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Robust H_2 and H_∞ Control of Discrete-time Systems with Polytopic Uncertainties via Dynamic Output Feedback

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***Abstract*— This paper addresses the problem of robust H_2 and H_∞ control of discrete linear time-invariant (LTI) systems with polytopic uncertainties via dynamic output feedback. The problem has been known to be difficult when a parameter dependent Lyapunov function is to be applied for a less conservative design due to non-convexity. Our approach is based on a novel bounding technique that converts the non-convex optimization into a convex one together with a line search, which is simple but may be conservative. To further reduce the design conservatism, the sequentially linear programming method (SLPMM) is applied. A numerical example is given which demonstrates the feasibility of the proposed design methods.**

I. Introduction

During the past two decades, robust control problems for uncertain systems have been extensively discussed. There have been a number of parametric uncertainty descriptions in robust control literature. Norm bounded uncertainty is one of the important descriptions of parametric uncertainty. Based on the norm bounded uncertainty, both the robust H_∞ state feedback and output feedback control problems have been addressed in [1], [7], [9], [13], [14], [15]. On the other hand, the robust static and dynamic output feedback control problems for systems with positive real uncertainty have been investigated in [3] and [6], respectively.

Another important description of uncertainty is the so-called polytopic uncertainty where the set of system parameters is a convex polyhedron and the nominal system is located at the center of this polyhedron. For systems with polytopic uncertainty, a robust D-stability condition was developed in [10] whereas [4] considers the robust filtering problem. Recently, the robust H_∞ state feedback control and robust H_∞ static output feedback control were studied in [11], [12].

In this paper, the problem of robust dynamic output feedback control of systems with polytopic uncertainty is addressed via an LMI (linear matrix inequality) method together with a line search. It is well known that the standard technique of change of variables cannot lead to a solution of dynamic output feedback controller via convex optimization. This is due to the fact that to linearize the matrix inequality that characterizes the H_2 or H_∞ performance of the closed-loop system the introduced new variables will have to be vertex-dependent and involve the controller parameters to be sought, which implies that the required controller parameters cannot be computed from the introduced variables. To overcome this difficulty, a technique is developed in this paper which allows us to solve the robust H_2 and H_∞ control problems for systems with polytopic uncertainty via dynamic output feedback. Our first solution is sufficient and involves solving two LMIs in conjunction with a line search. To further reduce the design conservatism, the sequentially linear programming method (SLPMM) is applied. Moreover, a new characterization of the H_2 and H_∞ norms of discrete-time LTI systems is applied in this paper, which is superior to the traditional characterization as it offers more free variables and thus a higher degree of freedom in optimization. Finally, a numerical example is given which demonstrates the feasibility of the design methods developed in this paper.

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II. Problem Formulation

Consider the following system:

$$x(k+1) = Ax(k) + B_1 w(k) + B_2 u(k) \qquad (1)$$

$$z(k) = C_1 x(k) + D_{11} w(k) + D_{12} u(k) \qquad (2)$$

$$y(k) = C_2 x(k) + D_{21} w(k) \qquad (3)$$

where $x(k) \in \mathcal{R}^n$ is the state vector, $w(k) \in \mathcal{R}^p$ is the disturbance input, $u(k) \in \mathcal{R}^m$ is the control input, $y(k) \in \mathcal{R}^r$ is the measurement, and $z(k) \in \mathcal{R}^q$ is the controlled output. The matrices A, B_1, B_2, C_1, C_2, D_{11}, D_{12} and D_{21} are appropriately dimensioned. They belong to the following uncertainty polytope:

$$\Omega = \Big\{(A, B_1, B_2, C_1, C_2, D_{11}, D_{12}, D_{21}) \mid (A, B_1, B_2, C_1, C_2, D_{11}, D_{12}, D_{21}) = \sum_{i=1}^{M} \theta_i (A^{(i)}, B_1^{(i)}, B_2^{(i)}, C_1^{(i)}, C_2^{(i)}, D_{11}^{(i)}, D_{12}^{(i)}, D_{21}^{(i)}), \theta_i \geq 0, \sum_{i=1}^{M} \theta_i = 1 \Big\} \qquad (4)$$

For the convenience of expression, it is assumed that $C_2^{(i)} = C_2$ and $D_{21}^{(i)} = D_{21}$ for $i = 1, 2, \cdots, M$. Without loss of generality, we shall also assume $p = q$, i.e., the disturbance input and the signal to be estimated have the same dimension. Note that, if this is not the case, some simple modification can render the requirement being satisfied. For example, if $p < q$, the matrices B_1, D_{21} and D_{11} can be augmented as $B_1' = [B_1 \ \ 0_{n\times(q-p)}]$, $D_{21}' = [D_{21} \ \ 0_{r\times(q-p)}]$ and $D_{11}' = [D_{11} \ \ 0_{q\times(q-p)}]$, respectively.

Let a controller for the system (1)-(3) be of the form:

$$\hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{B}y(k) \qquad (5)$$

$$u(k) = \hat{C}\hat{x}(k) + \hat{D}y(k) \qquad (6)$$

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where $\hat{x}(k) \in \mathcal{R}^n$ is the state of the controller, $\hat{A},\ \hat{B},\ \hat{C},\ \hat{D}$ are the controller matrices to be determined.

First, denote $\xi(k) = [\,x^T(k) \quad \hat{x}^T(k)\,]^T$. It follows from (1)-(3) and (5)-(6) that

$$\begin{aligned} \xi(k+1) &= \bar{A}\xi(k) + \bar{B}w(k) & (7) \\ z(k) &= \bar{C}\xi(k) + \bar{D}w(k) & (8) \end{aligned}$$

where

$$\begin{aligned} \bar{A} &= \begin{bmatrix} A + B_2\hat{D}C_2 & B_2\hat{C} \\ \hat{B}C_2 & \hat{A} \end{bmatrix},\ \bar{B} = \begin{bmatrix} B_2\hat{D}D_{21} + B_1 \\ \hat{B}D_{21} \end{bmatrix} \\ \bar{C} &= [\,C_1 + D_{12}\hat{D}C_2 \quad D_{12}\hat{C}\,],\ \ \bar{D} = D_{11} + D_{12}\hat{D}D_{21}. \end{aligned}$$

The H_2 or H_∞ control problem addressed in this paper can be stated as follows: *find a dynamic output feedback controller of the form of (5)-(6) such that the closed-loop system (7)-(8) is asymptotically stable and has an optimal upper bound for the H_2 or H_∞ performance for all uncertainties belonging to the polytope Ω, i.e.*

$$\min_{(\hat{A},\hat{B},\hat{C},\hat{D})} \ \max_{(A,B_1,B_2,C_1,D_{11},D_{12})\in\Omega} \|G\|_l$$

where $G : w \mapsto z := \bar{C}(zI - \bar{A})^{-1}\bar{B} + \bar{D}$ stands for the closed-loop transfer function, and $l = 2, \infty$ represent the H_2 norm and H_∞ norm of G, respectively.

III. Robust H_2 Controller Design

To find a robust H_2 dynamic output feedback controller for the system with polytopic uncertainties, we first give the following useful technical lemma.

Lemma 3.1: Let $\Delta \in \mathcal{R}^{n\times n} > 0$, $\Psi \in \mathcal{R}^{p\times p} > 0$, $W_1 \in \mathcal{R}^{n\times n}$, $W_2 \in \mathcal{R}^{n\times n}$, $\Xi \in \mathcal{R}^{n\times n}$, $\Omega_1 \in \mathcal{R}^{p\times n}$ and $\Omega_2 \in \mathcal{R}^{p\times n}$ with W_1 being nonsingular. There exists a matrix $H > 0$ such that

$$\begin{bmatrix} W_1^T H W_1 & \Xi^T & \Omega_1^T \\ \Xi & \Delta - W_2^T H W_2 & \Omega_2^T \\ \Omega_1 & \Omega_2 & \Psi \end{bmatrix} > 0 \qquad (9)$$

if there exist matrices $\bar{H} > 0$, $\hat{H} > 0$ and $\tilde{H}$ satisfying $\bar{H}\tilde{H} = I$ such that

$$\begin{bmatrix} \bar{H} & \Xi^T & \Omega_1^T \\ \Xi & \Delta - \hat{H} & \Omega_2^T \\ \Omega_1 & \Omega_2 & \Psi \end{bmatrix} > 0 \qquad (10)$$

$$\begin{bmatrix} \tilde{H} & W_1^{-1}W_2 \\ W_2^T W_1^{-T} & \hat{H} \end{bmatrix} > 0. \qquad (11)$$

Further, (11) holds if for some scalar $\varepsilon > 0$, the following holds:

$$\begin{bmatrix} 2\varepsilon I - \varepsilon^2\bar{H} & W_1^{-1}W_2 \\ W_2^T W_1^{-T} & \hat{H} \end{bmatrix} > 0. \qquad (12)$$

Proof Letting $\hat{H} = W_2^T H W_2$, (10) implies (9) if

$$\bar{H} < W_1^T H W_1 = W_1^T W_2^{-T}\hat{H}W_2^{-1}W_1. \qquad (13)$$

It can be seen easily that (13) holds if and only if $\bar{H}^{-1} > W_1^{-1}W_2\hat{H}^{-1}W_2^T W_1^{-T}$, which by Schur complement leads to (11).

Further, since $(\bar{H}^{-1} - \varepsilon I)^T\bar{H}(\bar{H}^{-1} - \varepsilon I) \geq 0$, we have $\bar{H}^{-1} \geq 2\varepsilon I - \varepsilon^2\bar{H}$. Therefore, we know that (13) holds if

$$2\varepsilon I - \varepsilon^2\bar{H} > W_1^{-1}W_2\hat{H}^{-1}W_2^T W_1^{-T},$$

which is equivalent to (12). □

In the case when the system (1)-(3) has no parameter uncertainty, given a controller that stabilizes the system, the H_2 norm square of the closed-loop system (7)-(8) can be obtained by solving the following optimization:

$$\min_{(Q^T=Q,S^T=S)} trace(S) \qquad (14)$$

subject to

$$\begin{aligned} \bar{A}^T Q\bar{A} - Q + \bar{C}^T\bar{C} &< 0 & (15) \\ \bar{B}^T Q\bar{B} + \bar{D}^T\bar{D} &< S & (16) \end{aligned}$$

An alternative characterization of the H_2 norm of a discrete-time LTI system (7)-(8) without parameter uncertainty is given below.

Lemma 3.2: The H_2 norm square of the system (7)-(8) can be obtained by the following minimization:

$$\min_{(Q^T=Q,S^T=S,\Sigma)} trace(S) \qquad (17)$$

subject to

$$\tilde{A}^T diag\{Q, I\}\tilde{A} - \begin{bmatrix} Q & \Sigma \\ \Sigma^T & S \end{bmatrix} < 0,\ \ \tilde{A} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix}. \qquad (18)$$

Proof First, (18) can be rewritten as

$$\begin{bmatrix} \bar{A}^T Q\bar{A} + \bar{C}^T\bar{C} - Q & \bar{A}^T Q\bar{B} + \bar{C}^T\bar{D} - \Sigma \\ \bar{B}^T Q\bar{A} + \bar{D}^T\bar{C} - \Sigma^T & \bar{B}^T Q\bar{B} + \bar{D}^T\bar{D} - S \end{bmatrix} < 0. \qquad (19)$$

It is then clear from (19) that if there exists a solution (Q, S, Σ) to (18), the (1,1)-th block and (2,2)-th block of (18) imply (15) and (16), respectively. On the other hand, if there exists a solution (Q, S) satisfying (15) and (16), then by letting $\Sigma = \bar{A}^T Q\bar{B} + \bar{C}^T\bar{D}$, (18) is also satisfied with the same Q and S.

Remark 3.1: The characterization of the H_2 norm in the above lemma has the advantage that the parameter Σ in (18) will give an additional freedom when designing controllers for uncertain systems. Furthermore, it provides a unified treatment of H_2 and H_∞ control via the LMI approach. In fact, the H_∞ norm of the closed-loop system (7)-(8) is less than γ if and only if (18) has a positive definite solution for $\Sigma = 0$ and $S = \gamma^2 I$ [15]. Furthermore, it is easy to know [2] that (18) holds for some matrices Q, S and Σ if and only if there exist matrices Q, S, Σ and Φ such that

$$\begin{bmatrix} -Q & -\Sigma & \bar{A}^T\Phi & \bar{C}^T \\ -\Sigma^T & -S & \bar{B}^T\Phi & \bar{D}^T \\ \Phi^T\bar{A} & \Phi^T\bar{B} & Q - (\Phi + \Phi^T) & 0 \\ \bar{C} & \bar{D} & 0 & -I \end{bmatrix} < 0, \qquad (20)$$

where an additional freedom Φ is introduced to the LMI.

An upper bound of the H_2 norm square of the uncertain closed-loop system (7)-(8) can be computed by

$$\min_{(Q^{(i)T}=Q^{(i)},S^T=S,\Sigma^{(i)})} trace(S) \qquad (21)$$

subject to

$$\begin{bmatrix} -Q^{(i)} & -\Sigma^{(i)} & \bar{A}^{(i)T}\Phi & \bar{C}^{(i)T} \\ -\Sigma^{(i)T} & -S & \bar{B}^{(i)T}\Phi & \bar{D}^{(i)T} \\ \Phi^T\bar{A}^{(i)} & \Phi^T\bar{B}^{(i)} & Q^{(i)} - (\Phi + \Phi^T) & 0 \\ \bar{C}^{(i)} & \bar{D}^{(i)} & 0 & -I \end{bmatrix} < 0. \qquad (22)$$

Since Φ is invertible due to $\Phi + \Phi^T > Q^{(i)} > 0$, we denote

$$\Phi = \begin{bmatrix} X & \bar{M} \\ M_1 & U \end{bmatrix},\ \Phi^{-1} = \begin{bmatrix} Y & \bar{N} \\ N_1 & V \end{bmatrix}$$

$$J = \begin{bmatrix} Y & I_n \\ N_1 & 0 \end{bmatrix},\ J_1 = diag\{J, I_n, J, I_n\}.$$

Multiplying from the left and the right of (22) by J_1^T and J_1 respectively, we obtain

$$\begin{bmatrix} -P_{11}^{(i)} & * & * & * & * & * \\ -P_{12}^{(i)T} & -P_{22}^{(i)} & * & * & * & * \\ -\Lambda_1^{(i)T} & -\Lambda_2^{(i)T} & -S & * & * & * \\ A^{(i)}Y + B_2^{(i)}V^{(i)} & A^{(i)} + B_2^{(i)}\hat{D}C_2^{(i)} & B_1^{(i)} + B_2^{(i)}\hat{D}D_{21}^{(i)} & P_{11}^{(i)} - Y^T - Y & * & * \\ U^{(i)} & X^TA^{(i)} + W^{(i)}C_2^{(i)} & X^TB_1^{(i)} + W^{(i)}D_{21}^{(i)} & P_{12}^{(i)T} - I - Z & P_{22}^{(i)} - X^T - X & * \\ C_1^{(i)}Y + D_{12}^{(i)}V^{(i)} & C_1^{(i)} + D_{12}^{(i)}\hat{D}C_2^{(i)} & D_{11}^{(i)} + D_{12}^{(i)}\hat{D}D_{21}^{(i)} & 0 & 0 & -I \end{bmatrix} < 0 \qquad (23)$$

where $P^{(i)} = [P_{lk}^{(i)}] = J^TQ^{(i)}J$, $[\Lambda_1^{(i)^T}\ \Lambda_2^{(i)^T}]^T = J^T\Sigma^{(i)}$ for $k, l, j = 1, 2$, and

$$\begin{aligned} U^{(i)} &= X^T(A^{(i)} + B_2^{(i)}\hat{D}C_2^{(i)})Y + M_1^T\hat{B}C_2^{(i)}Y \\ &\quad + X^TB_2^{(i)}\hat{C}N_1 + M_1^T\hat{A}N_1 && (24) \\ V^{(i)} &= \hat{C}N_1 + \hat{D}C_2^{(i)}Y && (25) \\ W^{(i)} &= X^TB_2^{(i)}\hat{D} + M_1^T\hat{B} && (26) \\ Z &= X^TY + M_1^TN_1. && (27) \end{aligned}$$

Although (23) is already an LMI, it is clear from (24)-(27) that one cannot obtain the fixed controller parameters $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$. This imposes the major difficulty in output feedback control of systems with polytopic uncertainty. In the following, we shall address this problem by invoking Lemma 3.1. To this end, pre- and post-multiplying (23) by $diag\{Y^{-T}, I, I, I, X^{-T}, I\}$ and $diag\{Y^{-1}, I, I, I, X^{-1}, I\}$, respectively, and then by some row-column changes, we have

$$\left[\begin{array}{cc|cc|cc} -\tilde{P}_{11}^{(i)} & * & * & * & * & * \\ -\tilde{P}_{12}^{(i)T} & -\tilde{P}_{22}^{(i)} & & * & * & * \\ \hline A^{(i)} + B_2^{(i)}V^{(i)}Y^{-1} & A^{(i)} + B_2^{(i)}\hat{D}C_2^{(i)} & \hat{P}_{11}^{(i)} - Y^T - Y & * & * & * \\ X^{-T}U^{(i)}Y^{-1} & A^{(i)} + X^{-T}W^{(i)}C_2^{(i)} & \hat{P}_{12}^{(i)T} - X^{-T} - \bar{Z} & * & * & * \\ \hline C_1^{(i)} + D_{12}^{(i)}V^{(i)}Y^{-1} & C_1^{(i)} + D_{12}^{(i)}\hat{D}C_2^{(i)} & 0 & \hat{P}_{22}^{(i)} - X^{-T} - X^{-1} & -I & * \\ -\bar{\Lambda}_1^{(i)T} & -\bar{\Lambda}_2^{(i)T} & (B_1^{(i)} + B_2^{(i)}\hat{D}D_{21}^{(i)})^T & (B_1^{(i)} + X^{-T}W^{(i)}D_{21}^{(i)})^T & (D_{11}^{(i)} + D_{12}^{(i)}\hat{D}D_{21}^{(i)})^T & -S \end{array}\right] < 0 \qquad (28)$$

where $\bar{\Lambda}_1^{(i)T} = \Lambda_1^{(i)T}Y^{-1}$, $\bar{\Lambda}_2^{(i)} = \Lambda_2^{(i)}$, $\bar{Z} = Y + X^{-T}M_1^TN_1$ and

$$\begin{aligned} \tilde{P}^{(i)} &= W_1^TP^{(i)}W_1 = \begin{bmatrix} \tilde{P}_{11}^{(i)} & \tilde{P}_{12}^{(i)} \\ \tilde{P}_{12}^{(i)T} & \tilde{P}_{22}^{(i)} \end{bmatrix} \\ &= \begin{bmatrix} Y^{-T} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P_{11}^{(i)} & P_{12}^{(i)} \\ P_{12}^{(i)T} & P_{22}^{(i)} \end{bmatrix} \begin{bmatrix} Y^{-1} & 0 \\ 0 & I \end{bmatrix}, \end{aligned} \qquad (29)$$

$$\begin{aligned} \hat{P}^{(i)} &= W_2^TP^{(i)}W_2 = \begin{bmatrix} \hat{P}_{11}^{(i)} & \hat{P}_{12}^{(i)} \\ \hat{P}_{12}^{(i)T} & \hat{P}_{22}^{(i)} \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & X^{-T} \end{bmatrix} \begin{bmatrix} P_{11}^{(i)} & P_{12}^{(i)} \\ P_{12}^{(i)T} & P_{22}^{(i)} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & X^{-1} \end{bmatrix}. \end{aligned} \qquad (30)$$

From the expressions of $B_2^{(i)}V^{(i)}Y^{-1}$, $X^{-T}W^{(i)}C_2^{(i)}$, $X^{-T}U^{(i)}Y^{-1}$ in (28) and the equations (24)-(27), it is clear that the controller parameters are separated from the plant matrices. Now, we define a new set of variables as follows:

$$\begin{aligned} \bar{X} &= X^{-1} && (31) \\ U &= \bar{X}^TM_1^T\hat{A}N_1Y^{-1} && (32) \\ V &= \hat{C}N_1Y^{-1} && (33) \\ W &= \bar{X}^TM_1^T\hat{B}. && (34) \end{aligned}$$

Then from (28) and (31)-(34), we can obtain

$$\begin{bmatrix} -\tilde{P}_{11}^{(i)} & * & * & * & * & * \\ -\tilde{P}_{12}^{(i)T} & -\tilde{P}_{22}^{(i)} & * & * & * & * \\ A^{(i)} + B_2^{(i)}V + B_2^{(i)}\hat{D}C_2^{(i)} & A^{(i)} + B_2^{(i)}\hat{D}C_2^{(i)} & \hat{P}_{11}^{(i)} - Y^T - Y & * & * & * \\ A^{(i)} + B_2^{(i)}\hat{D}C_2^{(i)} + WC_2^{(i)} + B_2^{(i)}V + U & A^{(i)} + B_2^{(i)}\hat{D}C_2^{(i)} + WC_2^{(i)} & \hat{P}_{12}^{(i)T} - \bar{X}^T - \bar{Z} & \hat{P}_{22}^{(i)} - \bar{X}^T - \bar{X} & * & * \\ C_1^{(i)} + D_{12}^{(i)}V + D_{12}^{(i)}\hat{D}C_2^{(i)} & C_1^{(i)} + D_{12}^{(i)}\hat{D}C_2^{(i)} & 0 & 0 & -I & * \\ -\bar{\Lambda}_1^{(i)T} & -\bar{\Lambda}_2^{(i)T} & (B_1^{(i)} + B_2^{(i)}\hat{D}D_{21}^{(i)})^T & (B_1^{(i)} + WD_{21}^{(i)} + B_2^{(i)}\hat{D}D_{21}^{(i)})^T & \Pi_i^T & -S \end{bmatrix} < 0, \qquad (35)$$

where $\Pi_i = D_{11}^{(i)} + D_{12}^{(i)}\hat{D}D_{21}^{(i)}$. Obviously, the above matrix inequality is linear in $(U, V, W, \hat{D}, \bar{\Lambda}_1^{(i)}, \bar{\Lambda}_2^{(i)}, Y, \bar{X}, \bar{Z}, S)$, however it cannot be solved directly by using the LMI approach since from (29)-(30) it is known that the variables $\tilde{P}^{(i)}$ and $\hat{P}^{(i)}$ are not independent variables. To this end, we apply Lemma 3.1 and have the following result.

Theorem 3.1: The robust output feedback H_2 control problem for the system (1)-(3) with polytopic uncertainties is solvable if for some scalar $\varepsilon > 0$, there exists a solution $(\bar{P}_{11}^{(i)}, \bar{P}_{12}^{(i)}, \bar{P}_{22}^{(i)}, \hat{P}_{11}^{(i)}, \hat{P}_{12}^{(i)}, \hat{P}_{22}^{(i)}, U, V, W, \hat{D}, \bar{\Lambda}_1^{(i)}, \bar{\Lambda}_2^{(i)}, Y, \bar{X}, \bar{Z}, S)$ to the following optimization:

$$\min trace(S)$$

subject to

$$\Psi(S, \bar{P}_{11}^{(i)}, \bar{P}_{12}^{(i)}, \bar{P}_{22}^{(i)}, \hat{P}_{11}^{(i)}, \hat{P}_{12}^{(i)}, \hat{P}_{22}^{(i)}, U, V, W, \hat{D}, \bar{\Lambda}_1^{(i)}, \bar{\Lambda}_2^{(i)}, Y, \bar{X}, \bar{Z}, \varepsilon) =$$

$$\begin{bmatrix} -\bar{P}_{11}^{(i)} & * & * & * & * & * \\ -\bar{P}_{12}^{(i)T} & -\bar{P}_{22}^{(i)} & * & * & * & * \\ A^{(i)} + B_2^{(i)} V + B_2^{(i)} \hat{D} C_2^{(i)} & A^{(i)} + B_2^{(i)} \hat{D} C_2^{(i)} & \hat{P}_{11}^{(i)} - Y^T - Y & * & * & * \\ A^{(i)} + B_2^{(i)} \hat{D} C_2^{(i)} + W C_2^{(i)} + B_2^{(i)} V + U & A^{(i)} + B_2^{(i)} \hat{D} C_2^{(i)} + W C_2^{(i)} & \hat{P}_{12}^{(i)T} - \bar{X}^T - \bar{Z} & \hat{P}_{22}^{(i)} - \bar{X}^T - \bar{X} & * & * \\ C_1^{(i)} + D_{12}^{(i)} V + D_{12}^{(i)} \hat{D} C_2^{(i)} & C_1^{(i)} + D_{12}^{(i)} \hat{D} C_2^{(i)} & 0 & 0 & -I & * \\ -\bar{\Lambda}_1^{(i)T} & -\bar{\Lambda}_2^{(i)T} & (B_1^{(i)} + B_2^{(i)} \hat{D} D_{21}^{(i)})^T & (B_1^{(i)} + W D_{21}^{(i)} + B_2^{(i)} \hat{D} D_{21}^{(i)})^T & (D_{11}^{(i)} + D_{12}^{(i)} \hat{D} D_{21}^{(i)})^T & -S \end{bmatrix} < 0 \quad (36)$$

$$\Gamma(\bar{P}_{11}^{(i)}, \bar{P}_{12}^{(i)}, \bar{P}_{22}^{(i)}, \hat{P}_{11}^{(i)}, \hat{P}_{12}^{(i)}, \hat{P}_{22}^{(i)}, Y, \bar{X}, \varepsilon) =$$

$$\begin{bmatrix} 2\varepsilon I - \varepsilon^2 \bar{P}_{11}^{(i)} & * & * & * \\ -\varepsilon^2 \bar{P}_{12}^{(i)T} & 2\varepsilon I - \varepsilon^2 \bar{P}_{22}^{(i)} & * & * \\ Y & 0 & \hat{P}_{11}^{(i)} & * \\ 0 & \bar{X} & \hat{P}_{12}^{(i)T} & \hat{P}_{22}^{(i)} \end{bmatrix} > 0 \quad (37)$$

for $i = 1, 2, \cdots, M$. The controller parameters are given by

$$\hat{A} = M_1^{-T} \bar{X}^{-T} U Y N_1^{-1} \quad (38)$$

$$\hat{B} = M_1^{-T} \bar{X}^{-T} W \quad (39)$$

$$\hat{C} = V Y N_1^{-1} \quad (40)$$

where M_1 and N_1 satisfy $\bar{Z} = Y + \bar{X}^T M_1^T N_1$. Note that $\hat{D}$ is obtained in the optimization.

Remark 3.2: If for a given $\varepsilon > 0$, there exists a solution to the LMIs (36)-(37), it is easy to see that

$$\begin{bmatrix} Y^T + Y & \bar{X} + \bar{Z}^T \\ \bar{X}^T + \bar{Z} & \bar{X}^T + \bar{X} \end{bmatrix} > 0.$$

Multiplying the above from the left by $[\,I \quad -I\,]$ and from the right by $[\,I \quad -I\,]^T$, we have

$$(Y - \bar{Z}) + (Y - \bar{Z})^T < 0.$$

It is clear that $\bar{Z} - Y$ is invertible. Since $\bar{Z} = Y + X^{-T} M_1^T N_1$, M_1 and N_1 are also invertible. Thus the controller parameters in (38)-(40) can be obtained from (31)-(34).

Remark 3.3: Theorem 3.1 involves a line search for the scaling parameter ε, which can be easily carried out. On the other hand, the result may be conservative due to that a lower bound of $(\bar{P}^{(i)})^{-1}$ is used (see Lemma 3.1), where

$$\bar{P}^{(i)} = \begin{bmatrix} \bar{P}_{11}^{(i)} & \bar{P}_{12}^{(i)} \\ \bar{P}_{12}^{(i)T} & \bar{P}_{22}^{(i)} \end{bmatrix}$$

To address the conservatism, we note from Lemma 3.1 that (37) can be replaced by

$$\begin{bmatrix} R_{11}^{(i)} & * & * & * \\ R_{12}^{(i)T} & R_{22}^{(i)} & * & * \\ Y & 0 & \hat{P}_{11}^{(i)} & * \\ 0 & \bar{X} & \hat{P}_{12}^{(i)T} & \hat{P}_{22}^{(i)} \end{bmatrix} > 0 \quad (41)$$

for $i = 1, 2, \cdots, M$, where

$$R^{(i)} = \begin{bmatrix} R_{11}^{(i)} & R_{12}^{(i)} \\ R_{12}^{(i)T} & R_{22}^{(i)} \end{bmatrix} = (\bar{P}^{(i)})^{-1} \quad (42)$$

Note that the condition $R^{(i)} \bar{P}^{(i)} = I$ is equivalent to

$$\begin{bmatrix} \bar{P}^{(i)} & I \\ I & R^{(i)} \end{bmatrix} \geq 0 \quad (43)$$

and $trace(\bar{P}^{(i)} R^{(i)}) = n_x$, where n_x is the dimension of $\bar{P}^{(i)}$ and $R^{(i)}$ satisfying (43). Thus, we further need to solve the following problem

$$\min \sum_{i=1}^{M} trace(\bar{P}^{(i)} R^{(i)})$$

subject to (43).

The above problem is not convex since the function $trace(P_i Q_i)$ is bilinear. This bilinear problem has been investigated by many researchers in static output control for continuous systems and many methods have been proposed such as the cone complementarity linearization method by L.E. Ghaoui in [5] and the sequential linear programming matrix method (SLPMM) developed by F. Leibfritz in [8]. We now extend the SLPMM to solve the robust H_2 output feedback control and have the following result.

The robust optimal output feedback H_2 control for the system (1)-(3) with polytopic uncertainties can be solved by the following optimization:

- Step 1: Obtain the initial values $(\bar{P}^{(i)0}, R^{(i)0}, S^0)$ satisfying (36), (41), (43).
- Step 2: Given $(\bar{P}^{(i)k}, R^{(i)k})$, obtain a solution of $(\bar{P}^{(i)}, R^{(i)}, S)$, denoted by $(P_T^{(i)k}, R_T^{(i)k}, S_T^k)$, together with $(\hat{P}^{(i)}, U, V, W, \hat{D}, \bar{\Lambda}_1^{(i)}, \bar{\Lambda}_2^{(i)}, \bar{X}, Y, \bar{Z})$, to the convex optimization

$$\min \Big(\sum_{i=1}^{M} trace(\bar{P}^{(i)} R^{(i)k} + \bar{P}^{(i)k} R^{(i)}) + trace(S) \Big)$$
$$subject\ to\ (36), (41), (43).$$

- Step 3: If

$$\left|\sum_{i=1}^{M} trace(P_T^{(i)k}R^{(i)k} + \bar{P}^{(i)k}R_T^{(i)k}) + trace(S_T^k - S^k) - 2\sum_{i=1}^{M} trace(\bar{P}^{(i)k}R^{(i)k})\right| \leq \epsilon$$

  then stop, where ϵ is a pre-defined sufficiently small positive scalar.
- Step 4: Compute $\alpha \in [0,1]$ by solving

$$\min_{\alpha\in[0,1]} \sum_{i=1}^{M} trace((\bar{P}^{(i)k} + \alpha(P_T^{(i)k} - \bar{P}^{(i)k})(R^{(i)k} + \alpha(R_T^{(i)k} - R^{(i)k})) + trace(S^k + \alpha(S_T^k - S^k)).$$

- Step 5: Set $\bar{P}^{(i)(k+1)} = (1-\alpha)\bar{P}^{(i)k} + \alpha P_T^{(i)k}$, $R^{(i)(k+1)} = (1-\alpha)R^{(i)k} + \alpha R_T^{(i)k}$, $S^{k+1} = (1-\alpha)S^k + \alpha S_T^k$, go to step 2.

Remark 3.4: Suppose the above optimization leads to solutions $U,\ V,\ W, \bar{X},\ Y$. Then, M_1 and N_1 and thus the controller can be computed from (38)-(40).

IV. Robust H_∞ Controller Design

In this section, we will extend the technique in the last section to study the H_∞ control problem. Recall that when the system (1)-(3) is known, it is stable and has an H_∞ norm less than γ if there exists a matrix $P^T = P$ such that [15]

$$\tilde{A}^T diag\{Q, I\}\tilde{A} - diag\{Q, \gamma^2 I\} < 0. \tag{44}$$

Note that (44) is a special case of (18) when $S = \gamma^2 I$ and $\Sigma = 0$. Thus, similar to the equivalence between (18) and (20), (44) is equivalent to

$$\begin{bmatrix} -Q^{(i)} & 0 & \bar{A}^{(i)T}\Phi & \bar{C}^{(i)T} \\ 0 & -\gamma^2 I & \bar{B}^{(i)T}\Phi & \bar{D}^{(i)T} \\ \Phi^T\bar{A}^{(i)} & \Phi^T\bar{B}^{(i)} & Q^{(i)} - (\Phi+\Phi^T) & 0 \\ \bar{C}^{(i)} & \bar{D}^{(i)} & 0 & -I \end{bmatrix} < 0. \tag{45}$$

The following result gives a solution to the H_∞ control problem.

Theorem 4.1: The H_∞ control of the system (1)-(3) via dynamic output feedback can be solved if for some scalar $\varepsilon > 0$, there exists a solution $(\bar{P}_{11}^{(i)}, \bar{P}_{12}^{(i)}, \bar{P}_{22}^{(i)}, \hat{P}_{11}^{(i)}, \hat{P}_{12}^{(i)}, \hat{P}_{22}^{(i)}, U, V, W, \hat{D}, Y, \bar{X}, \bar{Z})$ to the following LMIs:

$$\Psi(\gamma^2 I, \bar{P}_{11}^{(i)}, \bar{P}_{12}^{(i)}, \bar{P}_{22}^{(i)}, \hat{P}_{11}^{(i)}, \hat{P}_{12}^{(i)}, \hat{P}_{22}^{(i)}, U, V, W, \hat{D}, \bar{\Lambda}_1^{(i)}, \bar{\Lambda}_2^{(i)}, Y, \bar{X}, \bar{Z}, \varepsilon)|_{\bar{\Lambda}_1^{(i)}=\bar{\Lambda}_2^{(i)}=0} < 0$$

$$\Gamma(\bar{P}_{11}^{(i)}, \bar{P}_{12}^{(i)}, \bar{P}_{22}^{(i)}, \hat{P}_{11}^{(i)}, \hat{P}_{12}^{(i)}, \hat{P}_{22}^{(i)}, Y, \bar{X}, \varepsilon) > 0$$

for $i = 1, 2, \cdots, M$. In this situation, the parameters of a feasible H_∞ controller can be obtained by (38)-(40).

Remark 4.1: Observe that for a given ε, (36) and (37) are linear in $(S, \bar{P}_{11}^{(i)}, \bar{P}_{12}^{(i)}, \bar{P}_{22}^{(i)}, \hat{P}_{11}^{(i)}, \hat{P}_{12}^{(i)}, \hat{P}_{22}^{(i)}, U, V, W, \hat{D}, \bar{\Lambda}_1^{(i)}, \bar{\Lambda}_2^{(i)}, Y, \bar{X}, \bar{Z})$ and hence can be solved by employing the LMI Toolbox. Then the problem is how to find the optimal value of ε in order to optimize the H_2 or H_∞ norm. The numerical optimization algorithm **fminsearch** in the Optimization Toolbox of Matlab can be employed to give a locally convergent solution to the problem.

V. Illustrative Example

Consider a system which belongs to the 2-polytopic convex polyhedron in the form of (4), where

$$A^{(1)} = \begin{bmatrix} 0.1 & -0.2 \\ 0 & 1 \end{bmatrix},\ A^{(2)} = \begin{bmatrix} 0.2 & -0.2 \\ 0 & 1 \end{bmatrix},\ B_2^{(1)} = \begin{bmatrix} -2 \\ 2 \end{bmatrix},$$

$$B_2^{(2)} = \begin{bmatrix} 4 \\ 4 \end{bmatrix},\ B_1^{(1)} = \begin{bmatrix} 1 \\ 0.4 \end{bmatrix},\ B_1^{(2)} = \begin{bmatrix} 1 \\ 0.3 \end{bmatrix},$$

$$C_2^{(1)} = C_2^{(2)} = [\,1 \quad 2\,],\ D_{21}^{(1)} = D_{21}^{(2)} = [\,0.2\,],\ C_1^{(1)} = [\,1 \quad 0\,],$$

$$C_1^{(2)} = [\,1 \quad 0.5\,], D_{11}^{(1)} = 0.1, D_{11}^{(2)} = 0.2, D_{12}^{(1)} = 1, D_{12}^{(2)} = 2.$$

We first study the H_2 performance. Letting $\varepsilon = 0.2$ and following Theorem 3.1, we obtain the controller of the form (5)-(6) with

$$\hat{A} = \begin{bmatrix} 0.5098 & 0.0327 \\ -1.4807 & -0.0951 \end{bmatrix},\ \hat{B} = \begin{bmatrix} 0.4653 \\ -1.3515 \end{bmatrix},$$

$$\hat{C} = [\,0.0349 \quad 0.0204\,],\ \hat{D} = -0.0328,$$

which gives the H_2 norm bound of 1.8147.

In the following we will use the numerical optimization algorithm **fminsearch** in the Optimization Toolbox of Matlab to obtain a local optimal upper bound of the H_2 norm. Starting from the initial value $\varepsilon_0 = 0.2$, we arrive at the minimum value of the H_2 norm bound of 1.2435 with $\varepsilon = 0.6138$. And the resultant controller is given by

$$\hat{A} = \begin{bmatrix} 0.5390 & 0.0269 \\ -1.5951 & -0.0796 \end{bmatrix},\ \hat{B} = \begin{bmatrix} 0.1414 \\ -0.4185 \end{bmatrix},$$

$$\hat{C} = [\,0.1161 \quad 0.0745\,],\ \hat{D} = -0.0327.$$

The actual H_2 performance under the resulted controller above is shown in Figure 1 when θ_1 varies from 0 to 1.

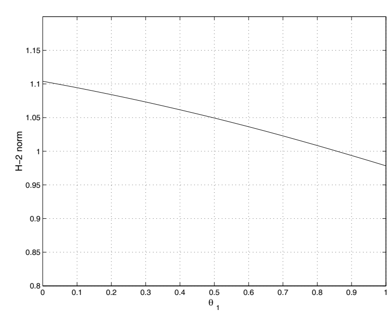

Fig. 1. Actual H_2 norm versus θ_1

To reduce the conservatism in the above results, we will use the SLPMM method in Section III. By setting $\epsilon = 1e-4$, we obtain the optimal H_2 performance with 1.0796 and the following resulted controller

$$\hat{A} = \begin{bmatrix} 1.0148 & 0.1749 \\ -3.2915 & -0.8678 \end{bmatrix},\ \hat{B} = \begin{bmatrix} 0.1683 \\ -0.5032 \end{bmatrix},$$

$$\hat{C} = [\,-0.0661 \quad -0.0305\,],\ \hat{D} = -0.0327.$$

The actual H_2 performance under the above resulted controller is shown in Figure 2 when θ_1 varies from 0 to 1, from which we can see that the SLPMM method achieves a better H_2 performance.

Letting $\varepsilon_0 = 0.4$ and applying Theorem 4.1 and the **fminsearch**, we can obtain an upper bound of the H_∞

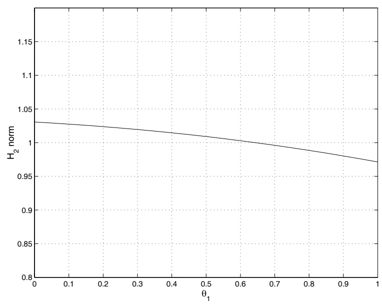

Fig. 2. Actual H_2 norm versus θ_1, SLPMM method

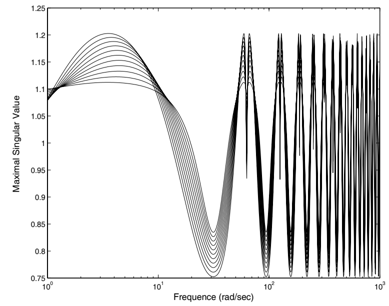

Fig. 4. The frequency responses, under the resulted controller, of 11 systems with $\theta_1 \in [0\ 1]$.

norm of 3.1366 with $\varepsilon = 0.2170$. The following controller parameters can be obtained:

$$\hat{A} = \begin{bmatrix} 0.1802 & -0.1156 \\ -0.3657 & 0.2611 \end{bmatrix},\ \hat{B} = \begin{bmatrix} 0.0588 \\ -0.5884 \end{bmatrix},$$

$$\hat{C} = [\,-0.0008 \quad 0.0037\,],\ \hat{D} = -0.0590.$$

The frequency responses are shown in Figure 3 when θ_1 varies from 0 to 1, which verifies that the H_∞ norm of each case is below the bound.

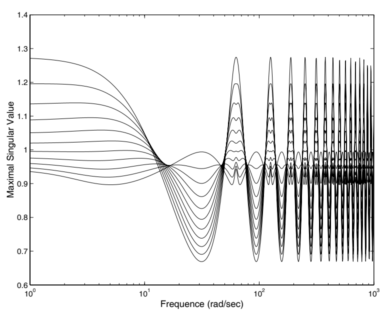

Fig. 3. The frequency responses, under the resulted controller, of 11 systems with $\theta_1 \in [0\ 1]$.

Now we will discuss the H_∞ performance by using the SLPMM method. Setting $\epsilon = 1e-4$, we obtain the optimal H_∞ bound of 1.2307 which is much smaller than 3.1366. The resultant controller parameters are given by

$$\hat{A} = \begin{bmatrix} 0.5305 & -0.0893 \\ -2.7839 & -0.0838 \end{bmatrix},\ \hat{B} = \begin{bmatrix} 0.0199 \\ -0.4772 \end{bmatrix},$$

$$\hat{C} = [\,0.0388 \quad 0.0072\,],\ \hat{D} = -0.0355.$$

The frequency responses under the above controller are shown in Figure 4, which shows that the SLPMM method is much less conservative than the method of using Fminsearch.

VI. CONCLUSION

In this paper, a technique of solving the H_2 and H_∞ control problems for discrete-time systems with polytopic uncertainties via dynamic output feedback was developed. Based upon a novel bounding technique, the problem of dynamic output feedback is converted to a convex optimization in conjunction with a line search. A further improvement based on the sequential linear programming method was also proposed. Simulation results have been given to demonstrate the applicability of the proposed design method.

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