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Further Results on Singular Time Delayed System Stability

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Abstract

This paper gives sufficient conditions for the stability of linear singular continuous delay systems of the form $E\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau)$. These new, delay–independent conditions are derived using approach based on Lyapunov's direct method.

1.Introduction

It should be noticed that in some systems we must consider their character of dynamic and static state at the same time. Singular systems (also referred to as degenerate, descriptor, generalized, differential - algebraic systems or semi – state) are those the dynamics of which are governed by a mixture of algebraic and differential equations. Recently many scholars have paid much attention to singular systems and have obtained many good consequences. The complex nature of singular systems causes many difficulties in the analytical and numerical treatment of such systems, particularly when there is a need for their control.

In the short overview, that follows, we shall be familiar only with results achieved in the area of Lyapunov stability of *linear, continuous singular time delay systems* (LCSTDS). In that sense we will not discuss contributions prsented in papers concerened with problem of robust stability, stabilization of this class of systems with parameter uncertainity, see the list of references, as well as with other questions in connection with stability of (LCSTDS) being necessarly, transformed by Lyapunov – Krasovski functional, to the state space model in the form of differential – integral equations.

Moreover in the last few years, a numerous papers have been published in the area of linear discrete descriptor time delay systems, but this discussion is out of the scope of this paper. To be familiar with this matter see the list of attached references. To the best of our knowledge, some attempts in stability investigation of (LCSTDS) was due to *Saric* (2001, 2002a, 2002b) where sufficent conditions for covergence of appropriate fundamental matrix were estabilshed.

Recently, in the paper of *Xu et al*. (2002) the problem of robust stability and stabilization for uncertain (LCSTDS) was addressed and necessary and sufficient conditions were obtained in terms of strict LMI. Moreover in the same paper, using suitable canonical description of (LCSTDS) a rather simple criteria for asymptotic stability testing was also proposed.

In the paper of *Debeljkovic et al.* (2004), for the first time, the problem of asymptotic stability for particular class of (LCSTDS) has been solved, based on some previous results documented in *Owens*, *Debeljkovic* (1985) using the benefits of so called *weak Ljapunov matrix equation* and its properties on the subspace of consistent initial conditions. Moreover the problem statement and its solution is given only over the known basic system matrices. Hence, the offered approach based on geometric theory of consistency leads to the natural class of positive definite quadratic forms on the subspace containing all solutions. This fact makes possible the construction of Lyapunov stability theory even for the (LCSTDS) in that sense that asymptotic stability is equivalent to the existence of symmetric, positive definite solutions to a weak form of Lyapunov matrix equation incorporating condition which reffer to time delay term.

In our paper we present quite another approach to this problem. Namely, our result is expressed directly in terms of matrices E and A_1 naturally occuring in the system model and avoid the need to introduce any canonical form into the statement of the main *Theorem* and is based on brilliant result presented by *Pandolfi* (1980).

2.Notations

$\mathbb{R}$	Real vector space
$\mathbb{C}$	Complex vector space
I	Unit matrix
F	$= (f_{ij}) \in \Re^{n\times n}$, real matrix
F^T	Transpose of matrix F
$F > 0$	Positive definite matrix
$F \geq 0$	Positive semi definite matrix
$\mathbb{N}(F)$	Null space (kernel) of matrix F

$\lambda(F)$	Eigenvalue of matrix F
$\sigma_{()}(F)$	Singular value of matrix F
$\sigma\{F\}$	Spectrum of matrix F
$\|F\|$	Euclidean matrix norm of F
F^D	Drazin inverse of matrix F
$\Rightarrow$	Follows, $\mapsto$ Such that

Generally, *the singular differential control systems with time delay* can be written as:

$$\begin{aligned} E(t)\dot{\mathbf{x}}(t) &= \mathbf{f}\left(t, \mathbf{x}(t), \mathbf{x}(t-\tau), \mathbf{u}(t)\right), \quad t \geq 0 \\ \mathbf{x}(t) &= \varphi(t), \ -\tau \leq t \leq 0 \end{aligned}, \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is a state vector, $\mathbf{u}(t) \in \mathbb{R}^l$ is a control vector, $E(t) \in \mathbb{R}^{n\times n}$ is a singular matrix, $\varphi \in C = C\left(\left[-\tau, 0\right], \Re^n\right)$ is an admissible initial state functional, $C = C\left(\left[-\tau, 0\right], \Re^n\right)$ is the Banach space of continuous functions mapping the interval $\left[-\tau, 0\right]$ into $\mathbb{R}^n$ with topology of uniform convergence.

3.Preliminaries

Consider a linear continuous singular system with state delay, described by

$$E\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau), \tag{2a}$$

with known compatibile vector valued function of initial conditions

$$\mathbf{x}(t) = \varphi(t), \ \ -\tau \leq t \leq 0 \ , \tag{2b}$$

where A_0 and A_1 are constant matrices of appropriate dimensions. Moreover we shall assume that $rank\, E = r < n$.

Definition 1. The matrix pair $\left(E, A_0\right)$ is said to be regular if $\det\left(sE - A_0\right)$ is not identically zero, Xu et al. (2002).

Definition 2. The matrix pair $\left(E, A_0\right)$ is said to be impulse free if $\deg\left(\det(sE - A_0)\right) = rang\, E$, Xu et al. (2002).

The linear continuous singular time delay system (2) may have an impulsive solution, however, the regularity and the apsence of impulses of the matrix pair $\left(E, A_0\right)$ ensure the existence and uniqueness of an impulse free solution to the system under consideration.

Definition 3.

a) *Linear continuous singular time delay system*, (2) is said to be *regular* and *impulsive free* if the matrix pair $\left(E, A_0\right)$ is regular and imulsive free.

b) *Linear continuous singulartime delay system*, (2), is said to be stable if for any $\varepsilon > 0$ there exist a scalar $\delta(\varepsilon) > 0$ such that, for any compatible initial conditions $\varphi(t)$, satisfying condition: $\sup_{-\tau\leq t\leq 0}\left\|\varphi(t)\right\| \leq \delta(\varepsilon)$, the solution $\mathbf{x}(t)$ of system (2) satisfies $\|\mathbf{x}(t)\| \leq \varepsilon, \ \forall t \geq 0$.

Moreover if $\lim_{t\to\infty}\|\mathbf{x}(t)\| \to 0$, system is said to be *asymptotically stable*, *Xu et al* (2002).

4.Main result

Our result is stated as follows:

Theorem 1. Suppose that the system matrix A_0 is nonsingular., e.i. $\det A_0 \neq 0$.

Then we can consider system of the form

$$E_0\dot{\mathbf{x}}(t) = \mathbf{x}(t) + A_1\mathbf{x}(t-\tau), \tag{3a}$$

instead of (2) with known compatibile vector valued function of initial conditions

$$\mathbf{x}(t) = \varphi(t), \ \ -\tau \leq t \leq 0 \ , \tag{3b}$$

where E_0 and A_1 are constant matrices of appropriate dimensions.

Moreover we shall assume that $rank\, E_0 = r < n$.

Matrix E_0 is defined in the following way

$$E_0 = A_0^{-1}E. \tag{4}$$

The system (3) is *asymptotically stable*, independent of delay, if

$$\left\|A_1\right\| < \sigma_{\min}\left(Q^{\frac{1}{2}}\right)\sigma_{\max}^{-1}(Q^{-\frac{1}{2}}E_0^TP), \tag{5}$$

and if there exist:

(i) $(n\times n)$ matrix P, being the solution of Lyapunov matrix

$$E_0^TP + PE_0 = -2I_\Omega, \tag{6}$$

with the following properties:

a) $P = P^T$ (7a)

b) $P\mathbf{q}(t) = \mathbf{0}, \quad \mathbf{q}(t) \in \Lambda$ (7b)

c) $\mathbf{q}^T(t)P\mathbf{q}(t) > 0, \quad \mathbf{q}(t) \neq \mathbf{0}, \quad \mathbf{q}(t) \in \Omega$, (7c) where:

$$\Omega = \mathbb{N}\left(I - EE^D\right), \tag{8}$$

$$\Lambda = \mathbb{N}\left(EE^D\right), \tag{9}$$

with matrix I_Ω reperesenting generalized operator on $\mathbb{R}^n$ and identity matrix on subspace Ω and zero operator on subspace Λ and matrix Q being any positive definite matrix. Moreover matrix P is symmetric and positive definite on the subspace of consistent initial conditions.

Here $\sigma_{\max}(\cdot)$ and $\sigma_{\min}(\cdot)$ are maximum and minimum singular values of matrix $(\cdot)$, respectively.

Proof. If (6) has solution P with the properties (7), then matrix E_0 cannot have eigenvalues with positive real parts, *Pandolfi* (1980). Hence, the system (3) *wihtout delay* is stable.Conversely, assume that system (3), *witout delay* is stable.

Let matrix P be defined in the following way

$$\mathbf{q}^T(t)P\mathbf{q}(t) = \int_0^\infty \left(\left\| e^{E_0 t} E\mathbf{q}(t) \right\|\right)^2 dt . \tag{10}$$

The integral is zero if $\mathbf{q}(t) \in \Lambda$, and is finite if $\mathbf{q}(t) \in \Omega$.

Then it is clear that P is a solution of (6) with properties (7), *Pandolfi* (1980).

Remark 1. Equations (6 - 7) are, in modify form, taken from *Pandolfi* (1980).

Remark 2. It is obvious that Ω corespond to the subspace of consistent initial conditions, *Campbell* (1980) or *Owens*, *Debeljkovic* (1985) there denoted with W_{k^*}

Remark 3. So the stability of (3) *without delay* is proved, *Pandolfi* (1980).

In the sequel we present the rest of proof, establishing conditions under which (LCSTDS) wiil be asymptotically stable. Let us consider functional

$$V\left(\mathbf{x}(t)\right) = \mathbf{x}^T(t)E^T PE\mathbf{x}(t) + \int_{t-\tau}^{t} \mathbf{x}^T(\kappa)Q\mathbf{x}(\kappa)\, d\kappa , \tag{11}$$

Note that result presented in *Owens*, *Debeljkovic* (1985), incdicates that

$$V\left(\mathbf{x}(t)\right) = \mathbf{x}^T(t)E^T PE\mathbf{x}(t) , \tag{12}$$

is *positive quadratic form* on $\Omega = W_{k^*}$, and it is obvious that all smoothe solutions $\mathbf{x}(t)$ envolve in W_{k^*}, so $V\left(\mathbf{x}(t)\right)$ can be used as a *Lyapunov function* for the system under consideration, *Owens*, *Debeljkovic* (1985).

It will be shown that the same argument can be used to declare the same property of another quadratic form present in (11).

Clearly, using the equation of motion of (3), we have

$$\begin{aligned}\dot{V}\left(\mathbf{x}(t)\right) = {} & \mathbf{x}^T(t)\left(E_0^T P + PE_0 - 2I_\Omega\right)\mathbf{x}(t) \\ & + 2\mathbf{x}^T(t)\left(E_0 P A_1^T\right)\mathbf{x}(t-\tau) \\ & - \mathbf{x}^T(t)Q\mathbf{x}(t)\mathbf{x}^T - 2\mathbf{x}^T(t)S\mathbf{x}(t) \\ & - \mathbf{x}^T(t-\tau)Q\mathbf{x}(t-\tau)\end{aligned} \tag{13}$$

where matrix I_Ω is defined by

$$I_\Omega = Q + S , \tag{14}$$

with symmetric matrix $S = S^T$, with the following property

$$\mathbf{x}^T(t)S\mathbf{x}(t) \geq 0,\ \ \forall \mathbf{x}(t) \in \Omega . \tag{15}$$

and after some manipulations, following ideas presented in *Tissir*, *Hmamed* (1996), yeilds to

$$\begin{aligned}\dot{V}\left(\mathbf{x}(t)\right) = {} & 2\mathbf{x}^T(t)\left(E_0^T PA_1\right)\mathbf{x}(t-\tau) \\ & - \mathbf{x}^T(t)Q\mathbf{x}(t) - \mathbf{x}^T(t-\tau)Q\mathbf{x}(t-\tau)\end{aligned}, \tag{16}$$

and based on well known inequality[1]:

$$\begin{aligned}2\mathbf{x}^T(t)E_0^T PA_1\mathbf{x}(t-\tau) &= 2\mathbf{x}^T(t)\left(E_0^T PA_1 Q^{-\frac{1}{2}}Q^{\frac{1}{2}}\right)\mathbf{x}(t-\tau) \\ &\leq \mathbf{x}^T(t)E_0^T PA_1 Q^{-1}A_1^T PE_0^T\mathbf{x}(t) \\ &\quad + \mathbf{x}^T(t-\tau)Q\mathbf{x}(t-\tau)\end{aligned} ,\tag{17}$$

and by substtituting into (16), it yeilds

$$\dot{V}\left(\mathbf{x}(t)\right) \leq -\mathbf{x}^T(t)Q\mathbf{x}(t) + \mathbf{x}^T(t)E_0^T PA_1 Q^{-1}A_1^T PE_0\mathbf{x}(t) ,\tag{18}$$

or

$$\dot{V}\left(\mathbf{x}(t)\right) \leq -\mathbf{x}^T(t)Q^{\frac{1}{2}}\Gamma Q^{\frac{1}{2}}\mathbf{x}(t) , \tag{19}$$

with matrix Γ defined by

$$\Gamma = \left(I - Q^{-\frac{1}{2}}E_0^T PA_1 Q^{-\frac{1}{2}}Q^{-\frac{1}{2}}A_1^T PE_0 Q^{-\frac{1}{2}}\right) . \tag{20}$$

$\dot{V}\left(\mathbf{x}(t)\right)$ is negative definite if

$$1 - \lambda_{\max}\left(Q^{-\frac{1}{2}}E_0^T PA_1 Q^{-\frac{1}{2}}Q^{-\frac{1}{2}}A_1^T PE_0 Q^{-\frac{1}{2}}\right) > 0 , \tag{21}$$

which is satisfied if

$$1 - \sigma_{\max}^2\left(Q^{-\frac{1}{2}}E_0^T PA_1 Q^{-\frac{1}{2}}\right) > 0 . \tag{22}$$

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[1] $2\mathbf{u}^T(t)\mathbf{v}(t) \leq \mathbf{u}^T(t)P\,\mathbf{u}(t) + \mathbf{v}^T(t)P^{-1}\mathbf{v}(t),\ P > 0$

Using the properties of the singular matrix values, *Amir-Moez* (1956), the condition (22) holds if

$$1-\sigma_{\max}^{2}\left(Q^{-\frac{1}{2}}E_0^T P\right)\sigma_{\max}^{2}\left(A_1 Q^{-\frac{1}{2}}\right)>0\,, \tag{23}$$

which is satisfied if

$$1-\frac{\left\|A_1\right\|^2\sigma_{\max}^{2}\left(Q^{-\frac{1}{2}}E_0^T P\right)}{\sigma_{\min}^{2}\left(Q^{\frac{1}{2}}\right)}>0\,, \tag{24}$$

what completes proof. Q.E.D.

In the sequel we give an example to show the effectiveness of proposed method.

Example. Consider the linear continuous singulatr time delay system with matrices as follows:

$$E_0=\begin{bmatrix}-1&0&0\\0&-1&0\\0&1&0\end{bmatrix},\quad A_0=I,\quad A_1=\begin{bmatrix}0{,}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

Based on given data, one can calculate:

$$E_0^D=\begin{bmatrix}-1&0&0\\0&-1&0\\0&1&0\end{bmatrix},\quad E_0E_0^D=\begin{bmatrix}1&0&0\\0&1&0\\0&-1&0\end{bmatrix},$$

$$I-E_0E_0^D=\begin{bmatrix}0&0&0\\0&0&0\\0&1&1\end{bmatrix}.$$

$$\mathbb{N}\left(I-E_0E_0^D\right)=\left(I-E_0E_0^D\right)\mathbf{x}_0=\begin{bmatrix}0&0&0\\0&0&0\\0&1&1\end{bmatrix}\mathbf{x}_0=\mathbf{0},\Rightarrow$$

$$\mathbb{N}\left(I-\hat{E}\hat{E}^D\right)=\Omega=\left\{\mathbf{x}:\,x_1\in\Re,\;x_2\in\Re,\;\;x_2=-x_3\right\}.$$

$$\mathbb{N}\left(E_0E_0^D\right)=\left(E_0E_0^D\right)\mathbf{x}_0=\begin{bmatrix}1&0&0\\0&1&0\\0&-1&0\end{bmatrix}\mathbf{x}_0=\mathbf{0},\quad\Rightarrow$$

$$\mathbb{N}\left(E_0E_0^D\right)=\Lambda=\left\{\mathbf{x}:\,x_1=0,\;x_2=0,\;\;x_3\in\Re^n,x_3=1\right\}$$

$$\det A_0\neq 0,\quad \exists\lambda\;\ni\;\det(\lambda E-A_0)\neq 0,$$
$$rang\,E=2,\;\;\deg\det(sE-A_0)=2.$$

One can addopt:

$$Q=\begin{bmatrix}2&0&0\\0&2&1\\0&1&1\end{bmatrix}=Q^T>0,\quad S=S^T=\begin{bmatrix}0&0&0\\0&0&-1\\0&-1&-1\end{bmatrix},$$

$$\Rightarrow\; I_\Omega=S+Q=\begin{bmatrix}2&0&0\\0&2&0\\0&0&0\end{bmatrix}\;\Rightarrow$$

$$\mathbf{x}^T(t)S\mathbf{x}(t)=\begin{bmatrix}x_1&x_2&x_3\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&-1\\0&-1&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=$$
$$=\left(-2x_2x_3-x_3^2\right)_{x_2=-x_3}=2x_3^2-x_3^2=x_3^2>0,\;\;\forall\mathbf{x}(t)\in\Omega$$

.

$$\mathbf{x}^T(t)Q\mathbf{x}(t)=\begin{bmatrix}x_1&x_2&x_3\end{bmatrix}\begin{bmatrix}2&0&0\\0&2&1\\0&1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$
$$=\left(2x_1^2+2x_2^2+2x_2x_3+x_3^2\right)$$
$$=\left(2x_1^2+x_2^2+x_2^2+2x_2x_3+x_3^2\right),$$
$$=\left(2x_1^2+x_2^2+\left(x_2+x_3\right)^2\right)_{x_2=-x_3}$$
$$=2x_1^2+x_2^2>0,\quad\forall\mathbf{x}(t)\in\Omega$$

$$\det Q=2\neq 0\,,$$

$$\begin{bmatrix}-1&0&0\\0&-1&1\\0&0&0\end{bmatrix}\begin{bmatrix}p_{11}&p_{12}&p_{13}\\p_{12}&p_{22}&p_{23}\\p_{13}&p_{23}&p_{33}\end{bmatrix}+$$
$$+\begin{bmatrix}p_{11}&p_{12}&p_{13}\\p_{12}&p_{22}&p_{23}\\p_{13}&p_{23}&p_{33}\end{bmatrix}\begin{bmatrix}-1&0&0\\0&-1&0\\0&1&0\end{bmatrix}=\begin{bmatrix}-2&0&0\\0&-2&0\\0&0&0\end{bmatrix},$$

with solution:

$$P=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},$$

In the sequel we check properties of matrix P.

a) $P=P^T$.

b)

$$P\mathbf{q}(t)=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\mathbf{q}(t)$$

$$=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}=\ \mathbf{0},\quad \forall\mathbf{q}(t)\in\Lambda$$

c)

$$\mathbf{q}^T(t)P\mathbf{q}(t)=\mathbf{q}^T(t)\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\mathbf{q}(t)$$

$$=\begin{bmatrix}0&q_2(t)&-q_3(t)\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}0\\q_2(t)\\-q_3(t)\end{bmatrix}.$$

$$=q_2^2(t)>0,\quad \mathbf{q}(t)\neq\mathbf{0},\quad \forall\mathbf{q}(t)\in\Omega$$

Moreover there is need to check (5).

Based on:

$$Q=\begin{bmatrix}2&0&0\\0&2&1\\0&1&1\end{bmatrix}\ \wedge\ A_1=\begin{bmatrix}0{,}1&0&0\\0&0&0\\0&0&0\end{bmatrix},$$

simple calculations yeilds to:

$$\|A_1\|=0.10\qquad \sigma\{Q\}=\left\{2.62\quad 2.00\quad 0.38\right\},$$

$$Q^{\frac{1}{2}}=\begin{bmatrix}1.41&0&0\\0&1.34&0.45\\0&0.45&0.90\end{bmatrix},$$

$$Q^{-\frac{1}{2}}=\begin{bmatrix}0.71&0&0\\0&0.90&-0.45\\0&-0.45&1.34\end{bmatrix},$$

$$Q^{-\frac{1}{2}}E_0^TP=\begin{bmatrix}0.71&0&0\\0&-0.90&0\\0&0.45&0\end{bmatrix},$$

$$\sigma_{\min}\left(Q^{\frac{1}{2}}\right)=0.62\ \wedge\ \sigma_{\max}=\left(Q^{-\frac{1}{2}}E_0^TP\right)=1.00\,.$$

$$0{,}10=\|A_1\|<\frac{\sigma_{\min}\left(Q^{\frac{1}{2}}\right)}{\sigma_{\max}(Q^{-\frac{1}{2}}E_0^TP)}<0.26\,,$$

so, the system under consideration is asymptotically stable.

Moreover we have

$$\mathbf{x}^T(t)E^TPE\mathbf{x}(t)=\begin{bmatrix}x_1&x_2&x_3\end{bmatrix}\begin{bmatrix}1&0&0\\0&3&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$$=\left(x_1^2+3x_2^2\right)_{x_2=-x_3}>0,\quad \forall\mathbf{x}(t)\in W_{k^*}\setminus\{0\}$$

so $V\left(\mathbf{x}(t)\right)$ can be used as a *Lyapunov function* for the system (3).■

5. Conclusion.

A quite new sufficient delay–independent criteria for asymptotic stability of (LCSTDS) is presented.

In some sense this result may be treated as the further extension of results derived in Debeljkovic et. al (2004).

In comparison with some other papers on this matter, there is no need for linar transformations of basic systema, as well there is no need of solving the systems of high order linear matrix inqualities.

A numerical example is presented to show the applicability of results derived.

Apendix – A

The fundamental geometric tool in the characterization of the subspace of consistent initial conditions, for *linear singular system without delay*, is the subspace sequence

$$W_0=\Re^n\,, \tag{A1}$$

$$\vdots$$

$$W_{j+1}=A_0^{-1}(EW_j),\ \ j\geq 0\,, \tag{A2}$$

gde $A_0^{-1}(\cdot)$ denotes inverse image of ( ) under the operator A_0.

Lemma A1. The subsequence $\{W_0,\ W_1,\ W_2,\ ...\ \}$is nested in the sense that

$$W_0\supset W_1\supset W_2\supset W_3\supset ... \tag{A3}$$

Moreover

$$\mathbb{N}(A)\subset W_j\,,\ \forall j\geq 0\,, \tag{A4}$$

and there exist an integer $k\geq 0$, such that

$$W_{k+1}=W_k\ . \tag{A5}$$

Then it is obvious that

$$W_{k+j}=W_k\,,\ \forall j\geq 1\,. \tag{A6}$$

If k^* is the smallest such integer with this property, then

$$W_k\cap\mathbb{N}(E)=\{\mathbf{0}\}\,,\ k\geq k^*\,, \tag{A7}$$

provide that $\left(\lambda E-A_0\right)$ is invertible for some $\lambda\in\Re$.

Theorem A.1. Under the conditions of *Lemma* A1, $\mathbf{x}_0$ is a consistent initial condition for the system under consideration if and only if $\mathbf{x}_0 \in W_{k^*}$.

Moreover $\mathbf{x}_0$ generates a unique solution $\mathbf{x}(t) \in W_k^*$, $t \geq 0$, that is real analytic on $\{t : t \geq 0\}$.

Proof. See *Owens*, *Debeljkovic* (1985).

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