On Coprime Factorization for Linear System with Finite Discrete Jumps

Xiao-jun Yang, Zheng-xin Weng and Zuo-hua Tian

Abstract—**Basic results on coprime factorization for linear systems with finite discrete jumps are established. Especially, we deal with the case where the D matrices in the system are not assumed to be zero. State-space realizations of doubly coprime factorizations are provided and one special co-inner factorization for linear system with finite discrete jumps is proposed which is crucial to the optimization of fault detection systems for sampled-data systems.**

***Keywords*: Coprime factorization, co-inner factorization, sampled-data system, Riccati equation**

I. Introduction

THE jump system arises in the area of mechanical systems, chemical processes and economic systems where impulsive inputs naturally appear [1]. The continuous-time system is just a special case of the jump system as well as the discrete-time system and the sampled-data system. Therefore the results on the jump system are also applicable to above other three systems. For sampled-data problems (they have attracted much attention in control community in the past decade), three techniques have been developed since the 1990s include lifting technique [2-4], descriptor systems approach [5] and jump systems approach [6]. Compared to the other two techniques, it is more appealing to investigate the sampled-data systems from the view point of the jump systems. Since the jump system is a natural state-space representation of sampled-data systems and original signals and parameters are maintained in the new system [1]. Another advantage lies in that H_∞ and H_2 sampled-data problems can be treated in a unified manner by jump systems approach.

As well known, the coprime factorization is very important in the control theory. Coprime factorization can be used to obtain alternative characterizations of internal stability conditions and to derive all stabilizing controller parameterization [7]. The description of systems using coprime factors has been shown to have fundamental connections to the robust stabilization and Hankel norm approximations [8]. On the other hand, inner-outer factorization is essential to solve the model matching problem in H_∞-based robust control [9].

Motivated by above analysis, in this paper we investigate the coprime factorization for linear systems with finite discrete jumps (LSFDJ). Some basic results on coprime factorization for LSFDJ are established. Especially, we deal with the case where the D matrices in the system are assumed to be nonzero. The case where the D matrices are zero has been investigated in Reference [8]. The case where the D matrices are nonzero arises in many control and filtering problems, where the control object includes the unknown inputs. Especially, the nonzero D matrices are unavoidable to be met in the observer-based fault detection systems design based on factorization approach (see Reference [10], e.g.). In this paper, basic results on coprime factorization for LSFDJ are established. First, inner and co-inner LSFDJ is defined and two lemmas are introduced in section 1 which characterize the inner and co-inner factorization for LSFDJ. In section 2, state –space realization of the doubly coprime factorization is proposed. These two sections are an extension of the results in Reference [8]. One special co-inner factorization for LSFDJ is given by in section 3, which is completely new and crucial to the optimization of fault detection systems for sampled-data systems. We emphasize that it is not completely parallel between the cases where the D matrices are zero and nonzero, when the coprime factorization of LSFDJ is under considering. For example, it can be shown that there is no corresponding normalized coprime factorization under the case where the D matrices are nonzero. In this conference version paper, all proof will be omitted due to the limited space.

Notation: *Through out this paper, subscript "T" denotes the transpose of matrix.* $\Re^n$ *and* $\Re^{m\times n}$ *denotes* n*-dimensional and* $m\times n$*-dimensional real Euclidean space, respectively.* L_2 *stands for square integrable vector function space and* l_2 *is square summable vector sequence space.*

II. Preliminary

In this section, we'll introduce some preliminary results on definitions of adjoint system, inner and co-inner factorization for LSFDJ, which are necessary to the rest part of this paper.

First, consider the following LSFDJ Σ :

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$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t), & t \neq kh, \quad x(0) = 0 \\ x(kh) = A_d(kh)x(kh^-) + B_d(kh)u(kh) \\ y(t) = C(t)x(t) + D(t)u(t), & t \neq kh \\ y(kh) = C_d(kh)x(kh^-) + D_d(kh)u(kh) \end{cases} \tag{1}$$

where $x(t), x(kh) \in \Re^n$ is continuous and discrete state vector respectively, $x(kh^-)$ denote the left limit of $x(kh)$. The state $x(t)$ is right continuous but may be left discontinuous with possibly finite jumps at $t = kh$. $y(t), y(kh) \in \Re^m$ is continuous and discrete output respectively, while $u(t), u(kh) \in \Re^p$ is continuous and discrete input respectively. $A(t)$, $B(t)$, $C(t)$, $D(t)$, $A_d(kh)$, $B_d(kh)$, $C_d(kh)$ and $D_d(kh)$ are known bounded matrices functions describing the system with $A(t)$, $B(t)$, $C(t)$, $D(t)$ are piecewise continuous. All matrices have compatible dimensions. k is an positive integer and h is sampling time.

The system Σ can be seen as an operator mapping the input $u(t)$, $u(kh)$ to the output $y(t)$, $y(kh)$. Note that the input space and the output space are both hybrid signals space, i.e. $u, y \in L_2 \oplus l_2$. In this paper, we focus on the finite horizon case, i.e. $t \in [0, T]$. Without loss of generality, it is assumed that $0 < kh < T$ holds. Next, the norm of a vector $f \in L_2 \oplus l_2$ is defined as follows

$$\|f\| = \sum_{kh \in [0,T]} \int_{kh}^{(k+1)h^-} [f^T(t) f(t)] dt + \sum_{kh \in (0,T)} f^T(kh) f(kh)$$

Consider another vector $h \in L_2 \oplus l_2$, the inner product is defined as

$$< f, h >= \sum_{kh \in [0,T]} \int_{kh}^{(k+1)h^-} [f^T(t) h(t)] dt + \sum_{kh \in (0,T)} f^T(kh) h(kh)$$

Definition 1. *The system* $\tilde{\Sigma}$ *is called the adjoint of the system* Σ *if and only if the following relation holds*

$$< \Sigma u, v >=< u, \tilde{\Sigma} v >, \ \forall u, v \in L_2 \oplus l_2$$

From the definition of adjoint system, routine algebraic operation provides the sate-space realization of $\tilde{\Sigma}$:

$$\begin{cases} -\dot{p}(t) = A^T(t)p(t) + C^T(t)v(t), & t \neq kh, \quad p(T) = 0 \\ p(kh^-) = A_d^T(kh)p(kh) + C_d^T(kh)v(kh) \\ w(t) = B^T(t)p(t) + D^T(t)v(t), & t \neq kh \\ w(kh) = B_d^T(kh)p(kh) + D_d^T(kh)v(kh) \end{cases} \tag{2}$$

Recalling the definitions of the inner and co-inner function in pure continuous- or discrete-time case, we can give the inner and co-inner definitions for the LSFDJ.

Definition 2. *Consider the system* Σ *described by (1) and its adjoint* $\tilde{\Sigma}$ *described by (2). Then the system* Σ *is said to be inner or co-inner if it satisfies the relation* $\tilde{\Sigma}\Sigma = I$ *or* $\Sigma\tilde{\Sigma} = I$*, respectively.*

Note that, in Reference [8] the system Σ is inner equivalent to that the system is all pass. Thus, we also introduce the definition of all-pass for LSFDJ.

Definition 3. *The system* Σ *described by (1) is said to be all pass if the relation* $\|\Sigma u\| = \|u\|$ *holds.*

The following two lemmas provide a complete characterization of the inner and co-inner factorization for the LSFDJ. Note that lemma 1 is appeared in Reference [8] except that the definition of all-pass is replaced by the definition of inner here. Lemma 2 is just a dual version of Lemma 1.

Lemma 1 [8]. *The following statements are equivalent*:
(*a*) *The system* Σ *is inner.*
(*b*) *The following equations are satisfied*

$$P(T) = 0 \tag{3}$$

$$-\dot{P}(t) = A^T(t)P(t) + P(t)A(t) + C^T(t)C(t) \tag{4}$$

$$B^T(t)P(t) + D^T(t)C(t) = 0 \tag{5}$$

$$D^T(t)D(t) = I, \ t \neq kh \tag{6}$$

$$P(kh^-) = A_d^T(kh)P(kh)A_d(kh) + C_d^T(kh)C_d(kh) \tag{7}$$

$$A_d^T(kh)P(kh)B_d(kh) + C_d^T(kh)D_d(kh) = 0 \tag{8}$$

$$B_d^T(kh)P(kh)B_d(kh) + D_d^T(kh)D_d(kh) = I \tag{9}$$

where the solution P *to the Riccati differential equation (RDE) with jumps is bounded, symmetric and positive semi-definite.*

Lemma 2. *The following statements are equivalent*:
(*a*) *The system* Σ *is co-inner.*
(*b*) *The following equations are satisfied*

$$Q(0) = 0 \tag{10}$$

$$\dot{Q}(t) = A(t)Q(t) + Q(t)A(t)^T + B(t)B(t)^T \tag{11}$$

$$C(t)Q(t) + D(t)B(t)^T = 0 \tag{12}$$

$$D(t)D(t)^T = I, \ t \neq kh \tag{13}$$

$$A_d(kh)Q(kh^-)A_d^T(kh) + B_d(kh)B_d^T(kh) - Q(kh) = 0 \tag{14}$$

$$A_d(kh)Q(kh^-)C_d^T(kh) + B_d(kh)D_d^T(kh) = 0 \tag{15}$$

$$C_d(kh)Q(kh)C_d^T(kh) + D_d(kh)D_d^T(kh) = I \tag{16}$$

where the solution Q *to the RDE with jumps is bounded, symmetric and positive semi-definite.*

Remark 1. *Note that the terminal condition of the RDE with jumps in lemma 1 is not provided in Reference* [8].

III. DOUBLY COPRIME FACTORIZATION

In this section, we first give the notion of doubly coprime factorization and then propose the state-space realization of the right and left doubly coprime factors.

Suppose there exist stable LSFDJ Σ_n, Σ_m, $\Sigma_{\tilde{n}}$, $\Sigma_{\tilde{m}}$ such that

$$\Sigma = \Sigma_n \Sigma_m^{-1} = \Sigma_{\tilde{m}}^{-1} \Sigma_{\tilde{n}} \quad (17)$$

holds, then both the system Σ_n and Σ_m comprise the right factorization, the system $\Sigma_{\tilde{n}}$ and $\Sigma_{\tilde{m}}$ comprise the left factorization of the system Σ, respectively.

Definition 4. *Right or left factorization for the system Σ is said to be coprime if and only if there exists stable LSFDJ Σ_x, Σ_y, $\tilde{\Sigma}_x$ and $\tilde{\Sigma}_y$, such that the following conditions are satisfied*

$$\Sigma_x \Sigma_n + \Sigma_y \Sigma_m = I, \qquad \Sigma_{\tilde{n}} \Sigma_{\tilde{x}} + \Sigma_{\tilde{m}} \Sigma_{\tilde{y}} = I \quad (18\text{-}19)$$

Moreover, the factorization is said to be doubly coprime if a stronger condition is satisfied

$$\begin{bmatrix} \Sigma_y & \Sigma_x \\ -\Sigma_{\tilde{n}} & \Sigma_{\tilde{m}} \end{bmatrix} \begin{bmatrix} \Sigma_m & -\Sigma_{\tilde{x}} \\ \Sigma_n & \Sigma_{\tilde{y}} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad (20)$$

Proposition 1. *Consider the system Σ. One of its right coprime factorization is given by*

$$\Sigma_n = \left\{ \begin{bmatrix} A-BK & B \\ C-DK & D \end{bmatrix}, \begin{bmatrix} A_d - B_d K_d & B_d \\ C_d - D_d K_d & D_d \end{bmatrix} \right\} \quad (21)$$

$$\Sigma_m = \left\{ \begin{bmatrix} A-BK & B \\ -K & I \end{bmatrix}, \begin{bmatrix} A_d - B_d K_d & B_d \\ -K_d & I \end{bmatrix} \right\} \quad (22)$$

and one of its left coprime factorization is given by

$$\Sigma_{\tilde{m}} = \left\{ \begin{bmatrix} A-LC & L \\ -C & I \end{bmatrix}, \begin{bmatrix} A_d - L_d C_d & L_d \\ -C_d & I \end{bmatrix} \right\} \quad (23)$$

$$\Sigma_{\tilde{n}} = \left\{ \begin{bmatrix} A-LC & B-LD \\ C & D \end{bmatrix}, \begin{bmatrix} A_d - L_d C_d & B_d - L_d D_d \\ C_d & D_d \end{bmatrix} \right\} \quad (24)$$

Moreover, these factorization are also doubly coprime, where the other systems in the relation (20) are defined as follows

$$\Sigma_x = \left\{ \begin{bmatrix} A-LC & L \\ K & 0 \end{bmatrix}, \begin{bmatrix} A_d - L_d C_d & L_d \\ K_d & 0 \end{bmatrix} \right\} \quad (25)$$

$$\Sigma_y = \left\{ \begin{bmatrix} A-LC & B-LD \\ K & I \end{bmatrix}, \begin{bmatrix} A_d - L_d C_d & B_d - L_d D_d \\ K_d & I \end{bmatrix} \right\} \quad (26)$$

$$\Sigma_{\tilde{x}} = \left\{ \begin{bmatrix} A-BK & L \\ K & 0 \end{bmatrix}, \begin{bmatrix} A_d - B_d K_d & L_d \\ K_d & 0 \end{bmatrix} \right\} \quad (27)$$

$$\Sigma_{\tilde{y}} = \left\{ \begin{bmatrix} A-BK & L \\ C-DK & I \end{bmatrix}, \begin{bmatrix} A_d - B_d K_d & L_d \\ C_d - D_d K_d & I \end{bmatrix} \right\} \quad (28)$$

IV. One Specialized Co-Inner Factorization

In this section, a specialized co-inner factorization is introduced which is particular useful in residual generator design for fault detection of sampled-data systems.

Lemma 3. *Given the system Σ described by (1) which is detectable, then there exists a left coprime factorization $\Sigma = \Sigma_{\tilde{m}}^{-1} \Sigma_{\tilde{n}}$ such that $\Sigma_{\tilde{n}}$ is co-inner. A particular realization of such a coprime factorization is given by*

$$\Sigma_{\tilde{n}} = \left\{ \begin{bmatrix} A-LC & B-LD \\ R^{-1/2} C & R^{-1/2} D \end{bmatrix}, \begin{bmatrix} A_d - L_d C_d & B_d - L_d D_d \\ R_d^{-1/2} C_d & R_d^{-1/2} D_d \end{bmatrix} \right\} \quad (29)$$

$$\Sigma_{\tilde{m}} = \left\{ \begin{bmatrix} A-LC & L \\ -R^{-1/2} C & R^{-1/2} \end{bmatrix}, \begin{bmatrix} A_d - L_d C_d & L_d \\ -R_d^{-1/2} C_d & R_d^{-1/2} \end{bmatrix} \right\} \quad (30)$$

$$R(t) = D(t) D(t)^T \quad (31)$$

$$R_d(kh) = C_d Q(kh^-) C_d^T + D_d D_d^T \quad (32)$$

$$L(t) = [Q(t) C^T + B D^T] R^{-1} \quad (33)$$

$$L_d(kh) = [A_d Q(kh^-) C_d^T + B_d D_d^T] R_d^{-1} \quad (34)$$

where bounded, symmetric and semi-positive matrices $Q(t)$ and $Q(kh)$, $\forall t \in [0,T]$, $0 < kh < T$, solve the RDE with jumps

$$Q(0) = 0 \quad (35)$$

$$\dot{Q}(t) = (A - BD^T R^{-1} C) Q(t) + Q(t)(A - BD^T R^{-1} C)^T - Q(t) C^T R^{-1} C Q(t) + B(I - D^T R^{-1} D) B^T, \; t \neq ih \quad (36)$$

$$Q(kh) = A_d Q(kh^-) A_d^T + B_d B_d^T - [A_d Q(kh^-) C_d^T + B_d D_d^T] R_d^{-1} [A_d Q(kh^-) C_d^T + B_d D_d^T]^T \quad (37)$$

Remark 2. *Lemma 3 combined with the results in Reference [10] (lemma 1 and theorem 3) provides a fault detection observer for sampled-data system which optimizes the fault detection systems by increasing the robustness to the unknown inputs and simultaneously enhancing the sensitivity to the faults.*

V. Conclusion

The purpose of this paper is to provide a full version about the coprime factorizations for LSFDJ. Especially, we deal with the case where the D matrices in the system are not assumed to be zero. Results on charactering the inner and co-inner factorization is introduced which is useful in the inner-outer factorization for sampled-data systems. State-space realizations of doubly coprime factorization for LSFDJ are provided. Furthermore, one special co-inner factorization is proposed which provides a simple state-space approach to deal with the observer-based sampled-data fault detection problems. At last, it is emphasized that the results on this paper encompass the pure continuous-time case as well as discrete-time case.

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