by considering an example where a resonance peak has been efficiently reduced, without acting on the bandwidth.

VII. Acknowledgement

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