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Global Finite-Time Stabilization of a Class of Nonsmooth Nonlinear Systems by Output Feedback

Ji Li and Chunjiang Qian

***Abstract*— This paper studies the problem of *global finite-time* stabilization by output feedback for a class of *nonsmooth* nonlinear systems which can be viewed as the dual of the high-order systems considered in [9], [10]. By extending the adding a power integrator technique [9] and nonsmooth observer design in [8], an output feedback controller is explicitly constructed to render the nonsmooth nonlinear systems globally finite-time stable. The novelty of the paper is the development of a recursive design procedure to construct the nonsmooth observer with rigorous gains.**

I. INTRODUCTION

In this paper, we consider the output feedback stabilization problem of a class of *nonsmooth* nonlinear systems of the form

$$\begin{aligned} \dot{x}_1 &= x_2^r + \phi_1(x_1) \\ \dot{x}_2 &= x_3^r + \phi_2(x_1, x_2) \\ &\vdots \\ \dot{x}_n &= u + \phi_n(x_1, \cdots, x_n) \\ y &= x_1 \end{aligned} \quad (1.1)$$

where $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$, $y \in \mathbb{R}$ and $u \in \mathbb{R}$ are the system state, output and control input, respectively. For $i = 1, \cdots, n$, $\phi_i : \mathbb{R}^i \to \mathbb{R}$, is a continuous function with $\phi_i(0, \cdots, 0) = 0$. The system is nonsmooth due to the presence of the nonlinearity x_i^r where the number $r \in (0, 1)$ is a fraction of two odd integers.

In the nonlinear control community, the global output feedback stabilization is one of the most fundamental and challenging problems. Over the past decade, a number of researchers have studied this difficult problem and obtained some interesting results (see the survey paper [7] and the references therein). Note that almost all the existing results in global output feedback stabilization are for nonlinear systems which are at least C^1.

When the nonlinear system is *nonsmooth* (e.g. (1.1) with $r < 1$), the problem of output feedback stabilization has been rarely considered. Nonsmooth system (1.1) is of interest because it can be viewed as the dual of the high-order nonlinear systems with $r > 1$ studied in [9]. The high-order nonlinear systems have been well-studied and recently a number of interesting results have been achieved including the global stabilization by state feedback [9] and by output feedback [10]. Compared to the advance for the high-order system, there is almost no existing result on the global output feedback stabilization of nonsmooth system (1.1). As a matter of fact, even the state-feedback stabilization of (1.1) has not been addressed. In this paper, we aim to tackle the output feedback stabilization problem of (1.1). Our objective is to design an output feedback controller which renders all the trajectories of (1.1) globally finite-time stable, rather than asymptotically stable.

The concept of finite-time stabilization by time-invariant feedback was a relatively new idea. Some seminal results on the *state feedback* finite-time stabilization of dynamic systems were achieved by a number of papers [1], [2], [3], [4], [5], [11]. Compared to the state feedback case, there are fewer results dealing with *output feedback* finite-time stabilization. An interesting result was given in [6], where the finite-time stabilization of the double integrator systems was achieved by coupling a finite-time convergent observer with a finite-time control law.

The novelty of the paper is the development of a nonsmooth output feedback controller which is comprised of a finite-time state feedback controller and a nonsmooth observer. The state feedback controller is constructed using a recursive design method which is inspired by the adding a power integrator technique [9]. With the help of this state feedback controller, the remaining major difficulty is the lack of constructive observer to estimate the unmeasurable states of the *nonsmooth* systems (1.1). Recently, we introduced a one-dimensional nonsmooth observer for a class of planar systems in [8]. Although the observer was first developed for smooth systems in [8], it actually has a *nonsmooth* structure which can also be applied to nonsmooth systems. In this paper, we extend this one-dimensional observer to an (n-1)-dimensional observer for systems (1.1) and develop a rigorous procedure to recursively select the observer gains. Combining this new observer with the nonsmooth state-feedback controller, we explicitly construct an output feedback controller globally stabilizing systems (1.1) in finite time.

The rest of the paper is organized as follows: Section II includes some useful preliminary results. The main result is contained in Section III, where a design algorithm is presented for the explicit construction of a reduced-order nonlinear observer as well as a nonsmooth state feedback control law. Concluding remarks are given in Section IV.

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The authors are with the Department of Electrical Engineering, The University of Texas at San Antonio, San Antonio, TX 78249, USA. Email: cqian@utsa.edu, jli@lonestar.utsa.edu

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Appendix collects the proofs of propositions that are used in the proof of the main theorem.

II. PROBLEM STATEMENT AND PRELIMINARIES

The objective of the paper is to find a *continuous output feedback* controller of the form

$$\begin{aligned} \dot{z} &= \eta(z,y), \qquad z \in \mathbb{R}^{n-1} \\ u &= u(z,y) \end{aligned} \tag{2.1}$$

such that the closed-loop system (1.1)-(2.1) is globally finite-time stable[1]. In other words, the closed-loop system should be globally stable and all the trajectories will converge to zero in finite-time.

Next, we adopt the following Lyapunov-like Theorem which has been frequently used to determine the finite-time stability of nonlinear systems.

Theorem 2.1: For a continuous system

$$\dot{x}(t) = f(x(t)) \tag{2.2}$$

suppose there exist a C^1 positive definite and proper function V: $\mathbb{R}^n \to \mathbb{R}$ and real numbers $k > 0$ and $\alpha \in (0,1)$, such that $\dot{V} + kV^{\alpha}$ is negative semi-definite, where $\dot{V}(x) = \dfrac{\partial V(x)}{\partial x} f(x)$. Then the origin is a globally finite-time stable equilibrium of (1).

In the remainder of the section, we introduce three lemmas which will serve as the basis for the development of an output feedback finite-time controller for (1.1). The first two lemmas are the key tools for adding a power integrator and their proofs can be found in [9].

Lemma 2.1: For $a \in \mathbb{R}$, $b \in \mathbb{R}$, and $q \geq 1$, the following inequalities hold:

$$|a+b|^q \leq 2^{q-1}|a^q + b^q|,$$
$$(|a| + |b|)^{\frac{1}{q}} \leq |a|^{\frac{1}{q}} + |b|^{\frac{1}{q}}.$$

Lemma 2.2: Suppose n and m are two positive real numbers, and $a \geq 0$, $b \geq 0$ and $\pi \geq 0$ are continuous functions. Then, for any constant $c > 0$,

$$a^n b^m \pi \leq c \cdot a^{n+m} + \frac{m}{n+m}\left[\frac{n}{c(n+m)}\right]^{\frac{n}{m}} b^{n+m}\pi^{\frac{n+m}{m}}.$$

The next lemma plays a key role in constructing the observer for the nonsmooth system.

Lemma 2.3: Let $0 < r < 1$ be a ratio of odd integers. Then the following inequality holds for any real numbers $0 < \varepsilon < 1$ and t

$$(t^r + (1-t)^r) + \varepsilon^2 t^{1+r} \geq (2^r - 1)\varepsilon^{1-r}. \tag{2.3}$$

[1]For a more precise definition of global finite-time stability, please refer to [1], [3], [5].

Lemma 2.3 was first introduced and proved in [8] to provide a rigorous design procedure of a nonsmooth observer for a class of planar systems. In this paper, this lemma will also be the main observer design tool which will be used for $n-1$ times to select $n-1$ observer gains.

III. GLOBAL FINITE-TIME STABILIZATION BY OUTPUT FEEDBACK

In this section, we develop a constructive design method for an output feedback controller, which globally stabilizes the system (1.1) in finite time.

To solve the problem of finite-time stabilization via output feedback, we assume the following condition for nonlinear system (1.1).

Assumption 3.1: For $i = 1, \cdots, n$,

$$|\phi_i(x_1, \cdots, x_i) - \phi_i(\hat{x}_1, \cdots, \hat{x}_i)| \leq \sum_{m=1}^{i} a_m |x_m - \hat{x}_m|^r$$

where a_m is a positive constant for $m = 1, \cdots, n$.

Remark 3.1: At the first glance, Assumption 3.1 seems to be quite restrictive. However, there are a number of nonlinear functions actually satisfying this assumption. For example, one such function is x^r since by Lemma 2.1,

$$|x^r - \hat{x}^r| \leq 2^{1-r}|x - \hat{x}|^r, \quad r \in (0,1).$$

Moreover, there are smooth functions, such as $\sin(x)$, $\ln(1+x^2)$ and $\arctan(x)$, also satisfying this property. In fact, it is easy to verify that

$$|\sin x - \sin \hat{x}| \leq 2|x - \hat{x}|^r, \quad \text{for any} \quad r \in (0,1).$$

Remark 3.2: Note that when $\hat{x}_i = 0$ for $i = 1, \cdots, n$, Assumption 3.1 reduces to the following condition

$$|\phi_i(x_1, \cdots, x_i)| \leq \sum_{m=1}^{i} a_m |x_m|^r$$

since $\phi_i(0) = 0$. This growth condition will be crucial in constructing a homogeneous state-feedback controller.

With the help of Assumption 3.1, now we are ready to present the main result of the paper.

Theorem 3.1: There is a nonsmooth output feedback controller of the form (2.1) rendering system (1.1) globally finite-time stable.

Proof: To prove the result, we first develop a recursive design method to explicitly construct a state feedback control law for system (1.1). Then, we construct a nonsmooth observer with a set of constant gains to be determined later. Finally, the observer gains will be carefully selected to guarantee that the closed-loop system is globally finite-time stable.

I. State Feedback Controller Design

Initial Step: Construct a Lyapunov function $V_1 = \frac{x_1^2}{2}$. By Remark 3.2, it can be verified that the following holds

$$\dot{V}_1 \leq x_1 x_2^r + a_1 x_1^{1+r}$$

where $|x_1|^{1+r} = x_1^{1+r}$ by the fact that the numerator of $1+r$ is even. Clearly, the following virtual controller

$$x_2^* = -(n + a_1)^{\frac{1}{r}} x_1$$

renders

$$\dot{V}_1 \leq -n\xi_1^{1+r} + \xi_1(x_2^r - x_2^{*r})$$

where $x_1 = \xi_1$.

Inductive Step: Suppose at step $i-1$, there exist a Lyapunov function $V_{i-1}(x_1, \cdots, x_{i-1})$, which is positive definite and proper, and a set of virtual controllers $x_1^*, \cdots, x_i^*$, defined by

$$\begin{array}{ll} x_1^* = 0 & \xi_1 = x_1 - x_1^* = x_1, \\ x_2^* = -b_1\xi_1 & \xi_2 = x_2 - x_2^* = x_2 + b_1 x_1 \\ \vdots & \quad \vdots \\ x_i^* = -b_{i-1}\xi_{i-1} & \xi_i = x_i - x_i^* = x_i + b_{i-1}\xi_{i-1} \end{array} \qquad (3.1)$$

with constants $b_1 > 0, \cdots, \ b_{i-1} > 0$ such that

$$\begin{aligned} \dot{V}_{i-1} \ \leq\ & -(n-i+2)(\xi_1^{1+r} + \xi_2^{1+r} + \cdots + \xi_{i-1}^{1+r}) \\ & +\xi_{i-1}(x_i^r - x_i^{*r}). \end{aligned} \qquad (3.2)$$

It will be shown that (3.2) still holds at step i. To see how this can be done, construct the following new Lyapunov function

$$V_i(x_1, \cdots, x_i) = V_{i-1}(x_1, \cdots, x_{i-1}) + \frac{\xi_i^2}{2}.$$

Hence, the time derivative of V_i is

$$\begin{aligned} \dot{V}_i \ =\ & \dot{V}_{i-1} + \xi_i(\dot{x}_i - \dot{x}_i^*) \\ \leq\ & -(n-i+2)(\xi_1^{1+r} + \xi_2^{1+r} + \cdots + \xi_{i-1}^{1+r}) \\ & +\xi_{i-1}(x_i^r - x_i^{*r}) + \xi_i(x_{i+1}^r + \phi_i - \dot{x}_i^*). \end{aligned} \qquad (3.3)$$

Using Lemmas 2.1 and 2.2, one has the following estimate:

$$\xi_{i-1}(x_i^r - x_i^{*r}) \leq |\xi_{i-1}| 2^{1-r} |\xi_i|^r \leq \frac{1}{2}\xi_{i-1}^{1+r} + \rho\xi_i^{1+r} \qquad (3.4)$$

for a constant $\rho > 0$.

By the change of coordinates (3.1), we have

$$\begin{aligned} \xi_i(\phi_i - \dot{x}_i^*) \ \leq\ & |\xi_i||\phi_i + b_{i-1}\dot{\xi}_{i-1}| \\ =\ & |\xi_i||\phi_i + c_{i-1}\phi_{i-1} + \cdots + c_1\phi_1 + \\ & c_{i-1}x_i^r + c_{i-2}x_{i-1}^r + \cdots + c_1 x_2^r| \end{aligned}$$

where $c_1, \cdots, c_{i-1}$ are positive constants. This, together with Remark 3.2, yields

$$\begin{aligned} \xi_i(\phi_i - \dot{x}_i^*) \ \leq\ & |\xi_i|(\sum_{m=1}^{i} a_m |x_m|^r + c_{i-1}\sum_{m=1}^{i-1} a_m |x_m|^r \\ & + \cdots + c_1 a_1 |x_1|^r + c_{i-1}|x_i|^r + \\ & c_{i-2}|x_{i-1}|^r + \cdots + c_1|x_2|^r) \\ \leq\ & \tilde{c}|\xi_i|(|x_1|^r + |x_2|^r + \cdots + |x_i|^r) \end{aligned} \qquad (3.5)$$

with a constant $\tilde{c} \geq 0$.

Next, we introduce a proposition whose proof can be found in the appendix.

Proposition 1. There is a constant $h > 0$ such that

$$|x_{k+1}| \leq h(|\xi_{k+1}| + |\xi_k|), \quad k = 1, \cdots, i-1. \qquad (3.6)$$

With the help of Proposition 1, we know that there is a positive constant $\hat{\rho}$ such that

$$\begin{aligned} \xi_i(\phi_i - \dot{x}_i^*) \ \leq\ & \hat{\rho}|\xi_i|(|\xi_1|^r + |\xi_2|^r + \cdots + |\xi_i|^r) \\ \leq\ & \frac{1}{2}(\xi_1^{1+r} + \xi_2^{1+r} + \cdots + \xi_{i-1}^{1+r}) + \tilde{\rho}\ \xi_i^{1+r} \end{aligned} \qquad (3.7)$$

where $\tilde{\rho} > 0$ is a constant obtained by applying Lemma 2.2 with $b = \xi_i$ and $a = \xi_k, \ k = 1, \cdots, i-1$ respectively.

Substituting (3.4) and (3.7) into (3.3) yields

$$\begin{aligned} \dot{V}_i \ \leq\ & -(n-i+2)(\xi_1^{1+r} + \xi_2^{1+r} + \cdots + \xi_{i-1}^{1+r}) \\ & +\frac{1}{2}\xi_{i-1}^{1+r} + \rho\xi_i^{1+r} + \frac{1}{2}(\xi_1^{1+r} + \xi_2^{1+r} \\ & + \cdots + \xi_{i-1}^{1+r}) + \tilde{\rho}\ \xi_i^{1+r} + \xi_i x_{i+1}^r \\ \leq\ & -(n-i+1)(\xi_1^{1+r} + \xi_2^{1+r} + \cdots + \xi_{i-1}^{1+r}) \\ & +(\rho + \tilde{\rho})\xi_i^{1+r} + \xi_i(x_{i+1}^r - x_{i+1}^{*r}) + \xi_i x_{i+1}^{*r}. \end{aligned}$$

Hence, the virtual controller of the form

$$x_{i+1}^{*r} = -(n-i+1+\rho+\tilde{\rho})\xi_i^r$$

yields

$$\begin{aligned} \dot{V}_i \ \leq\ & -(n-i+1)(\xi_1^{1+r} + \xi_2^{1+r} + \cdots + \xi_i^{1+r}) \\ & +\xi_i(x_{i+1}^r - x_{i+1}^{*r}). \end{aligned} \qquad (3.8)$$

This completes the inductive proof.

Therefore, (3.8) is true for $i = 1, 2, \cdots, n$. In other words, when $i = n$, there is a controller

$$x_{n+1}^{*r} = -b_n\xi_n^r, \quad \text{for a constant } b_n > 0$$

and a Lyapunov function $V_n(x_1, \cdots, x_n)$ such that

$$\dot{V}_n \leq -(\xi_1^{1+r} + \xi_2^{1+r} + \cdots + \xi_n^{1+r}) + \xi_n(u - x_{n+1}^{*r}) \qquad (3.9)$$

where ξ_i is defined as (3.1) for $i = 1, \cdots, n$.

II. Construction of a Nonsmooth Observer

For an n dimensional system, we design an $n-1$ dimensional observer of the form

$$\begin{aligned} \dot{\hat{z}}_2 \ =\ & \hat{x}_3^r + \hat{\phi}_2 - \ell_1(\hat{x}_2^r + \hat{\phi}_1) \\ \dot{\hat{z}}_3 \ =\ & \hat{x}_4^r + \hat{\phi}_3 - \ell_2(\hat{x}_3^r + \hat{\phi}_2) \\ & \vdots \\ \dot{\hat{z}}_n \ =\ & u + \hat{\phi}_n - \ell_{n-1}(\hat{x}_n^r + \hat{\phi}_{n-1}) \end{aligned} \qquad (3.10)$$

where $\ell_i > 1$, $i = 1, \cdots, n-1$, are gains to be determined later and

$$\begin{aligned} \hat{x}_1 \ =\ & x_1, \quad \hat{x}_{i+1} = \hat{z}_{i+1} + \ell_i\hat{x}_i, \ i = 1, \cdots, n-1 \\ \hat{\phi}_i \ =\ & \phi_i(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_i), \ i = 1, \cdots, n. \end{aligned} \qquad (3.11)$$

Define the error

$$e_{i+1} = z_{i+1} - \hat{z}_{i+1} \tag{3.12}$$

where

$$z_{i+1} = x_{i+1} - \ell_i x_i, \ i = 1, \cdots, n-1.$$

A direct calculation yields

$$\begin{aligned} \dot{e}_2 &= x_3^r - \hat{x}_3^r + \phi_2 - \hat{\phi}_2 - \ell_1 (x_2^r - \hat{x}_2^r) \\ \dot{e}_3 &= x_4^r - \hat{x}_4^r + \phi_3 - \hat{\phi}_3 - \ell_2 \left(x_3^r - \hat{x}_3^r + \phi_2 - \hat{\phi}_2 \right) \\ &\vdots \\ \dot{e}_n &= \phi_n - \hat{\phi}_n - \ell_{n-1} \left(x_n^r - \hat{x}_n^r + \phi_{n-1} - \hat{\phi}_{n-1} \right). \end{aligned}$$

According to the definition of e_i, we have

$$e_{i+1} = x_{i+1} - \ell_i x_i - \hat{x}_{i+1} + \ell_i \hat{x}_i,$$

which in turn implies

$$x_{i+1} - \hat{x}_{i+1} = e_{i+1} + \ell_i (x_i - \hat{x}_i). \tag{3.13}$$

With this in mind, it is easy to have

$$\begin{aligned} x_2 - \hat{x}_2 &= e_2 \\ x_3 - \hat{x}_3 &= e_3 + \ell_2 e_2 \\ &\vdots \\ x_n - \hat{x}_n &= e_n + \ell_{n-1}(e_{n-1} + \ell_{n-2}(e_{n-2} \\ &\quad + \cdots + \ell_3(e_3 + \ell_2 e_2) \cdots)). \end{aligned} \tag{3.14}$$

Construct the following Lyapunov function which apparently is positive definite and proper

$$W(e_2, \cdots, e_n) = \frac{e_2^2}{2} + \frac{e_3^2}{2} + \cdots + \frac{e_n^2}{2}. \tag{3.15}$$

It is clear that the derivative of the Lyapunov function (3.15) is as follows

$$\begin{aligned} \dot{W} &= -\sum_{i=1}^{n-1} \ell_i e_{i+1} \left(x_{i+1}^r - \hat{x}_{i+1}^r \right) - \sum_{i=2}^{n-1} \ell_i e_{i+1} \left(\phi_i - \hat{\phi}_i \right) \\ &\quad + \sum_{i=2}^{n-1} e_i \left(x_{i+1}^r - \hat{x}_{i+1}^r \right) + \sum_{i=2}^{n} e_i \left(\phi_i - \hat{\phi}_i \right). \end{aligned} \tag{3.16}$$

To estimate the first term in (3.16), we introduce a proposition whose proof is included in the Appendix.

Proposition 2. There is a function $f(\ell_i) = (2^r - 1)\ell_i^{\frac{2r}{1+r}}$ such that

$$\begin{aligned} &-\sum_{i=1}^{n-1} \ell_i e_{i+1} \left(x_{i+1}^r - \hat{x}_{i+1}^r \right) \\ &\le \sum_{i=1}^{n-1} \frac{x_{i+1}^{1+r}}{\ell_i^{\frac{1-r}{1+r}}} - \sum_{i=1}^{n-1} f(\ell_i) e_{i+1}^{1+r} + \sum_{i=2}^{n-1} 2^{1-r} \ell_i^{1+r} \\ &\quad \times |e_{i+1}||x_i - \hat{x}_i|^r. \end{aligned} \tag{3.17}$$

For the term $\sum_{i=2}^{n-1} e_i (x_{i+1}^r - \hat{x}_{i+1}^r)$, applying Lemma 2.1 yields

$$\sum_{i=2}^{n-1} e_i (x_{i+1}^r - \hat{x}_{i+1}^r) \le \sum_{i=2}^{n-1} 2^{1-r} |e_i||x_{i+1} - \hat{x}_{i+1}|^r. \tag{3.18}$$

According to Assumption 3.1, the remaining terms in (3.16) can be estimated as

$$\begin{aligned} -\sum_{i=2}^{n-1} \ell_i e_{i+1} (\phi_i - \hat{\phi}_i) &\le \sum_{i=2}^{n-1} \ell_i |e_{i+1}| \sum_{m=2}^{i} a_m |x_m - \hat{x}_m|^r \\ \sum_{i=2}^{n} e_i \left(\phi_i - \hat{\phi}_i \right) &\le \sum_{i=2}^{n} |e_i| \sum_{m=2}^{i} a_m |x_m - \hat{x}_m|^r. \end{aligned}$$

Substituting the above estimates into (3.16), it is not difficult to obtain

$$\begin{aligned} &\dot{W}(e_2, \cdots, e_n) \\ &\le \sum_{i=1}^{n-1} \frac{x_{i+1}^{1+r}}{\ell_i^{\frac{1-r}{1+r}}} - \sum_{i=1}^{n-1} f(\ell_i) e_{i+1}^{1+r} + |e_2| \big[2^{1-r} |x_3 - \hat{x}_3|^r + \\ &\quad a_2 |x_2 - \hat{x}_2|^r \big] + |e_3| \big[2^{1-r} |x_4 - \hat{x}_4|^r + a_3 |x_3 - \hat{x}_3|^r \\ &\quad + (2^{1-r} \ell_2^{1+r} + \ell_2 a_2 + a_2) |x_2 - \hat{x}_2|^r \big] + \cdots \\ &\quad + |e_{n-1}| \big[2^{1-r} |x_n - \hat{x}_n|^r + a_{n-1} |x_{n-1} - \hat{x}_{n-1}|^r \\ &\quad + 2^{1-r} \ell_{n-2}^{1+r} |x_{n-2} - \hat{x}_{n-2}|^r \\ &\quad + (\ell_{n-2} + 1) \sum_{m=2}^{n-2} a_m |x_m - \hat{x}_m|^r \big] \\ &\quad + |e_n| \big[a_n |x_n - \hat{x}_n|^r + 2^{1-r} \ell_{n-1}^{1+r} |x_{n-1} - \hat{x}_{n-1}|^r) \\ &\quad + (\ell_{n-1} + 1) \sum_{m=2}^{n-1} a_m |x_m - \hat{x}_m|^r \big]. \end{aligned} \tag{3.19}$$

Recall that by (3.14) $x_k - \hat{x}_k$ can be represented by a linear combination of $e_2, \cdots, e_k$. With this in mind, we can verify that (3.19) can be further expressed as

$$\begin{aligned} &\dot{W}(e_2, \cdots, e_n) \\ &\le \sum_{i=1}^{n-1} \frac{x_{i+1}^{1+r}}{\ell_i^{\frac{1-r}{1+r}}} - \sum_{i=1}^{n-1} f(\ell_i) e_{i+1}^{1+r} \\ &\quad + |e_2| \big[2^{1-r} |e_3|^r + f_{2,2}(\ell_2) |e_2|^r \big] + \cdots \\ &\quad + |e_{n-1}| \big[2^{1-r} |e_n|^r + f_{n-1,n-1}(\ell_{n-1}) |e_{n-1}|^r \\ &\quad + \cdots + f_{n-1,2}(\ell_2, \cdots, \ell_{n-1}) |e_2|^r \big] \\ &\quad + |e_n| \big[a_n |e_n|^r + f_{n,n-1}(\ell_{n-1}) |e_{n-1}|^r + \cdots \\ &\quad + f_{n,2}(\ell_2, \cdots, \ell_{n-1}) |e_2|^r \big] \end{aligned} \tag{3.20}$$

where $f_{i,j}(\cdot)$ is a polynomial function of its variables.

Note that by Lemma 2.2,

$$|e_{n-1}||f_{n-1,2}(\cdot) e_2^r| \le e_{n-1}^{1+r} + f_{n-1,2}^{1+1/r}(\ell_2, \cdots, \ell_{n-1}) e_2^{1+r}.$$

Applying the similar argument to other terms in (3.20), we can obtain

$$\dot{W}(e_2,\cdots,e_n) \le \sum_{i=1}^{n-1}\frac{x_{i+1}^{1+r}}{\ell_i^{\frac{1-r}{1+r}}}-\sum_{i=1}^{n-1}f(\ell_i)e_{i+1}^{1+r} +\hat{c}e_n^{1+r}+\bar{f}_{n-1}(\ell_{n-1})e_{n-1}^{1+r}+\cdots +\bar{f}_2(\ell_2,\cdots,\ell_{n-1})e_2^{1+r} \quad (3.21)$$

where $\bar{f}_i(\ell_i,\cdots,\ell_{n-1})$, $i=2,\cdots,n-1$ are continuous function of $\ell_i,\cdots,\ell_{n-1}$ and $\hat{c}$ is a positive constant.

On the other hand, by Proposition 1 there is a constant c such that

$$\begin{aligned}&\dot{W}(e_2,\cdots,e_n)\\ &\le(-f(\ell_{n-1})+\hat{c})e_n^{1+r}+(-f(\ell_{n-2})+\bar{f}_{n-1}(\ell_{n-1}))\\ &\quad\times e_{n-1}^{1+r}+\cdots+(-f(\ell_1)+\bar{f}_2(\ell_2,\cdots,\ell_{n-1}))e_2^{1+r}+\\ &\quad\frac{c}{\ell_1^{\frac{1-r}{1+r}}}\xi_1^{1+r}+\cdots+(\frac{c}{\ell_{n-2}^{\frac{1-r}{1+r}}}+\frac{c}{\ell_{n-1}^{\frac{1-r}{1+r}}})\xi_{n-1}^{1+r}+\frac{c}{\ell_{n-1}^{\frac{1-r}{1+r}}}\xi_n^{1+r}.\end{aligned} \quad (3.22)$$

III. Recursive Determination of the Observer Gains

Since $x_2,\cdots,x_n$ are not measurable, we use $\hat{x}_i$, $i=2,\cdots,n$ to construct the following controller

$$u(\hat{x})=-b_n\left(\hat{x}_n+b_{n-1}(\hat{x}_{n-1}+\cdots+b_2(\hat{x}_2+b_1x_1))\right)^r. \quad (3.23)$$

To estimate the redundant term $\xi_n(u(\hat{x})-x_{n+1}^{*r})$ in (3.9), we first use Lemma 2.1 to have the following estimate

$$\begin{aligned}&|\xi_n(u(\hat{x})-x_{n+1}^{*r})|\\ &\le b_n2^{1-r}|\xi_n||\xi_n(\hat{x})-\xi_n(x)|^r\\ &=b_n2^{1-r}|\xi_n||\hat{x}_n-x_n+b_{n-1}(\hat{x}_{n-1}-x_{n-1})\\ &\quad+\cdots+b_{n-1}\cdots b_3b_2(\hat{x}_2-x_2)|^r.\end{aligned} \quad (3.24)$$

By using (3.14), it is easy to see that there are continuous functions $\hat{\gamma}_i(\ell_i,\cdots,\ell_{n-1})$ with $i=2,\cdots,n-1$ such that

$$\begin{aligned}&|\xi_n(u(\hat{x})-x_{n+1}^{*r})|\\ &\le b_n2^{1-r}|\xi_n|(|e_n|^r+\hat{\gamma}_{n-1}(\ell_{n-1})|e_{n-1}|^r+\cdots\\ &\quad+\hat{\gamma}_2(\ell_2,\cdots,\ell_{n-1})|e_2|^r).\end{aligned} \quad (3.25)$$

Similar to (3.21), applying Lemma 2.2 to (3.25) yields

$$|\xi_n(u(\hat{x})-x_{n+1}^{*r})|\le\frac{\xi_n^{1+r}}{4}+\check{c}e_n^{1+r}+\gamma_{n-1}(\ell_{n-1})e_{n-1}^{1+r} +\cdots+\gamma_2(\ell_2,\cdots,\ell_{n-1})e_2^{1+r} \quad (3.26)$$

where $\gamma_i(\ell_i,\cdots,\ell_{n-1})$, $i=2,\cdots,n-1$ are functions of $\ell_i,\cdots,\ell_{n-1}$ and $\check{c}$ is a positive constant.

Combining (3.22), (3.9) and (3.26) together, one has

$$\begin{aligned}&\dot{W}+\dot{V}\\ &\le(-f(\ell_{n-1})+\hat{c}+\check{c})e_n^{1+r}+(-f(\ell_{n-2})\\ &\quad+\bar{f}_{n-1}(\ell_{n-1})+\gamma_{n-1}(\ell_{n-1}))e_{n-1}^{1+r}+\cdots+(-f(\ell_1)\\ &\quad+\bar{f}_2(\ell_2,\cdots,\ell_{n-1})+\gamma_2(\ell_2,\cdots,\ell_{n-1}))e_2^{1+r}+(-1\\ &\quad+\frac{c}{\ell_1^{\frac{1-r}{1+r}}})\xi_1^{1+r}+\cdots+(-1+\frac{c}{\ell_{n-2}^{\frac{1-r}{1+r}}}+\frac{c}{\ell_{n-1}^{\frac{1-r}{1+r}}})\xi_{n-1}^{1+r}\\ &\quad+(-\frac{3}{4}+\frac{c}{\ell_{n-1}^{\frac{1-r}{1+r}}})\xi_n^{1+r}\end{aligned} \quad (3.27)$$

With the help of (3.27), we are now ready to choose appropriate ℓ_i. Recall $f(l_i)=(2^r-1)\ell_i^{\frac{2r}{1+r}}$, then we can select a large enough gain ℓ_{n-1} such that

$$-f(\ell_{n-1})+\hat{c}+\check{c}\le-1/2,\quad c/\ell_{n-1}^{\frac{1-r}{1+r}}\le 1/4.$$

By now we have fixed the gain ℓ_{n-1}. Next, we choose the constant ℓ_{n-2} which is sufficiently large enough to satisfy the following conditions

$$-f(\ell_{n-2})+\bar{f}_{n-1}(\ell_{n-1})+\gamma_{n-1}(\ell_{n-1})\le-1/2,\quad c/\ell_{n-2}^{\frac{1-r}{1+r}}\le 1/4.$$

Following the same line, at the last step, we choose ℓ_1 based on the fixed gains $\ell_2\cdots,\ell_{n-1}$

$$-f(\ell_1)+\bar{f}_2(\ell_2,\cdots,\ell_{n-1})+\gamma_2(\ell_2,\cdots,\ell_{n-1})\le-1/2,\quad c/\ell_1^{\frac{1-r}{1+r}}\le 1/4.$$

Under the above choice of gains, it is easy to verify that

$$\dot{V}+\dot{W}\le-\frac{1}{2}(\xi_1^{1+r}+\cdots+\xi_n^{1+r}+e_2^{1+r}+\cdots+e_n^{1+r}) \quad (3.28)$$

which is negative definite. Note that by Lemma 2.1 the following holds

$$\begin{aligned}&(V+W)^{\frac{1+r}{2}}\le|V|^{\frac{1+r}{2}}+|W|^{\frac{1+r}{2}}\\ &\le|\frac{\xi_1^2}{2}+\cdots+\frac{\xi_n^2}{2}|^{\frac{1+r}{2}}+|\frac{e_2^2}{2}+\cdots+\frac{e_n^2}{2}|^{\frac{1+r}{2}}\\ &\le(\xi_1^{1+r}+\cdots+\xi_n^{1+r}+e_2^{1+r}+\cdots+e_n^{1+r})(\frac{1}{2})^{\frac{1+r}{2}}.\end{aligned}$$

With this in mind, it can be deduced from (3.28) that

$$\dot{V}+\dot{W}+k(V+W)^{\frac{1+r}{2}}\le 0$$

for a positive constant $k>0$. Therefore, by Theorem 2.1 with $\alpha=\frac{1+r}{2}<1$ the closed-loop system (1.1)-(3.10)-(3.23) is globally finite-time stable. ∎

Clearly, when $\phi_i(\cdot)=0$, Assumption 3.1 is automatically satisfied. As a direct consequence, Theorem 3.1 reduces to the following corollary.

Corollary 3.1: There is an output feedback controller rendering

$$\dot{x}_i=x_{i+1}^r,\quad i=1,\cdots,n-1,\quad \dot{x}_n=u,\quad y=x_1$$

globally finite-time stable.

Example 3.1: Consider a controlled dynamic

$$f(\dot{y},\ddot{y})=3\ddot{y}\dot{y}^2=u.$$

This system can be modelled as a nonsmooth system

$$\dot{x}_1=x_2^{1/3},\quad \dot{x}_2=u,\quad y=x_1. \quad (3.29)$$

By Corollary 3.1, there is a global finite-time stabilizer by output feedback for system (3.29).

We conclude the section by a numerical example which illustrates the effectiveness of the proposed finite-time stabilizer constructed by output feedback.

Example 3.2: Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2^{\frac{3}{5}} \\ \dot{x}_2 &= u + \sin(\frac{x_2}{8}), \quad y = x_1. \end{aligned} \tag{3.30}$$

As shown in Remark 3.1, Assumption 3.1 is satisfied. Hence, by Theorem 3.1, system (3.30) is globally finite-time stabilizable by output feedback. In fact, the output feedback controller can be constructed as follows

$$\begin{cases} u = -4.25(\hat{x}_2 + y)^{\frac{3}{5}} \\ \dot{\hat{z}} = u + \sin(\frac{\hat{x}_2}{8}) - 16 * (\hat{x}_2)^{\frac{3}{5}} \end{cases} \tag{3.31}$$

where $\hat{x}_2 = \hat{z} + 16y$ and the gain $l_1 = 16$. The computer simulation is shown in Figure 1.

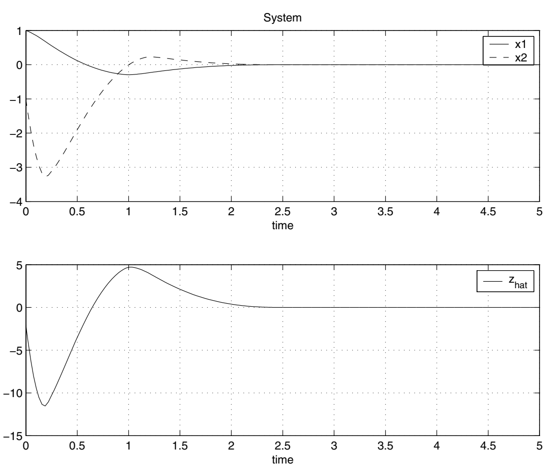

Figure 1: State trajectories of the closed-loop system (3.30)-(3.31) with $(x_1(0), x_2(0), \hat{z}(0)) = (1, -1, -2)$

IV. Conclusions

This paper investigates the problem of output feedback stabilization for a class of nonsmooth nonlinear systems which are the dual of the inherently nonlinear systems with null linearization. To solve the problem, we develop a new design approach which enables us to recursively construct a nonsmooth observer and a nonsmooth controller. An interesting feature of the proposed output feedback stabilizer is that the resulting closed-loop system will converge to the equilibrium in finite time.

Appendix

This section collects the proofs of the propositions.

Proof of Proposition 1:

By the change of coordinates (3.1), it is clear that

$$x_{k+1} = \xi_{k+1} - b_k\xi_k \le |\xi_{k+1}| + |b_k\xi_k| \le h(|\xi_{k+1}| + |\xi_k|)$$

where h is a positive constant. ∎

Proof of Proposition 2: Note that

$$\begin{aligned} &-\sum_{i=1}^{n-1} \ell_i e_{i+1}\left(x_{i+1}^r - \hat{x}_{i+1}^r\right) \\ &= -\sum_{i=1}^{n-1} \ell_i e_{i+1}\left[x_{i+1}^r - (x_{i+1} - e_{i+1})^r\right] \\ &-\sum_{i=2}^{n-1} \ell_i e_{i+1}[(x_{i+1} - e_{i+1})^r - (x_{i+1} - e_{i+1} \\ &-\ell_i(x_i - \hat{x}_i))^r]. \end{aligned} \tag{A.1}$$

By Lemma 2.1, it is easy to see that

$$\begin{aligned} &\ell_i e_{i+1}[(x_{i+1} - e_{i+1})^r - (x_{i+1} - e_{i+1} - \ell_i(x_i - \hat{x}_i))^r] \\ &\le 2^{1-r}\ell_i|e_{i+1}||\ell_i(x_i - \hat{x}_i)|^r \end{aligned} \tag{A.2}$$

When $e_{i+1} \neq 0$, substituting $t = \frac{x_{i+1}}{e_{i+1}}$, $\varepsilon = \ell_i^{\frac{-1}{1+r}}$ into (2.3) yields

$$-\ell_i e_{i+1}\left[x_{i+1}^r - (x_{i+1} - e_{i+1})^r\right] \le \frac{x_{i+1}^{1+r}}{\ell_i^{\frac{1-r}{1+r}}} - f(\ell_i)e_{i+1}^{1+r} \tag{A.3}$$

where $f(\ell_i) = (2^r - 1)\ell_i^{\frac{2r}{1+r}}$. In addition, when $e_{i+1} = 0$, (A.3) holds automatically.

Applying (A.2), (A.3) to (A.1) gives

$$\begin{aligned} &-\sum_{i=1}^{n-1} \ell_i e_{i+1}\left(x_{i+1}^r - \hat{x}_{i+1}^r\right) \\ &\le \sum_{i=1}^{n-1} \frac{x_{i+1}^{1+r}}{\ell_i^{\frac{1-r}{1+r}}} - \sum_{i=1}^{n-1} f(\ell_i)e_{i+1}^{1+r} + \sum_{i=2}^{n-1} 2^{1-r}\ell_i^{1+r}|e_{i+1}| \\ &\times \quad |(x_i - \hat{x}_i)|^r. \end{aligned}$$

∎

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