Design of worst spatial distribution of disturbances for a class of parabolic partial differential equations

Michael A. Demetriou
Dept. of Mechanical Engineering
Worcester Polytechnic Institute
Worcester, MA 01609
Email: mdemetri@wpi.edu

Jeff Borggaard
Interdisciplinary Center for Applied Math
Virginia Tech
Blacksburg VA 24061-0531, USA
Email: jborggaard@vt.edu

Abstract

The aim of the present research study is to provide a computational framework for computing worst-case distributed disturbances for distributed parameter control problems. This algorithm couples constrained optimization along with robustness measures for the control problem. We describe the algorithm utilizing a parabolic partial differential equation with variable dissipation coefficients. Numerical results demonstrate the ability of the algorithm to find spatial locations where disturbances have maximal influence on the system.

1. Introduction

One of the design issues considered in the control of systems governed by partial differential equations, is the problem of actuator and sensor selection. Most work to date has dealt with selecting actuator and sensor location (and possibly type) based on enhanced controllability/observability criteria; for instance, actuator locations where chosen to ensure that the first few (dominant) modes are rendered at least controllable, and if possible, increase a certain controllability index. Based on these enhanced controllability notions, further work considers actuator locations that not only guarantee a certain level of controllability of certain modes, but also provide some measures of enhanced performance. Specifically, a set of admissible actuator locations was chosen as the set of locations that ensure a certain level of controllability of specific modes and an optimization search was performed over this set that optimized an associated cost of an LQR performance measure. This performance measure was expressed in terms of the solution of an Algebraic Riccati equation parameterized by the actuator locations. However this performance measure had no contribution from the spatial distribution of process disturbances. A remedy for that was the consideration of the $\mathcal{H}^2$ control problem whose optimal solution was expressed in terms of both the location-parameterized Riccati solution and the disturbance distribution. However, this considered an a priori assumed spatial distribution of process disturbances.

The effects of the spatial distribution of process disturbances on both the sensor and actuator locations was observed in [3] and [4], respectively. In the latter, an associated norm of the estimation error was optimized in order to provide the sensor location that would yield the "best" estimator subject to different spatial distributions of process disturbances. A few "representative" spatial distributions were considered, as for example based on a uniform distribution and on the first eigenfunction of the system, and each resulted in different optimal actuator/sensor locations. As a continuation of that effort, we undertake a study in which a "worst" distribution of disturbances can be found subject to a certain bound of the distribution "mass." Once such a worst spatial disturbance is found, one may then proceed with a performance-based placement of actuators and sensors. The resulting distribution of disturbances would provide a *s*patial robustness which when combined with a standard $\mathcal{H}^\infty$ control/estimation scheme would result in *both spatial and temporal robustness* of the actuator/sensor placement scheme.

Motivated by the above, in the first stage of the optimization procedure, we develop a computational scheme for the estimation of the worst spatial distribution of disturbances for a class of systems governed by parabolic partial differential equations. Two different performance indices are considered that minimize different norms of the disturbance-to-distributed output transfer function. For the special case of Riesz-spectral systems, one may utilize an eigenfunction expansion and obtain closed-form solutions to the associated norms. Utilizing the notion of *spatial* $\mathcal{H}^2/\mathcal{H}^\infty$ norms, one may subsequently arrive at computationally tractable schemes.

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2. Mathematical Model

The class of systems under consideration is described by the following SISO parabolic partial differential equation

$$\begin{aligned}\frac{\partial x}{\partial t}(t,\xi) &= A(\xi)x(t,\xi) + b(\xi)u(t) + e(\xi)w(t)\\ y(t) &= \int_\Omega c(\xi)x(t,\xi)\,d\xi + \nu(t),\end{aligned} \tag{1}$$

where $A(\xi)$ is a strongly elliptic operator [2], $\xi \in \Omega \subset \mathbb{R}^n$ $(n = 1,2,3)$ denotes the spatial variable, $x(t,\xi)$ is the solution, $u(t)$ the control signal, $b(\xi)$ the distribution function of the actuator, $e(\xi)$ the distribution function of the disturbance entering the system and $w(t), \nu(t)$ the temporal components of the process and output disturbances, respectively. Associated with (1) are the boundary conditions which may be taken as Dirichlet ($x(t,\xi)|_{\partial\Omega} = 0$), Neumann ($\partial x(t,\xi)/\partial n = 0$) or mixed.

Following [1], the above system may be placed in an abstract form via the general evolution equation

$$\begin{aligned}\dot{x}(t) &= \mathcal{A}x(t) + \mathcal{B}_1 w(t) + \mathcal{B}_2 u(t)\\ y(t) &= \mathcal{C}x(t) + \nu(t),\end{aligned} \tag{2}$$

where $x \in \mathcal{X}$ $(= L^2(\Omega))$ is the state of the system, $\mathcal{X}$ the state space (Hilbert space), u denotes the control signal, w denotes the system disturbance, $y \in \mathcal{Y}$ the measured output signal and ν the measurement noise. The state is given by the mild solution of (2)

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau)\Big[\mathcal{B}_1 w(\tau) + \mathcal{B}_2 u(\tau)\Big]\,d\tau. \tag{3}$$

Please note that the operators $\mathcal{B}_1, \mathcal{B}_2, \mathcal{C}$ are given by

$$\langle \mathcal{B}_1 w, \varphi\rangle = \int_\Omega e(\xi)\varphi(\xi)\,d\xi\, w(t),$$

$$\langle \mathcal{B}_2 u, \varphi\rangle = \int_\Omega b(\xi)\varphi(\xi)\,d\xi\, u(t)$$

$$\mathcal{C}x(t) = \int_\Omega c(\xi)x(t,\xi)\,d\xi.$$

and $T(t)$ denotes the exponentially C_0 semigroup generated by the infinitesimal generator $\mathcal{A}$. The design goal here is to choose the *worst* distribution of disturbances $e(\xi)$ subject to some "mass" constraints. The rationale for such a design is to generate the worst such distribution subject to some design criteria so that the resulting system would exhibit *spatial robustness*. Indeed, when the worst disturbance operator $\mathcal{B}_1$ is found, then combining with the worst temporal component of the disturbances $w(t)$, an $\mathcal{H}^\infty$ control formulation would yield a truly robust (worst temporal and worst spatial components of the disturbance input) controller signal $u(t)$. Continuing, when combined with an optimal actuator selection scheme, one may obtain an actuator location that has the property of providing enhanced performance and exhibiting robustness to worst spatiotemporal disturbances.

3. Design Criteria

In order to proceed with the design of the worst such $e(\xi)$ one must consider a performance index. We consider different performance indices reflecting key system-theoretic properties. Prior to this, we revisit the above system and by taking advantage of the properties of the system operator, we consider a closed form solution to (2). Using the fact that the Riesz-spectral operator is self-adjoint and the generator of an exponentially stable C_0 semigroup, we have

$$x(t,\xi) = \sum_{i=1}^{\infty} x_i(t)\varphi_i(\xi), \quad Ax(t,\xi) = -\sum_{i=1}^{\infty}\lambda_i x_i(t)\varphi_i(\xi). \tag{4}$$

Then the Laplace transform of $x(t,\xi)$ is given by

$$X(s,\xi) = \sum_{i=1}^{\infty}\frac{E_i}{s+\lambda_i}\varphi_i(\xi)\; W(s), \tag{5}$$

where

$$E_i \triangleq \int_\Omega e(\xi)\varphi_i(\xi)\,d\xi = \langle e, \varphi_i\rangle_{L^2(\Omega)}. \tag{6}$$

Therefore, the associated transfer function from the disturbance $w(t)$ to the infinite dimensional (distributed) output $x(t,\xi)$ is given by

$$G(s,\xi) = \sum_{i=1}^{\infty}\frac{E_i}{s+\lambda_i}\varphi_i(\xi). \tag{7}$$

Since we will be considering spatial distributions with "finite mass," we now define the space of admissible functions Θ_{ad} via

$$\Theta_{ad} = \left\{ e \in \mathcal{C}(\Omega) : \int_\Omega e(\xi)\,d\xi \le M_e \right\}.$$

The optimization will be restricted to the admissible set Θ_{ad} and thus one will consider a constrained optimization problem. Two approaches will be concerned with optimizing the impulse response of the system.

3.1. $\mathcal{H}^2$ norm optimization-enhanced controllability

The design objective is to find an admissible $e \in \Theta_{ad}$ such that the optimal $e(\xi)$ makes the system "more" controllable. Recall from [1] that the controllability Gramian

L_c for the system under examination is the unique self-adjoint solution to the Lyapunov operator equation

$$\langle L_c\phi, \mathcal{A}^*\psi\rangle + \langle \mathcal{A}^*\phi, L_c\psi\rangle = -\langle \mathcal{B}_1\phi, \mathcal{B}_1\psi\rangle, \ \phi,\psi \in \mathbf{D}(A). \tag{8}$$

The above equation satisfies

$$\langle L_c\phi, \phi\rangle \geq \gamma(e)\,\|\phi\|^2. \tag{9}$$

Therefore, the criterion now becomes that of finding $e \in \Theta_{ad}$ such that the coercivity bound γ is maximized:

$$(P_2) \qquad \max_{e\in\Theta_{ad}} \gamma(e). \tag{10}$$

Alternatively, one may consider optimizing the $\mathcal{H}^2$ norm of the system

$$(P_2) \qquad \max_{e\in\Theta_{ad}} \|(Is-\mathcal{A})^{-1}\mathcal{B}_1\|_2. \tag{11}$$

Remark 3.1 *In the finite dimensional case, the $\mathcal{H}^2$ norm is in fact given by the trace of the controllability Gramian. Thus one has*

$$\|(I^n s-\mathcal{A}^n)^{-1}\mathcal{B}_1^n\|_2 = \sqrt{trace(L_c^n)}, \quad where \tag{12}$$

$$L_c^n(\mathcal{A}^n)^T + \mathcal{A}^n L_c^n = -\mathcal{B}_1^n(\mathcal{B}_1^n)^T. \tag{13}$$

3.2. $\mathcal{H}^\infty$ norm optimization

In this case, one chooses the spatial distribution that maximizes the effect of the disturbances on the output, and specifically the peak of the singular values of the transfer function, and therefore

$$(P_\infty) \qquad \max_{e\in\Theta_{ad}} \|(s-\mathcal{A})^{-1}\mathcal{B}_1\|_\infty. \tag{14}$$

4. Computational Considerations for Riesz-spectral Systems Using Spatial Norms

For the special case where one has a Riesz-spectral system with the single input-infinite output transfer function given by $G(s,\xi)$, one may utilize the $\mathcal{H}^2$ and $\mathcal{H}^\infty$ spatial norms of the system. We summarize here the definition of these spatial norms as taken from [5, 6].

Definition 4.1 (Spatial $\mathcal{H}^2$ norm) *The spatial $\mathcal{H}^2$ norm of a system is defined as*

$$\|G(s,\xi)\|^2_{\mathcal{H}_2} \triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_\Omega G^*(j\omega,\xi)G(j\omega,\xi)\,d\xi\, d\omega. \tag{15}$$

Definition 4.2 (Spatial $\mathcal{H}^\infty$ norm) *The spatial $\mathcal{H}^\infty$ norm of a system is defined as*

$$\|G(s,\xi)\|_{\mathcal{H}_\infty} \triangleq \sup_{\omega\in\mathbb{R}} \sqrt{\int_\Omega G^*(j\omega,\xi)G(j\omega,\xi)\,d\xi}. \tag{16}$$

Using the properties of the Riesz-spectral system and specifically the orthonormality of the eigenfunctions, one has

$$\begin{aligned}\|G(s,\xi)\|^2_{\mathcal{H}_2} &= \sum_{i=1}^{\infty}\left\|\frac{E_i}{s+\lambda_i}\right\|_2^2 \\ &= \sum_{i=1}^{\infty}\frac{|E_i|^2}{2\lambda_i},\end{aligned} \tag{17}$$

and

$$\begin{aligned}\|G(s,\xi)\|_{\mathcal{H}_\infty} &= \sup_{\omega\in\mathbb{R}}\sum_{i=1}^{\infty}\frac{E_i}{j\omega+\lambda_i} \\ &= \sum_{i=1}^{\infty}\left\|\frac{E_i}{s+\lambda_i}\right\|_\infty\end{aligned} \tag{18}$$

Based on the above, the two optimization criteria become

$$(P_2) \qquad \max_{e\in\Theta_{ad}} \sum_{i=1}^{\infty}\frac{|E_i|^2}{2\lambda_i}, \tag{19}$$

and, with λ_i real and s of the form $j\omega$, we pose

$$(P_\infty) \qquad \max_{e\in\Theta_{ad}} \sum_{i=1}^{\infty}\left|\frac{E_i}{\lambda_i}\right|. \tag{20}$$

Remark 4.1 *In the event that one chooses the expansion*

$$e(\xi) = \sum_{i=1}^{\infty}\varepsilon_i\phi_i(\xi) \tag{21}$$

then

$$E_i = \int_\Omega\left(\sum_{j=1}^{\infty}\varepsilon_j\phi_j(\xi)\right)\phi_i(\xi)\,d\xi = \varepsilon_i. \tag{22}$$

When truncated sums are considered, then this simplifies the optimization to

$$(P_2) \qquad \begin{cases}\max\limits_{E_i} \sum\limits_{i=1}^{n}\dfrac{|E_i|^2}{2\lambda_i}, \\ subject\ to\ \ \Phi_n^T E < M_e\end{cases} \tag{23}$$

and

$$(P_\infty) \qquad \begin{cases}\max\limits_{E_i} \sum\limits_{i=1}^{n}\left|\dfrac{E_i}{\lambda_i}\right|, \\ subject\ to\ \ \Phi_n^T E < M_e\end{cases} \tag{24}$$

where

$$\Phi_n^T = \left[\ \int_\Omega \phi_1(\xi)\,d\xi \quad \dots \quad \int_\Omega \phi_n(\xi)\,d\xi\ \right]$$

$$E^T = [\ E_1 \quad \dots \quad E_n\ ].$$

5. Numerical results

We consider the 1-D parabolic PDE where the assumption of readily available eigenfunctions/eigenvalues no longer holds

$$\frac{\partial x}{\partial t}(t,\xi) = \frac{\partial}{\partial \xi}\left(\kappa(\xi)\frac{\partial x}{\partial \xi}(t,\xi)\right) + e(\xi)w(t),$$
$$x(t,0) = 0 = x(t,Ł).$$

We thus use Galerkin approximation and consider an approximate solution of the form

$$x(t,\xi) = \sum_{i=1}^{n} x_i(t)\psi_i^n(\xi) \tag{25}$$

where $\psi_i^n \in V^n$ are the trial functions and we define the finite dimensional subspace $V^n = \text{span}\,\{\psi_i^n, 1 \le i \le n\}$. By considering the PDE in variational form with the test function taken as ψ_j^n yields

$$M^n \dot{x}^n(t) = -K^n x^n + E^{nm}\theta w(t) \tag{26}$$

where the spatial distribution of disturbances is parameterized as a piecewise constant function over m subintervals

$$e(\xi) = \sum_{i=1}^{m} \theta_i \chi_{(\xi_{i-1},\xi_i)}(\xi). \tag{27}$$

The weights θ_i represent the distribution over the i^{th} subinterval of size $\Delta\xi = \xi_i - \xi_{i-1} = \frac{L}{m}$. The set of admissible functions then becomes

$$\Theta_{ad} = \left\{ \theta \in \mathbb{R}^m \,:\, -2 \le \theta_i \le 2 \;\text{ and }\; \sum_{i=1}^{m} \theta_i \left(\frac{L}{m}\right) \le 2 \right\}.$$

The matrices in the above finite dimensional dynamical system are given by

$$M_{ij}^n = \int_0^L \psi_i^n(\xi)\psi_j^n(\xi)\,d\xi, \;\; i,j = 1,\dots,n,$$
$$K_{ij}^n = \int_0^L \kappa(\xi)(\psi_i^n)'(\xi)(\psi_j^n)'(\xi)\,d\xi, \;\; i,j = 1,\dots,n,$$
$$E_{ik}^{nm} = \int_0^L \psi_i^n(\xi)\chi_{(\xi_{k-1},\xi_k)}(\xi)\,d\xi, \;\; k = 1,\dots,m.$$

The resulting finite dimensional system is then given by

$$\begin{aligned} \dot{x}^n(t) &= -(M^n)^{-1}K^n x^n + (M^n)^{-1}E^{nm}\theta w(t) \\ &= A^n x^n(t) + B_1^n \theta w(t) \end{aligned}$$

and the optimization seeks to find the entries of the vector $\theta = \{\theta_1,\dots,\theta_m\} \in \Theta_{ad}$ such that the norm

$$\left\| I_n(sI_n - A^n)^{-1}B_1^n\theta \right\|_p, \quad p = 2 \text{ or } \infty,$$

is maximized.

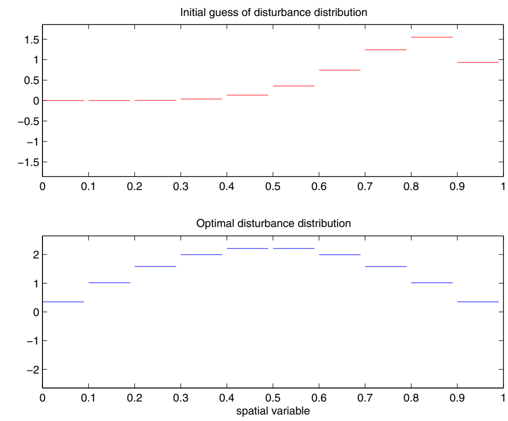

Figure 1. ℓ_2 Constrained: Initial and Final Worst-Case Disturbance

5.1. Preliminary Studies

In this section, we compute the optimum distributed disturbances using very coarse discretizations of e. To obtain meaningful results, we place some natural constraints on e (by constraining the coefficients in the optimization). We note that a simple $\mathcal{L}_infty$ constraint on e always produced a constant function at those bounds regardless of our choice of $\kappa(\cdot)$ as expected. More interesting disturbances were obtained with $\mathcal{L}_1$ and $\mathcal{L}_2$ norm constraints.

We first consider the computed disturbances for a uniform dissipation function $\kappa = 1$. When constraining the ℓ_2-norm of e^n, we find the usual (first mode) disturbance distribution (see Figure 1). With this disturbance fixed, we use the techniques developed in [4] (with b a short unit patch centered at the actuator location) to solve the optimal actuator placement problem. This disturbance leads to the usual optimal actuator placement in the center of the domain, cf. Figure 2, when minimizing the norm of the Riccati operator.

Similarly intuitive results hold when constraints are placed on the ℓ_1-norm, namely, that an approximation to the delta function at the center is found as shown in Figures 3 and 4. Note that the asymmetry of the initial guess decides which minimum is found.

More interesting results are obtained once we consider more general dissipation functions. Here, we present the worst-case disturbance for $\kappa(\xi) = 1 - .75\sin(2\pi\xi)$ with ℓ_2 constraints in Figures 5 and 6. Note that the disturbance is concentrated in the region where the dissipation is smaller. Since the solution decays slower here, disturbances in this region persist for a longer time. It is natural that the optimal LQR cost would be minimized by locating the actuator in this region.

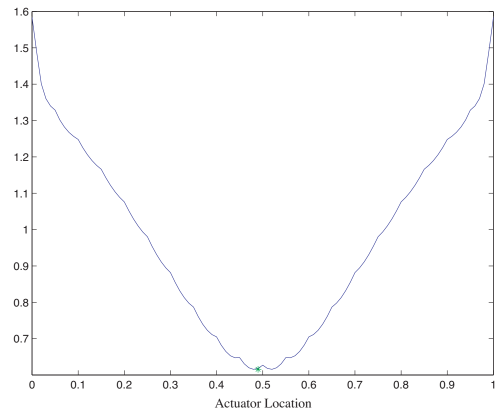

Figure 2. ℓ_2 Constrained: Optimal LQR Cost

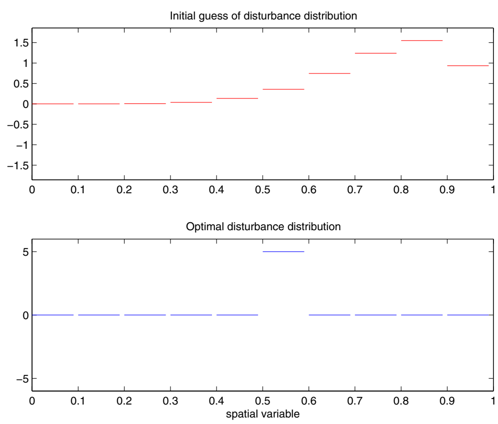

Figure 3. ℓ_1 Constrained: Initial and Final Worst-Case Disturbance

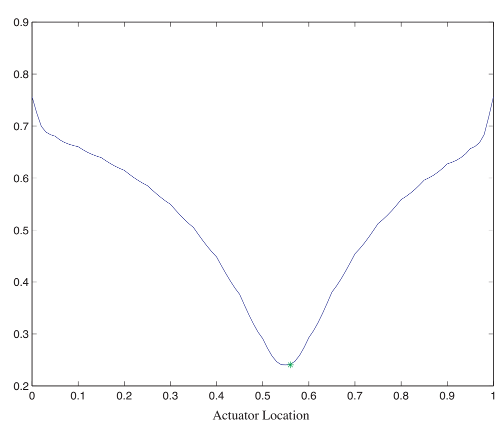

Figure 4. ℓ_1 Constrained: Optimal LQR Cost

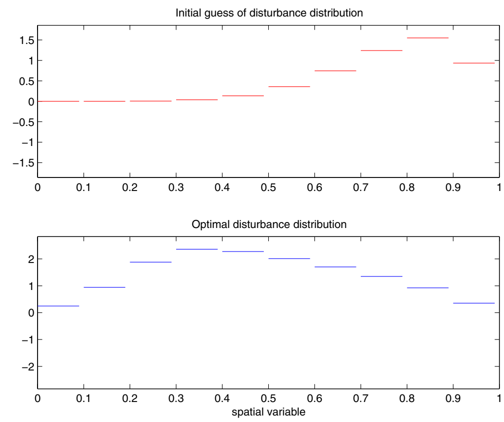

Figure 5. ℓ_2 Constrained: Worst-Case Disturbance with Varying κ

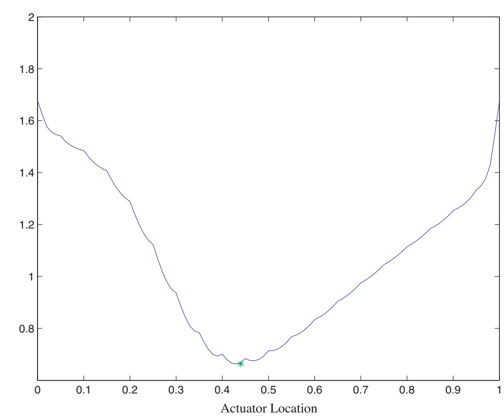

Figure 6. ℓ_2 Constrained: Optimal LQR Cost with Varying κ

6. Conclusions

We point out that the use of the ℓ_2 constraints actually lead to faster converging optimization algorithms due to the smoothness in the constraint. The ℓ_1 constraint does pinpoint the optimal location of the actuator mass, predicting a delta function for a disturbance with a sharper LQR cost function. We note that there is a strong correlation with the maximum value of e and the optimal actuator location. This optimal disturbance may be used to place multiple actuators (trying to position these actuators by, for instance, considering the disturbance to be a weighting function in some integration scheme). There exists an analogous approach for computing sensor locations by considering functional gains.

Acknowledgments

This work was supported in part by the Air Force Office of Scientific Research under grant F49620-00-1-0299.

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