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Observer design for a class of stochastic bilinear systems with multiplicative noise

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Abstract— **In this paper, the purpose is to design an observer for a stochastic bilinear system such that the estimation error is mean-square stable. The bilinear system is bilinear in control and with multiplicative noise. The approach is based on the resolution of LMI and is then easily implementable.**

Index Terms— **Observer, Stochastic systems, Bilinear systems, Itô equation, Lyapunov stochastic function.**

INTRODUCTION

The bilinear system is sometimes a good mean to represent system when the linear representation is not sufficiently significant. The stochastic systems get a great importance the last decades as shown by numerous references [1], [2], [3], [4]. Generally, the bilinear stochastic system designs a stochastic system with multiplicative noise instead of additive one [1]. In this paper, the bilinearity is also between the state and the control input. The challenge is to design a full order observer for such a system. The approach is based on a change of variable on the control input to transform the problem into a robust stochastic filtering one. Then the Itô formula and LMI method permit to get a condition to verify for the existence of the observer gains.

I. PROBLEM STATEMENT

Let us consider the Itô stochastic bilinear system [3], [2]

$$\mathrm{d}\,x(t) = (Ax(t) + u(t)N_2 x(t))\mathrm{d}\,t + Nx(t)\,\mathrm{d}\rho(t) \quad (1a)$$

$$y(t) = Cx(t) \quad (1b)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^p$ is the output and $u(t) \in \mathbb{R}$ is the control input. $\rho(t)$ is a zero mean scalar Wiener process verifying [3]

$$\mathbf{E}(\mathrm{d}\,\rho(t)) = 0 \text{ and } \mathbf{E}(\mathrm{d}\,\rho^2(t)) = \mathrm{d}\,t. \quad (2)$$

To simplify the notation and without loss of generality, we consider only the single input case.

As in the most cases for physical processes, we assume that the stochastic bilinear system (1) has known bounded control input, i.e. $u(t) \in \Omega \subset \mathbb{R}$, where

$$\Omega := \{u(t) \in \mathbb{R} \mid u_{\min} \leqslant u(t) \leqslant u_{\max}\}. \quad (3)$$

We want to design a state observer in the following form

$$\mathrm{d}\,\hat{x}(t) = (A\hat{x}(t) + u(t)N_2\hat{x}(t))\,\mathrm{d}\,t + K(y(t) - C\hat{x}(t))\,\mathrm{d}\,t + \overline{K}u(t)(y(t) - C\hat{x}(t))\,\mathrm{d}\,t \quad (4)$$

where K and $\overline{K}$ are the gains to design in order to ensure that the estimation error $x - \hat{x}$ is mean-square stable.

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For this, the following definition and assumption are used.

Definition 1: [3] The stochastic system (1) is said to be mean-square stable if all initial states $x(0)$ yield

$$\lim_{t\to\infty} \mathbf{E}\,\|x(t)\|^2 = 0. \quad (5)$$

Assumption 1: The stochastic bilinear system (1) is assumed to be mean-square stable.

Notice that the estimation error $x(t) - \hat{x}(t)$ has the following dynamics

$$\mathrm{d}\,e(t) = \mathrm{d}\,x(t) - \mathrm{d}\,\hat{x}(t) = (Ae(t) + u(t)N_2 e(t) - KCe(t) - u(t)\overline{K}Ce(t))\,\mathrm{d}\,t + Nx(t)\,\mathrm{d}\,\rho(t). \quad (6)$$

Let us consider the augmented state vector $\xi = \begin{bmatrix} x^T & e^T \end{bmatrix}^T$. Then the dynamics of the augmented system is given by

$$\mathrm{d}\,\xi^T(t) = (\mathcal{A}\xi(t) + u(t)\mathcal{N}\xi(t))\,\mathrm{d}\,t + \mathcal{M}\xi(t)\,\mathrm{d}\,\rho(t) \quad (7)$$

with

$$\mathcal{A} = \begin{bmatrix} A & 0 \\ 0 & A - KC \end{bmatrix},\ \mathcal{N} = \begin{bmatrix} N_2 & 0 \\ 0 & N_2 - \overline{K}C \end{bmatrix},\ \mathcal{M} = \begin{bmatrix} N & 0 \\ N & 0 \end{bmatrix}. \quad (8)$$

II. ANALYSIS OF THE MEAN-SQUARE STABILITY

We study the conditions to ensure the mean-square stability of system (7). The gains K and $\overline{K}$ are considered to be known to avoid bilinearity in the derivation of the stability condition. Consider the following Lyapunov function

$$V(\xi) = \xi^T \mathcal{P} \xi. \quad (9)$$

Using the Itô's formula [3] we have

$$\mathrm{d}\,V(\xi(t)) = LV\,\mathrm{d}\,t + 2\xi^T(t)\mathcal{P}\mathcal{M}\xi\,\mathrm{d}\,\rho(t) \quad (10)$$

$$\text{with} \quad LV(\xi(t)) = 2\xi^T(t)\mathcal{P}\mathcal{A}\xi(t) + \xi^T(t)u(\mathcal{N}^T\mathcal{P} + \mathcal{P}\mathcal{N})\xi(t) + \xi^T(t)\mathcal{M}^T\mathcal{P}\mathcal{M}\xi(t). \quad (11)$$

To study the stability of this system, we introduce a change of variable on the control $u(t)$ in order to reduce the conservatism introduced by the assumption that $u(t)$ is bounded (see (3)). We have

$$u(t) = \alpha + \sigma\varepsilon(t) \quad (12)$$

where $\alpha \in \mathbb{R}$ and $\sigma \in \mathbb{R}$ are given by

$$\alpha = 0.5(u_{\min} + u_{\max}),\ \sigma = 0.5(u_{\max} - u_{\min}). \quad (13)$$

The new "uncertain" variable is $\varepsilon(t) \in \overline{\Omega} \subset \mathbb{R}$ where the polytope $\overline{\Omega}$ is defined by

$$\overline{\Omega} := \{\varepsilon(t) \in \mathbb{R} \mid \varepsilon_{\min} = -1 \leqslant \varepsilon(t) \leqslant \varepsilon_{\max} = 1\}. \quad (14)$$

Equation (10) is rewritten as

$$\mathrm{d}\,V(\xi(t)) = \xi^T(t)\{\mathcal{P}\mathcal{A}_t + \mathcal{A}_t^T\mathcal{P} + \mathcal{M}^T\mathcal{P}\mathcal{M} + \mathcal{P}\Delta\mathcal{A}(t) + \Delta\mathcal{A}(t)^T\mathcal{P}\}\xi(t)\,\mathrm{d}\,t + 2\xi^T(t)\mathcal{P}\mathcal{M}\xi\,\mathrm{d}\,\rho(t) \quad (15)$$

where

$$\mathcal{A}_t = (\mathcal{A} + \alpha\mathcal{N}) \ , \ \Delta\mathcal{A}(t) = H_1 \Delta_\xi(\varepsilon(t)) H_2 \quad (16)$$

$$H_1 = \sigma\mathcal{N} \ , \ \Delta_\xi(\varepsilon(t)) = \varepsilon(t) \text{ and } H_2 = I_{2n}. \quad (17)$$

From (14), we have $\|\Delta_\xi(\varepsilon(t))\| \leqslant 1$ and the majoration lemma [4] gives

$$2\xi^T \mathcal{P}\Delta\mathcal{A}(t)\xi \leqslant \xi^T \left[\mu_1 + \mu_1^{-1}\mathcal{P}\sigma\mathcal{N}\mathcal{N}^T\sigma\mathcal{P}\right]\xi. \quad (18)$$

So,

$$\mathrm{d}\, V(\xi(t)) \leqslant \xi^T(t)\{\mathcal{P}\mathcal{A}_t + \mathcal{A}_t^T\mathcal{P} + \mathcal{M}^T\mathcal{P}\mathcal{M} + \mu_1 I_{2n} + \mu_1^{-1}\mathcal{P}\sigma\mathcal{N}\mathcal{N}^T\sigma\mathcal{P}\}\xi(t)\mathrm{d}\, t + 2\xi^T(t)\mathcal{P}\mathcal{M}\xi(t)\mathrm{d}\, \rho(t). \quad (19)$$

Then the following theorem is given to ensure the mean-square stability of the augmented system (7)

Theorem 1: If the following LMI is verified with $\mathcal{P} > 0$ and μ_1 a given positive real,

$$\begin{bmatrix} \mathcal{P}\mathcal{A}_t + \mathcal{A}_t^T\mathcal{P} + \mathcal{M}^T\mathcal{P}\mathcal{M} + \mu_1 H_2^T H_2 & \mathcal{P}H_1 \\ H_1^T\mathcal{P} & -\mu_1 I_{2n} \end{bmatrix} < 0 \quad (20)$$

then the system (7) is mean-square stable.

Proof: Using the Schur lemma, theorem 1 gives

$$\mathcal{P}\mathcal{A}_t + \mathcal{A}_t^T\mathcal{P} + \mathcal{M}^T\mathcal{P}\mathcal{M} + \mu_1 H_2^T H_2 + \mu_1^{-1}\mathcal{P}H_1 H_1^T\mathcal{P} = -\mathcal{K} < 0. \quad (21)$$

Note that $\lambda_{\min}(\mathcal{K}) > 0$ where $\lambda_{\min}$ is the smallest eigenvalue of $\mathcal{K}$. This and (19) yield to

$$\mathrm{d}V(\xi(t)) \leqslant -\lambda_{\min}(\mathcal{K})\|\xi(t)\|^2 \mathrm{d}\, t + 2\xi^T(t)\mathcal{P}\mathcal{M}\xi(t)\mathrm{d}\rho(t). \quad (22)$$

Let $\beta > 0$ be given, using the integration-by-part formula [4], we obtain

$$\mathrm{d}\left[e^{\beta t}V(\xi(t))\right] \leqslant e^{\beta t}([-\beta\lambda_{\max}(\mathcal{P}) - \lambda_{\min}(K)]\|\xi(t)\|^2)\mathrm{d}\, t + 2e^{\beta t}\xi^T(t)\mathcal{P}\mathcal{M}\xi(t)\mathrm{d}\rho(t). \quad (23)$$

As $[-\beta\lambda_{\max}(\mathcal{P}) - \lambda_{\min}(K)]\,\|\xi(t)\|^2 \leqslant 0$ then inequality (23) implies that

$$\mathrm{d}\left[e^{\beta t}V(\xi(t))\right] \leqslant 2e^{\beta t}\xi^T(t)\mathcal{P}\mathcal{M}\xi(t)\,\mathrm{d}\,\rho(t). \quad (24)$$

Integrating both sides from 0 to $t > 0$ and then taking expectation give

$$e^{\beta t}\,\mathbf{E}\left[\xi^T(t)\mathcal{P}\xi(t))\right] - e^{\beta\times 0}\,\mathbf{E}\left[\xi^T(0)\mathcal{P}\xi(0))\right] \leqslant \int_0^T 2e^{\beta t}\xi^T(t)\mathcal{P}\mathcal{M}\xi(t)\,\mathbf{E}\left[\mathrm{d}\,\rho(t)\right]. \quad (25)$$

But due to (2), the right term of inequality (25) is given by

$$\int_0^T 2e^{\beta t}\xi^T(t)\mathcal{P}\mathcal{M}\xi(t)\,\mathbf{E}\left[\mathrm{d}\,\rho(t)\right] = 0. \quad (26)$$

Then (25) can be rewritten as

$$\lambda_{\min}(\mathcal{P})\,\mathbf{E}\,\|\xi(t)\|^2 \leqslant \mathbf{E}\left[\xi^T(t)\mathcal{P}\xi(t)\right] \leqslant ce^{-\beta t} \quad (27)$$

where $c = \mathbf{E}\left[\xi^T(0)\mathcal{P}\xi(0))\right]$ is a given constant.

Finally, from (27) the following inequality

$$\mathbf{E}\,\|\xi(t)\|^2 \leqslant \frac{c}{\lambda_{\min}(\mathcal{P})}e^{-\beta t} \quad (28)$$

ensures that the system (7) is mean square stable. □

III. Synthesis of the Observer Gains

From section II, the following theorem is then given for the observer synthesis.

Theorem 2: The stochastic system (4) is a full order observer for the bilinear stochastic system (1) such that the estimation error (6) is mean square stable if there exist matrices $P_1 > 0$, $P_2 > 0$, $P_3 > 0$, Y_2, $\overline{Y}_2$, Y_3 and $\overline{Y}_3$ such that the following LMI hold

$$\begin{bmatrix} (1,1) & (1,2) & P_1 N_{2\sigma} & P_3 N_{2\sigma} - \sigma\overline{Y}_3 C \\ (1,2)^T & (2,2) & P_3^T N_{2\sigma} & P_2 N_{2\sigma} - \sigma\overline{Y}_2 C \\ N_{2\sigma}^T P_1 & N_{2\sigma}^T P_3 & -\mu_1 I_n & 0 \\ (1,4)^T & (2,4)^T & 0 & -\mu_1 I_n \end{bmatrix} < 0 \quad (29)$$

$$\mathcal{P} = \begin{bmatrix} P_1 & P_3 \\ P_3^T & P_2 \end{bmatrix} > 0 \quad (30)$$

where

$$(1,1) = P_1 A_\alpha + A_\alpha^T P_1 + \mu_1 I_n + N^T P_1 N + N^T P_3^T N + N^T P_3 N + N^T P_2 N, \quad (31a)$$

$$(1,2) = P_3 A_\alpha + A_\alpha^T P_3 - Y_3 C - \overline{Y}_3 C_\alpha, \quad (31b)$$

$$(2,2) = P_2 A_\alpha + A_\alpha^T P_2 - Y_2 C - \overline{Y}_2 C_\alpha - C^T Y_2^T - C_\alpha^T \overline{Y}_2^T + \mu_1 I_n, \quad (31c)$$

$$A_\alpha = A + \alpha N_2 \ , \ C_\alpha = \alpha C \ , \ N_{2\sigma} = \sigma N_2, \quad (31d)$$

$$Y_i = P_i K \text{ and } \overline{Y}_i = P_i \overline{K} \quad i = 2,3. \quad (31e)$$

Proof: Consider the Lyapunov matrix $\mathcal{P}$ given by (30). As in [5], we assume without loss of generality, that P_3 is nonsingular. Using inequality (20) of theorem 1 and putting $\mathcal{P}$ as in (30) lead easily to the LMI (29) of the theorem. □

The gains K and $\overline{K}$ are then obtained by solving the following equation

$$\begin{bmatrix} P_2 \\ P_3 \end{bmatrix} \begin{bmatrix} K & \overline{K} \end{bmatrix} = \begin{bmatrix} Y_2 & \overline{Y}_2 \\ Y_3 & \overline{Y}_3 \end{bmatrix}. \quad (32)$$

Note that K and $\overline{K}$ exist if and only if the following rank condition is satisfied.

$$\operatorname{rank}\begin{bmatrix} P_2 \\ P_3 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} Y_2 & \overline{Y}_2 & P_2 \\ Y_3 & \overline{Y}_3 & P_3 \end{bmatrix}. \quad (33)$$

This condition may be difficult to fulfill. We then present, in the following section, a result which is somewhat conservative but very useful to solve the problem under consideration.

IV. Conclusion

In this paper, a method has been proposed to solve the problem of observer design for bilinear stochastic system.

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