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Unbiased Minimum Variance Estimator Design for Scalar Quadratic Maps

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Abstract- **In this paper, we consider the state estimation problem for scalar discrete-time nonlinear systems with second degree polynomial nonlinearities. This research is a follow up to our previous work on suboptimal minimum variance estimator design for quadratic maps. A novel state estimator is proposed where the Extended Kalman filter structure is generalized to include quadratic terms and two consecutive measurements. Gains for unbiased minimum variance estimation are derived. It is shown both mathematically and by simulations that the new estimator achieves lower mean square estimation error than the Extended Kalman filter. It is also shown in simulations that the new estimator performs better than the recently developed suboptimal estimator.**

1. INTRODUCTION

The main purpose of this work is the introduction of a novel nonlinear state estimator design for scalar discrete-time nonlinear systems with second degree polynomial nonlinearities. The problem of estimation of state of a stochastic nonlinear dynamical system in a noisy environment is of central importance in engineering. Furthermore it has a wide range of applications in chaotic synchronization [1] which is of primary application focus of the authors. A classical state estimation technique for nonlinear stochastic system is Extended Kalman Filter (EKF) [2], which is the extension of Kalman Filter for linear systems and was designed based on a local linearization around the current state estimate. The estimation performance and convergence conditions for EKF have been investigated by many researchers, e.g. [3, 4, 5]. Although EKF is a well defined recursive locally optimal estimator for systems with differentiable nonlinearities, its performance may not be desirable. In the context of chaotic synchronization using nonlinear observers, due to the deterministic nature of the chaotic system, one can possibly use a technique based on the many available nonlinear state observation methods: feedback linearization [6]-[8], variable structure technique [9]-[12], extended linearization [13] etc. In general, each of these approaches has been successful in producing satisfactory results for particular classes of problems.

In this paper, we will focus on discrete-time scalar dynamical systems with second order nonlinearities. An unbiased state estimator design with quadratic terms and two consecutive measurements is proposed for this class of systems. The designed filter scheme achieves the minimum state estimation variance for the system and measurement model and the given filter structure. This paper is part of our wider research project on chaotic synchronization and secure communication using chaotic signals. In this sense, the results are a natural follow up of those in [14]-[15].

This paper is organized as follows: in Section 2 the estimation problem is formulated and the new unbiased estimator for the dynamical systems with second order nonlinearities is driven. Section 3 contains the simulation results for the new filter applied to two chaotic systems, the Quadratic map and the Logistic Map to illustrate the theoretical results of Section 2. The comparison of the error behavior between Extended Kalman Filter (EKF) and this new unbiased filter is presented in the same section. Also, an additional comparison is presented with a suboptimal estimator that has been proposed recently by the authors [16]. Finally, conclusions are drawn in section 4.

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2. OPTIMAL FILTER DESIGN FOR ONE-DIMENSIONAL QUADRATIC MAPS

Consider the following first order nonlinear system with a linear measurement equation:

$$x_{n+1} = f(x_n) \tag{2.1}$$

$$y_n = x_n + w_n \tag{2.2}$$

where x_n is the state, y_n is the output, w_n is the zero-mean noise with variance $\sigma_w{}^2$ with an arbitrary distribution. The nonlinearity is described by:

$$f(x) = ax^2 + bx + c$$

So, using Taylor expansion of $f(x)$ at $\hat{x}$, we have:

$$f(x) = f(\hat{x}) + \frac{\partial f}{\partial x}\bigg|_{x=\hat{x}} (x-\hat{x}) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\bigg|_{x=\hat{x}} (x-\hat{x})^2, \tag{2.3}$$

which means that we can represent the state nonlinearity as:

$$f(x_n) = f(\hat{x}_n) + A_n e_n + \frac{1}{2}\partial_2 e_n^2 \tag{2.4}$$

where $e_n = x_n - \hat{x}_n$ is state estimation error between x_n and $\hat{x}_n$, $A_n = \frac{\partial f}{\partial x}\Big|_{x=\hat{x}_n} = 2a\hat{x}_n + b$ and $\partial_2 = \frac{\partial^2 f}{\partial x^2}\Big|_{x=\hat{x}_n} = 2a.$

Note that this class of nonlinear maps includes some of the simplest chaotic systems, such as the quadratic map $f(x) = x^2 + c$ and the logistic map $f(x) = \lambda x(1-x)$ for suitable parameter values c and λ, respectively.

Assumption 1*: *Assume that the nonlinear function f has a finite derivative for all its arguments and is bounded away from 0, such that:

$$0 < \left| A_n \right| < \infty \tag{2.5}$$

This assumption has been commonly made in the convergence studies of the EKF, e.g. in [10].

Assumption 2: *The estimator has the general structure:*

$$\hat{x}_{n+1} = K_n^1(\hat{x}_n) + K_n^2(y_{n+1} - f(\hat{x}_n)) + K_n^3(y_{n+1} - f(\hat{x}_n))^2 + K_n^4 + K_n^5(y_n - \hat{x}_n) + K_n^6(y_n - \hat{x}_n)^2 \tag{2.6}$$

where $K_n^1, K_n^2, K_n^3, K_n^4, K_n^5, K_n^6$ *are unknown time-varying gains.*

Note that the structure chosen for the estimator is compatible with that of the nonlinear system as well as taking two successive measurements into account and more general than that of the Extended Kalman filter. In this sense, it is also an extension of the second order suboptimal filter in [16] and the second order filters in [17].

Theorem: Let filter (2.6) be used in estimating the state of system (2.1) from its noisy measurements. The unbiased minimum variance filter is given by the following choice of gains:

$$K_n^1 = f(\hat{x}_n)$$

$$K_n^2 = \frac{A_n^2 \sigma_w^2 P_n + \partial_2^2 \sigma_w^2 P_n (P_n + \sigma_w^2)}{A_n^2 \sigma_w^2 P_n + (P_n + \sigma_w^2)(\partial_2^2 \sigma_w^2 P_n + \sigma_w^2)}$$

$$K_n^3 = 0 \tag{2.7}$$

$$K_n^4 = -K_n^6 \sigma_w^2$$

$$K_n^5 = \frac{A_n P_n}{P_n + \sigma_w^2}(1 - K_n^2)$$

$$K_n^6 = \frac{1}{2}\partial_2 (1 - K_n^2)$$

and the mean square error is given by

$$P_{n+1} = \frac{A_n^2 \sigma_w^2 P_n + (P_n + \sigma_w^2)\partial_2^2 \sigma_w^2 P_n}{A_n^2 \sigma_w^2 P_n + (P_n + \sigma_w^2)(\partial_2^2 \sigma_w^2 P_n + \sigma_w^2)} \sigma_w^2 \tag{2.8}$$

where $P_n = E\{e_n^2\}$.

Proof:

Under the two assumptions, the error dynamics from (2.1)-(2.4) become:

$$e_{n+1} = x_{n+1} - \hat{x}_{n+1} = f(\hat{x}_n) + A_n e_n + \frac{1}{2}\partial_2 e_n^2 - K_n^1(\hat{x}_n) - K_n^2(y_{n+1} - f(\hat{x}_n)) - K_n^3(y_{n+1} - f(\hat{x}_n))^2 - K_n^4 - K_n^5(y_n - \hat{x}_n) - K_n^6(y_n - \hat{x}_n)^2 \tag{2.9}$$

The estimation error is evaluated as follows:

$$\begin{aligned} e_{n+1} &= f(\hat{x}_n) - K_n^1(\hat{x}_n) + A_n e_n - K_n^2(y_{n+1} - f(\hat{x}_n)) - K_n^3(y_{n+1} - f(\hat{x}_n))^2 - K_n^4 - K_n^5(y_n - \hat{x}_n) \\ &- K_n^6(y_n - \hat{x}_n)^2 + \frac{1}{2}\partial_2 e_n^2 = [f(\hat{x}_n) - K_n^1(\hat{x}_n)] + [A_n e_n - K_n^2(x_{n+1} + w_{n+1} - f(\hat{x}_n))] \\ &- [K_n^3(x_{n+1} + w_{n+1} - f(\hat{x}_n))^2] - [K_n^4] - [K_n^5(x_n + w_n - \hat{x}_n)] - [K_n^6(x_n + w_n - \hat{x}_n)^2 - \frac{1}{2}\partial_2 e_n^2] \\ &= [f(\hat{x}_n) - K_n^1(\hat{x}_n)] + [(A_n - A_n K_n^2)e_n - \frac{1}{2}K_n^2 \partial_2 e_n^2 - K_n^2 w_{n+1}] - [K_n^3(x_{n+1} + w_{n+1} - f(\hat{x}_n))^2] \\ &- [K_n^4] - [K_n^5 e_n + K_n^5 w_n] - [(K_n^6 + \frac{1}{2}\partial_2 + \frac{1}{2}K_n^2 \partial_2)e_n^2 + 2K_n^6 w_n e_n + K_n^6 w_n^2] = [f(\hat{x}_n) - K_n^1(\hat{x}_n)] \\ &+ (A_n - A_n K_n^2 - K_n^5)e_n - [K_n^3(x_{n+1} + w_{n+1} - f(\hat{x}_n))^2] - [K_n^4 + K_n^6 w_n^2] - [(K_n^6 - \frac{1}{2}\partial_2 + \frac{1}{2}K_n^2 \partial_2)e_n^2] \\ &- [K_n^2 w_{n+1} + K_n^5 w_n + 2K_n^6 w_n e_n] \end{aligned} \tag{2.10}$$

In order to achieve an unbiased estimator, we let,

$$K_n^1 = f(\hat{x}_n)$$

$$K_n^3 = 0 \tag{2.11}$$

$$K_n^4 = -K_n^6 \sigma_w^2$$

$$K_n^6 = \frac{1}{2}\partial_2 (1 - K_n^2)$$

Thus, the expected value of the estimation error will be:

$$E\{e_{n+1}\} = (A_n - A_n K_n^2 - K_n^5)E\{e_n\} - K_n^2 E\{w_{n+1}\} - K_n^5 E\{w_n\} - 2K_n^6 E\{w_n\}E\{e_n\} = 0 \tag{2.12}$$

So, we achieve unbiased estimation.

We pick the remaining gains K_n^2 and K_n^5 to minimize the mean square error: Substitute equation (2.12) into error equation (2.10), we get:

$$\begin{aligned} e_{n+1}^2 &= (A_n - A_n K_n^2 - K_n^5)^2 e_n^2 + (K_n^2)^2 w_{n+1}^2 + (K_n^5)^2 w_n^2 \\ &+ \partial_2^2 (1 - K_n^2)^2 w_n^2 e_n^2 - 2K_n^2 (A_n - A_n K_n^2 - K_n^5) e_n w_{n+1} \\ &- 2K_n^5 (A_n - A_n K_n^2 - K_n^5) e_n w_n - 2\partial_2 (1 - K_n^2)(A_n - A_n K_n^2 - K_n^5) w_{n+1} e_n^2 \\ &+ 2K_n^2 K_n^5 w_{n+1} w_n + 2K_n^2 \partial_2 (1 - K_n^2) w_{n+1} w_n e_n + 2\partial_2 (1 - K_n^2) K_n^5 w_n^2 e_n \end{aligned} \tag{2.13}$$

So

$$E\{e_{n+1}^2\} = (A_n - A_n K_n^2 - K_n^5)^2 E\{e_n^2\} + (K_n^2)^2 E\{w_{n+1}^2\} + (K_n^5)^2 E\{w_n^2\} + \partial_2^2 (1 - K_n^2)^2 E\{w_n^2\} E\{e_n^2\} \tag{2.14}$$

After simplification, we have:

$$\begin{aligned} E\{e_{n+1}^2\} &= (K_n^5)^2 (E\{e_n^2\} + \sigma_w^2) + K_n^5 (2A_n K_n^2 E\{e_n^2\} - 2A_n E\{e_n^2\}) + (A_n^2 + A_n^2 (K_n^2)^2 \\ &- 2A_n^2 K_n^2) E\{e_n^2\} + (1 + (K_n^2)^2 - 2K_n^2) \partial_2^2 \sigma_w^2 E\{e_n^2\} + (K_n^2)^2 \sigma_w^2 \end{aligned} \tag{2.15}$$

In order to achieve the smallest mean square error with respect to K_n^5, we complete the squares and find that the following condition must be satisfied.

$$K_n^5 = \frac{A_n E\{e_n^2\}}{E\{e_n^2\} + \sigma_w^2}(1 - K_n^2) \tag{2.16}$$

After substitute (2.16) into (2.15), we obtain,

$$\begin{aligned} E\{e_{n+1}^2\} &= \frac{A_n^2 (E\{e_n^2\})^2}{E\{e_n^2\} + \sigma_w^2}(1 - K_n^2)^2 + \frac{A_n E\{e_n^2\}}{E\{e_n^2\} + \sigma_w^2}(1 - K_n^2)(2A_n K_n^2 E\{e_n^2\} \\ &- 2A_n E\{e_n^2\}) + (A_n^2 + A_n^2 (K_n^2)^2 - 2A_n^2 K_n^2) E\{e_n^2\} + (1 - K_n^2)^2 \partial_2^2 \sigma_w^2 E\{e_n^2\} + K_n^2 \sigma_w^2 \end{aligned} \tag{2.17}$$

To minimize the mean square error with respect to K_n^2, we complete the squares and find that the following condition must be satisfied.

$$K_n^2 = \frac{A_n^2 \sigma_w^2 P_n + \partial_2^2 \sigma_w^2 P_n (P_n + \sigma_w^2)}{A_n^2 \sigma_w^2 P_n + (P_n + \sigma_w^2)(\partial_2^2 \sigma_w^2 P_n + \sigma_w^2)} \tag{2.18}$$

Since the initial condition, $E\{e_0^2\} = P_0$, we can update the $P_n = E\{e_n^2\}$ recursively as follows:

$$P_{n+1} = \frac{A_n^2\sigma_w^2 P_n + (P_n + \sigma_w^2)\partial_2^2\sigma_w^2 P_n}{A_n^2\sigma_w^2 P_n + (P_n + \sigma_w^2)(\partial_2^2\sigma_w^2 P_n + \sigma_w^2)}\sigma_w^2 \qquad (2.19)$$

which is the generalized Riccati difference equation for this problem.

Note that, no matter what P_0 we start with, in one step,

$$P_{n+1} = \frac{A_n^2\sigma_w^2 P_n + (P_n + \sigma_w^2)\partial_2^2\sigma_w^2 P_n}{A_n^2\sigma_w^2 P_n + (P_n + \sigma_w^2)(\partial_2^2\sigma_w^2 P_n + \sigma_w^2)}\sigma_w^2 < \sigma_w^2 \qquad (2.20)$$

So, the estimation error variance is less than the measurement noise variance for $n \geq 1$.

When $K_n^2 = \frac{A_n^2\sigma_w^2 P_n + \partial_2^2\sigma_w^2 P_n(P_n + \sigma_w^2)}{A_n^2\sigma_w^2 P_n + (P_n + \sigma_w^2)(\partial_2^2\sigma_w^2 P_n + \sigma_w^2)}$

and $K_n^5 = \frac{A_n E\{e_n^2\}}{E\{e_n^2\} + \sigma_w^2}(1 - K_n^2)$, we get the minimum value of this quadratic function. By substituting this gain into the expressions for other gains, we get optimal values of all the other coefficients in (2.7).

Remark 1. Even if the third and higher order terms of the Taylor series of $f(x)$ are not all zero but relatively small, this filter will still achieve satisfactory performance.

Remark 2. In [16], the second order filter is designed based on the assumption that $E\{e_0\} \neq 0$, So the co-efficient of $E\{e_n\}$ is set to zero in (2.12) to keep the filter unbiased. This results in suboptimal performance compared to the above filter design as will be shown later.

3. SIMULATION RESULTS

3.1 Optimal Estimator Performance

In this section, our new estimator is applied to two kinds second order maps for performance evaluation.

3.1.1 Quadratic Map

The Quadratic Map is defined as:

$x_{n+1} = x_n^2 + c$

When c< -1.76, the system has chaotic behavior.

In this experiment, we use the following system.

$x_{n+1} = x_n^2 + c, c = -2$

$y_n = x_n + w_n$

$\sigma_w^2 = 0.5$, $e_0^2 = 1.2596$

Figure 1 shows the simulation result for mean square error vs iteration time n. The mean square error is smaller that the noise variance σ_w^2.

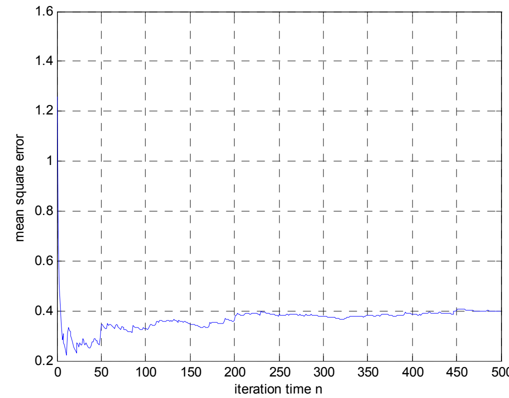

Figure 1. Mean square error vs iteration time n-Quadratic Map

3.1.2 Logistic Map

The Logistic Map is defined as:

$x_{n+1} = \lambda x_n(1 - x_n)$

When $3.6 \leq \lambda \leq 4.0$, the system has chaotic behavior.

In this experiment, we use the following system:

$x_{n+1} = \lambda x_n(1 - x_n), \lambda = 4$

$y_n = x_n + w_n$

$\sigma_w^2 = 0.25,\ e_0^2 = 0.7165$

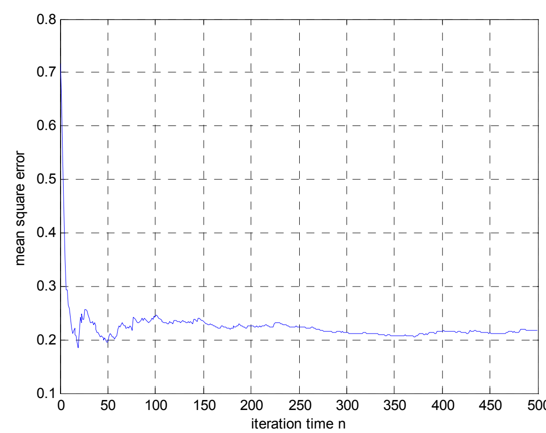

Figure 2. Mean square error vs iteration time n-Logistic Map

Figure 2 shows the simulation result for mean square error vs iteration time n. The mean square error is again smaller that the noise variance σ_w^2.

3.2 EKF-based Nonlinear State Estimation

Consider the same first order nonlinear system with the linear measurement equation in (2.1) and (2.2):

The EKF-based state estimation is described as follows:

$$\hat{x}_{n+1} = f(\hat{x}_n) + K_n(y_n - \hat{x}_n) \qquad (3.1)$$

The gain K_n value is updated at each step as follows:

$$K_n = \frac{A_n P_n}{P_n + \sigma_w^2} \quad (3.2)$$

where $A_n = \frac{df(x)}{dx}|_{x=\hat{x}_n}$,

$$P_{n+1} = A_n^2 P_n - \frac{A_n^2 P_n^2}{P_n + \sigma_w^2} = \frac{A_n^2 \sigma_w^2}{P_n + \sigma_w^2} P_n \quad (3.3)$$

In the following, we look at the performance of the EKF under smaller initial estimation error and smaller measurement noise variance to favor the EKF.

3.2.1 Quadratic Map

We use the quadratic map as in 3.1.1, where $\sigma_w^2 = 0.01$, $e_0^2 = 0.9860$

Figure 3 shows the simulation result for mean square error vs iteration time n for EKF. The mean square error is divergent in spite of the fact that the initial estimation error and the noise variance are smaller than those of the new filter.

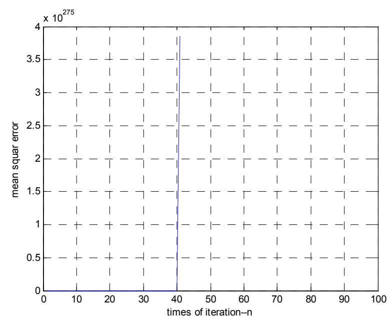

Figure 3. Mean square error vs iteration time n for EKF -Quadratic Map

3.2.2 Logistic Map

The same logistic map in 3.1.2 is used, where $\sigma_w^2 = \frac{1}{300}$, $e_0^2 = 0.0405$

Figure 4 shows the simulation result for mean square error vs iteration time n. The mean square error for this map is also divergent.

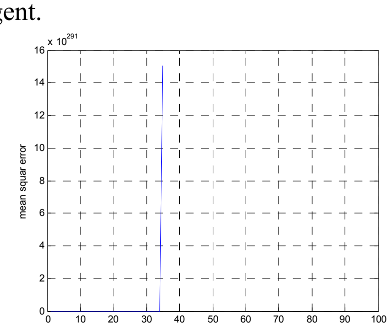

Figure 4. Mean square error vs iteration time n for EKF (Logistic Map)

3.3 Comparison Between the New Optimal Filter and the Previous Suboptimal Filter

The new unbiased filter compares favorably with Suboptimal Filter in Fig. 5 for the Quadratic map. The same result is shown for the logistic map in Fig. 6.

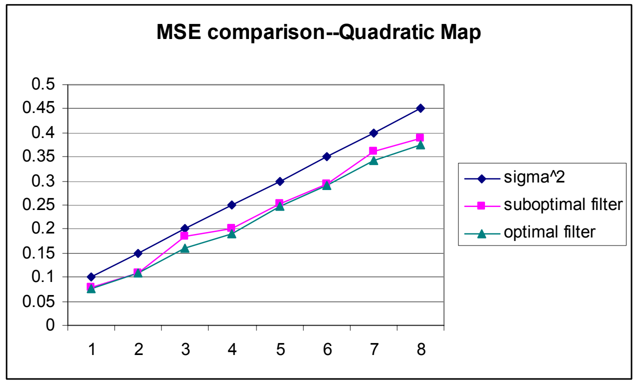

Figure 5. Comparison of Filters—Quadratic Map

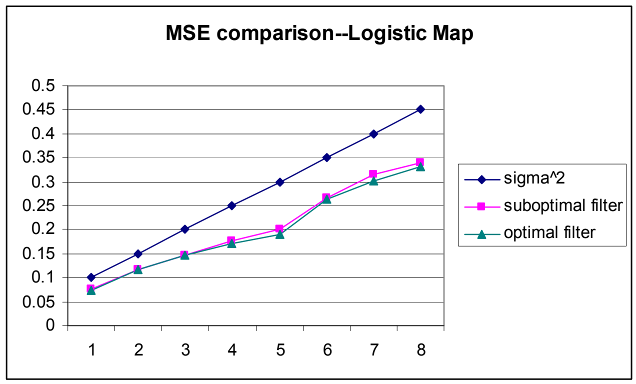

Figure 6. Comparison of Filters—Logistic Map

These two figures show the comparison of mean square error between the new Optimal Filter that has a more general structure involving two consecutive measurements and the previous Suboptimal Filter [16] under different noise levels. From the simulations, we can see that new Optimal Filter works better for second order maps.

4. CONCLUSION

In this paper, a new solution is proposed for state estimation of systems with second degree polynomial nonlinearities. This new unbiased filter achieves minimum error variance for this class of dynamical systems and this structure of filters. It is shown that this estimator is convergent under conditions where the EKF diverges. The new optimal filter has better performance compared to the previously designed suboptimal estimator. Further work will include the application of this new estimation scheme to chaotic synchronization and secure communication problems.

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