2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

Practical Attractivity and Practical Asymptotic Stabilizability of a Class of Switched Systems

Xuping Xu
Department of Electrical and Computer Engineering
Penn State Erie, Erie, PA 16563, USA

Abstract— **In this paper, we study the practical asymptotic stabilizability of a class of switched systems with autonomous subsystems. We first introduce the notions of practical attractivity and practical asymptotic stabilizability. Then we propose sufficient conditions for practical attractivity and practical asymptotic stabilizability of such systems.**

I. Introduction

In our recent papers [3], [4], we point out that, under appropriate switching laws, switched systems without common equilibria may still exhibit interesting behavior similar to that of a conventional stable system near an equilibrium. Such behavior is defined to be practical stability.

The notion of practical stability concerns the local behavior of the system near the origin. However, in many cases, we are also interested in the behavior of the system in a larger region. For example, we may want to know whether the system can exhibit "convergent behavior" similar to that of a conventional asymptotically stable system. The first contribution of this paper is the formal introduction of the notions of practical attractivity and practical asymptotic stabilizability that can describe such "convergent behavior" for switched systems with autonomous subsystems. Such notions are different from the notions of Lyapunov asymptotic stabilizability for switched systems [1], [2]. The second contribution consists of sufficient conditions for practical attractivity and practical asymptotic stabilizability of such switched systems. In the proof of the conditions, a switching law leading to ϵ-practical attractivity is constructed.

II. Preliminaries

A. Switched Systems and Switching Laws

In this paper, we consider switched systems consisting of autonomous subsystems

$$\dot{x} = f_i(x), \quad i \in I \stackrel{\triangle}{=} \{1, 2, \cdots, M\}. \tag{1}$$

In (1), every $f_i : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous. The active subsystem at each time instant is orchestrated by a switching law. Given any initial t_0 and $x(t_0)$, the law generates a switching sequence $\sigma = \big((t_0, i_0), (t_1, i_1), \cdots, (t_k, i_k), \cdots\big)$ ($t_0 \le t_1 \le \cdots \le t_k \le \cdots$, $i_k \in I$) indicating that subsystem i_k is active in $[t_k, t_{k+1})$. We only consider **nonZeno** sequences which switch finite times in any finite time interval.

Definition 1 (Switching Law over $[0,\infty)$): For switched system (1), a switching law $\mathcal{S}$ over $[0,\infty)$ is defined to be a mapping $\mathcal{S} : \mathbb{R}^n \to \Sigma_{[0,\infty)}$ which specifies a nonZeno switching sequence $\sigma \in \Sigma_{[0,\infty)}$ for any initial state $x(0)$. Here $\Sigma_{[0,\infty)} \stackrel{\triangle}{=} \{\text{switching sequence } \sigma \text{ over } [0,\infty)\}$. □

B. Review of Practical Stabilizability Results

Now we review some notions and results reported in [4]. We use $\|\cdot\|$ to denote the 2-norm; $B[x,r]$ to denote the closed ball $\{y \in \mathbb{R}^n : \|y - x\| \le r\}$ and $B(0,r)$ the open ball. Unlike the classical stability concept, we do not assume $f_i(0) = 0$, $\forall i \in I$. Without loss of generality, we only discuss the case of the origin and let the initial $t_0 = 0$.

Definition 2 (ϵ-Practical Stability): Given a switching law $\mathcal{S}$ for switched system (1) and given an $\epsilon > 0$, the system is said to be *ϵ-practically stable under $\mathcal{S}$* if there exists a $\delta = \delta(\epsilon) > 0$ such that $x(t) \in B[0,\epsilon]$, $\forall t > 0$, whenever $x(0) \in B[0,\delta]$. □

Definition 3 (Practical Stabilizability): Switched system (1) is said to be *practically stabilizable* if for any $\epsilon > 0$, there exists a switching law $\mathcal{S} = \mathcal{S}(\epsilon)$ such that the system is ϵ-practically stable under $\mathcal{S}$. □

Lemma 1 ([4]): Switched system (1) is practically stabilizable if the condition $C = \mathbb{R}^n$ holds. Here C is the convex cone $C = \{\sum_{i=1}^M \lambda_i f_i(0) | \lambda_1 \ge 0, \cdots, \lambda_M \ge 0\}$. □

III. Practical Attractivity and Practical Asymptotic Stabilizability

Practical stabilizability concerns the local behavior of the system trajectory within given bounds. In many cases, we are also interested in the behavior of the trajectory in a larger region. For example, we may want to know whether the system can exhibit "convergent behavior" similar to that of a conventional asymptotically stable system. In this section, we will formally define such behavior. In the sequel, by a region D around the origin, we mean an open connected subset of $\mathbb{R}^n$ containing the origin along with some, none, or all of its boundary points.

Definition 4 (ϵ-Practical Attractivity): Assume that a region D around the origin is given. Also assume that a switching law $\mathcal{S}$ is given for switched system (1). Given an $\epsilon > 0$, the system is said to be *ϵ-practically attractive on D under $\mathcal{S}$* if for any $x(0) \in D$, there exists a finite $T = T\big(x(0)\big) \ge 0$ such that $x(T) \in B[0,\epsilon]$. □

Definition 5 (Practical Attractivity): Assume that a region D around the origin is given. Switched system (1) is said to be *practically attractive on D* if for any $\epsilon > 0$, there exists a switching law $\mathcal{S} = \mathcal{S}(\epsilon)$ such that the system is ϵ-practically attractive on D under $\mathcal{S}$. □

Definition 6 (Practical Asymptotic Stabilizability): Assume a region D around the origin is given. Switched system (1) is said to be *practically asymptotically stabilizable on* D if it is both practically attractive on D and practically stabilizable. □

Remark 1: For a practically asymptotically stabilizable system, given an $\epsilon > 0$, $\exists \delta > 0$ as in Definition 2 and a switching law $\mathcal{S}_1$ that keeps any trajectory starting in $B[0, \delta]$ to be within $B[0, \epsilon]$. Furthermore, $\exists$ a switching law $\mathcal{S}_0$ that can drive the state trajectory starting in D into $B[0, \delta]$. Concatenating $\mathcal{S}_0$ and $\mathcal{S}_1$, a switching law $\mathcal{S}$ can be obtained to bring the trajectory into and keep it within $B[0, \epsilon]$. □

The results below can easily be extended to the case of any positive definite energy function and closed region D.

Theorem 1: Assume that $D = B[0, r], r > 0$. Let $V(x) = x^T x$ be an energy function. Switched system (1) is practically attractive on D if the following condition holds

$$\min_{i \in I} \frac{\partial V}{\partial x} f_i(x) = \min_{i \in I} 2x^T f_i(x) < 0, \ \forall x \in D - \{0\}. \quad (2)$$

Proof: Given any $\epsilon > 0$, we will construct a switching law $\mathcal{S}$ to make the system ϵ-practically attractive on D.

First note that the local Lipschitz continuity of all $f_i(x)$'s leads to the continuity of all functions $g_i(x) \triangleq 2x^T f_i(x)$, $i \in I$ and the function $h(x) \triangleq \min_{i \in I} \frac{\partial V}{\partial x} f_i(x) = \min_{i \in I} g_i(x)$ on D. Since the set $D - B(0, \epsilon)$ is compact, $\exists \gamma > 0$ such that $\max_{x \in D - B(0,\epsilon)} h(x) = -\gamma$.

Next note $g_i(x)$'s are uniformly continuous on the compact set $D - B(0, \epsilon)$. So $\exists r_1 > 0$ s.t. $|g_i(x_a) - g_i(x_b)| \leq \frac{\gamma}{2}$, $\forall i \in I$ whenever $x_a, x_b \in D - B(0, \epsilon)$ and $\|x_a - x_b\| \leq r_1$.

Let $L = \max_{i \in I}\{\max_{x \in D - B(0,\epsilon)} \|f_i(x)\|\}$ and define $T_d \triangleq \frac{r_1}{L}$, we construct a switch law as follows.

Switching Law (for system (1) with $x(0) \in D - B[0, \epsilon]$):

(1). Set $k = 0$.

(2). Repeat the following step until at some time instant $t \in (0, \infty)$, the state trajectory intersects $B[0, \epsilon]$:

(2a). At $t_k = kT_d$, activate subsystem $i_k = \arg\min_{i \in I} g_i\big(x(t_k)\big)$ in $[kT_d, (k+1)T_d)$. At time instant $(k+1)T_d$, set $k = k + 1$.

This switching law will drive the state to intersect $B[0, \epsilon]$ in finite time for any initial $x(0) \in D - B[0, \epsilon]$. We show this by contradiction as follows. Assume the trajectory never intersects $B[0, \epsilon]$. Then at any instant kT_d, we have $g_{i_k}\big(x(kT_d)\big) = h\big(x(kT_d)\big) \leq -\gamma$ due to step (2a). Moreover, from the definition of T_d, we have $\|x(t) - x(kT_d)\| = \|\int_{kT_d}^t f_{i_k}\big(x(\tau)\big)\ d\tau\| \leq r_1$, $\forall t \in [kT_d, (k+1)T_d)$. This leads to $\|g_{i_k}\big(x(t)\big) - g_{i_k}\big(x(kT_d)\big)\| \leq \frac{\gamma}{2}$, $\forall t \in [kT_d, (k+1)T_d)$. Consequently $g_{i_k}\big(x(t)\big) \leq -\frac{\gamma}{2}$, $\forall t \in [kT_d, (k+1)T_d)$. Due to this, we have for any $t \in [KT_d, (K+1)T_d)$,

$V\big(x(t)\big) = V\big(x(0)\big) + \sum_{k=0}^{K-1} \int_{kT_d}^{(k+1)T_d} g_{i_k}\big(x(\tau)\big)\ d\tau$
$+ \int_{KT_d}^t g_{i_K}\big(x(\tau)\big)\ d\tau \leq V\big(x(0)\big) - \frac{\gamma}{2} t.$

The right-hand side of the above inequality becomes negative if t is large enough. This leads to a contradiction. □

Corollary 1: Assume that $D = B[0, r], r > 0$. Switched system (1) is practically asymptotically stable on D if the conditions of both Theorem 1 and Lemma 1 hold. □

Remark 2: (2) may remind us of conditions for asymptotic stability of switched systems with a common equilibrium at 0. Yet for general nonlinear subsystems, such a condition may not guarantee the existence of nonZeno asymptotically stabilizing switching laws. Up to now, the condition can only help us construct nonZeno asymptotically stabilizing switching laws for limited classes of systems such as those with subsystems $\dot{x} = A_i x$. □

IV. An Example

Example 1: Consider a switched system (1) in $\mathbb{R}^2$ which consists of 4 subsystems with $f_1(x) = [0.6x_1 - 1,\ 0.4x_2 - 1]^T$, $f_2(x) = [-0.4x_1 + 1,\ 0.6x_2 - 1]^T$, $f_3(x) = [-0.6x_1 + 1,\ -0.4x_2 + 1]^T$, $f_4(x) = [0.4x_1 - 1,\ -0.6x_2 + 1]^T$. By Corollary 1, this system is practically asymptotically stable on $B[0, 1]$. Given $\epsilon = 0.1$, a switching law can be constructed to bring the state into and keep it within $B[0, \epsilon]$ (see Remark 1). Fig. 1 shows $x_1(t)$ and $x_2(t)$ generated by the law (with initial $x_1(0) = -0.5$, $x_2(0) = 0.6$). □

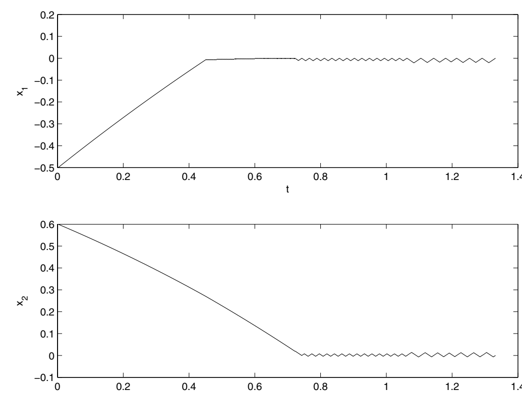

Fig. 1. $x_1(t)$ and $x_2(t)$ generated by the switching law (with initial condition $x_1(0) = -0.5$ and $x_2(0) = 0.6$).

V. Conclusion

This paper reports some preliminary results on practical attractivity and practical asymptotic stabilizability of a class of switched systems. Sufficient conditions for practical attractivity and practical asymptotic stabilizability are proposed. Future research includes studies of the practical asymptotic stabilizability of general classes of switched systems and applications in asymptotic tracking problems.

References

[1] R. DeCarlo, M.S. Branicky, S. Pettersson, and B. Lennartson. Perspectives and results on the stability and stabilizability of hybrid systems. *Proc. of the IEEE*, 88(7): 1069-1082, 2000.

[2] D. Liberzon. *Switching in Systems and Control*, Birkhäuser.

[3] X. Xu and G. Zhai. On practical stability and stabilization of hybrid and switched systems. *Hybrid Systems: Computation and Control*, LNCS 2993, pp. 615-630, Springer, 2004.

[4] X. Xu. Practical stabilizability of a class of switched systems. In *Proc. of the 2004 ACC*, pp. 4537-4542, 2004.