2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

Nonlinear Observer Design with General Criteria

Edwin Engin Yaz, Chung Seop Jeong, Adil Bahakeem, and Yvonne Ilke Yaz

***Abstract*—A class of nonlinear system and measurement equations involving incrementally conic nonlinearities with finite energy disturbances is considered. A linear matrix inequality based observer design approach is presented that guarantees the satisfaction of a variety of performance criteria ranging from simple estimation error boundedness to dissipativity. Simple simulation examples are included to explore the freedom in design and to illustrate and provide support to the proposed design methodology.**

I. INTRODUCTION

The recent research in nonlinear systems theory has resulted in the emergence of many new nonlinear state observation techniques: feedback linearization [1]-[3], variable structure techniques [4]-[8], extended linearization [9], high gain observers [10]-[11], and Lyapunov-based observer design [12]-[17], among others.

In this paper, we introduce a novel design of observers for a class of continuous-time nonlinear systems with incrementally conic nonlinearities and finite energy (L_2) type disturbances. Linear matrix inequalities (LMIs) [18] are used as the main mathematical tool. This result is a natural follow up to the LMI-based robust observer design method presented in [19] and its control counterpart in [20]. Initial simulation examples are presented at the end to explore design freedoms and to provide validation for the theoretical results.

The following notation is utilized in this work: $x \in R^n$ denotes an n-dimensional vector with real elements and with the associated norm $\|x\| = (x^T x)^{1/2}$ where $(\cdot)^T$ represents the transpose. $A \in R^{m\times n}$ denotes an $m \times n$ matrix with real elements. A^{-1} is the inverse of matrix A, $A > 0$ ($A < 0$) means A is a positive (negative) definite matrix, and I_m is an identity matrix of dimension m. L_2 is the space of vector valued signals with finite energy.

Edwin Engin Yaz and Chung Seop Jeong are with Marquette University, Department of Electrical and Computer Engineering, Milwaukee, Wisconsin 53210, USA (telephone: +1-414-288-6820, e-mail:, edwin.yaz@marquette.edu, chung.jeong@marquette.edu).
Adil Bahakeem is with the University of Arkansas, Department of Electrical Engineering, Fayetteville, Arkansas 72701 (E-mail : abahake@uark.edu)
Yvonne Ilke Yaz is with Milwaukee School of Engineering, Math Department, Milwaukee, Wisconsin 53202, USA (E-mail: yaz@msoe.edu).

II. PROBLEM FORMULATION

Consider a state space representation of a non-linear system of the general form:

$$\begin{aligned} \dot{x} &= f(x,u,w) \\ y &= h(x,u,w) \end{aligned} \tag{1}$$

where $x \in R^n$ is the state to be estimated from knowledge of the applied input $u \in R^m$ and the measurement output $y \in R^p$. w is an L_2 disturbance input. The nonlinear functions f and h are assumed to be measurable functions of their arguments.

We assume the following incrementally conic condition on the nonlinearities :

$$\left\| \begin{bmatrix} F(e,w) \\ H(e,w) \end{bmatrix} \right\| \stackrel{\bullet}{=} \left\| \begin{bmatrix} f(x,u,w) - f(\hat{x},u,0) - (Ae+Bw) \\ h(x,u,w) - h(\hat{x},u,0) - (Ce+Dw) \end{bmatrix} \right\| \le \left\| A_f e + B_f w \right\| \tag{2}$$

for some matrices A,B,C,D, A_f, and B_f, where (A,C) is a detectable pair. Note that inequality (2) represents the maximum derivation of the nonlinearities F and H from the linear model

$$\begin{aligned} \dot{x} &= Ax + Bw \\ y &= Cx + Dw \end{aligned}$$

by the matrix parameters A_f and B_f. Note also that the Lipschitz condition is a special case of the condition (2) where $A=0$, $B=0$, $C=0,D=0$, $A_f=\alpha I$, $B_f=0$ where $(A,C)=(0,0)$ is trivially detectable. Later on, it will be shown that for given nonlinear functions f and h , there maybe a multitude of parameters satisfying (2), each of which would result in a different response for the observer.

Let $\hat{x}$, the estimate of the true state, obey the following nonlinear Luenberger observer equation

$$\dot{\hat{x}} = f(\hat{x},u,0) + K(y - h(\hat{x},u,0)) \tag{3}$$

and $e = x - \hat{x}$ denote the estimation error.
Substituting from equation (1) and (3), we find that the error dynamics obey

$$\begin{aligned} \dot{e} &= f(x,u,w) - f(\hat{x},u,0) \pm (Ae + Bw) \\ &\quad - K(h(x,u,w) - h(\hat{x},u,0)) \pm (Ce + Dw)) \\ &= (A - KC)e + (B - KD)w + F - KH \\ &= (A - KC)e + [I \ , -K]\begin{bmatrix} Bw + F \\ Dw + H \end{bmatrix} \end{aligned} \tag{4}$$

Let z denote the performance output where

$$z = C_z e + D_z w \tag{5}$$

and consider the general performance objective

$$\dot{V} + \delta\|z\|^2 + \in \|w\|^2 - \beta z^T w \le 0 \tag{6}$$

for an energy function $V = e^T Pe$ where P>0.
Notice that upon integration, inequality (6) yields

$$\begin{aligned} e(t)^T Pe(t) &\le e_0{}^T Pe_0 \\ &- \int_0^t (\delta\|z(\tau)\|^2 + \in \|w(\tau)\|^2 - \beta z(\tau)^T w(\tau))d\tau \end{aligned} \tag{7}$$

or by using Rayleigh's inequalities,
($\lambda_{\min}(P)\|e\|^2 \le e^T Pe \le \lambda_{\max}(P)\|e\|^2$), we obtain

$$\begin{aligned} \lambda_{\min}(P)\|e(t)\|^2 &\le \lambda_{\max}(P)\|e_0\|^2 \\ &- \int_0^t (\delta\|z(\tau)\|^2 + \in \|w(\tau)\|^2 - \beta z(\tau)^T w(\tau))d\tau \end{aligned} \tag{8}$$

that allows several optimization possibilities in a unified eigenvalue problem [15] framework. We can design different observers for a variety of performance criteria for this system.

First of all, in the absence of noise $w(t) \equiv 0,\ t \ge 0$ if we take $\delta = 0$, $\beta = 0$, and $\in$=0, (8) yields

$$\|e(t)\|^2 \le \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\|e(0)\|^2$$

This means that by minimizing $\lambda_{\max}(P)$ and maximizing $\lambda_{\min}(P)$, we can lower the bound on the norm of the estimation error, which will guarantee a faster response for the observer. Note that this implies boundedness of the estimation error (stability).

By taking $\delta > 0$, $\beta = 0$, and $\in = 0$, (7) will yield a bound on the energy of the performance output in terms of the initial estimation error $e(0)$

$$\int_0^t \|z(\tau)\|^2 d\tau \le \frac{1}{\delta}\lambda_{\max}(P)\|e(0)\|^2$$

Minimizing $\lambda_{\max}(P)$ and maximizing δ will give us a smaller bound on the energy of the performance output. This is sub-optimal H_2 observer [15].

In the noisy case, by setting $\delta = 1$, $\beta = 0$, and, $\in < 0$ for e_0 = 0, gives the result

$$\int_0^t \|z(\tau)\|^2 d\tau \le - \in \int_0^t \|w(\tau)\|^2 d\tau$$

which means a bound on the L_2 to L_2 gain of the estimator.
When e_0 =0, if we use this formulation, we can design several dissipative controllers by using different values of δ, β, and $\in$.

If we take $\delta = 0$, $\beta = 1$, and $\in > 0$, it yields the input strict passivity result:

$$\int_0^t z(\tau)^T w(\tau)d\tau \ge \in \int_0^t \|w(\tau)\|^2 d\tau$$

Similarly, other dissipativity results can be obtained by changing δ, β, and $\in$ values. For example, taking e_0 =0, $\delta = 0$, $\beta = 1$, and $\in = 0$ gives passivity

$$\int_0^t z(\tau)^T w(\tau)d\tau \ge 0$$

If we set $\delta > 0$, $\beta = 1$, and $\in = 0$, we get output strict passivity:

$$\int_0^t z(\tau)^T w(\tau)d\tau \ge \delta \int_0^t \|z(\tau)\|^2 d\tau$$

Very strict passivity, which is the strict passivity both in the terms of the input and the output, can be obtained if we set $\delta > 0$, $\beta = 1$, and $\in > 0$:

$$\int_0^t z(\tau)^T w(\tau)d\tau \ge \in \int_0^t \|w(\tau)\|^2 d\tau + \delta \int_0^t \|z(\tau)\|^2 d\tau \ 2$$

As described above, this LMI formulation enables us to design various observers according to different performance criteria in a common framework.

III. LMI SOLUTION

Substituting for the terms in inequality (6), we obtain

$$\begin{aligned} &2e^T P\left[(A - KC)e + [I \ , -K]\begin{bmatrix} Bw + F \\ Dw + H \end{bmatrix}\right] \\ &+ \delta(C_z e + D_z w)^T (C_z e + D_z w) + \in w^T w \\ &- \beta(C_z e + D_z w)^T w \le 0 \end{aligned} \tag{9}$$

The following is true for any α>0

$$\begin{aligned} &e^T P[I \ , K]\begin{bmatrix} F \\ H \end{bmatrix} + [F^T \ , H^T]\begin{bmatrix} I \\ -K^T \end{bmatrix} Pe \\ &\le \alpha e^T P[I \ , K]\begin{bmatrix} I \\ -K^T \end{bmatrix} Pe + \alpha^{-1}[F^T \ , H^T]\begin{bmatrix} F \\ H \end{bmatrix} \\ &\le \alpha e^T P(I + KK^T)Pe + \alpha^{-1}(A_f e + B_f w)^T (A_f e + B_f w) \end{aligned} \tag{10}$$

where we have used (2).

Case I. There is no noise, $w(t) \equiv 0,\ t \geq 0$. This is the case with $B = 0, D = 0, B_f = 0, D_z = 0, \in = 0$, and $\beta=0$. Using (10) in (9) yields

$$e^T(-P(A-KC)-(A-KC)^T P - \alpha P(I+KK^T)P - \delta C_Z{}^T C_Z - \alpha^{-1} A_f{}^T A_f)e \geq 0 \quad (11)$$

Using the Schur complement [15] for the quadratic terms in (11), we obtain

$$Q = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ * & q_{22} & q_{23} \\ * & * & q_{33} \end{bmatrix} \geq 0 \quad (12)$$

where

$$q_{11} = -PA - A^T P + YC + C^T Y^T - \delta C_z{}^T C_z - \gamma A_f{}^T A_f$$

$$q_{12} = P,\ q_{13} = Y,\ q_{22} = q_{33} = \gamma I,\ q_{23} = 0$$

for any $\gamma = \alpha^{-1}$, and $Y=PK$. The LMI (12) needs to be solved for $P>0$, and $\gamma>0$ in the non-noisy case and K is found as $K=P^{-1}Y$.

Case II. Noise is present, $w(t) \neq 0,\ t \geq 0$. Substituting (10) into (9) yields

$$\begin{bmatrix} e^T & w^T \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ * & r_{22} \end{bmatrix} \begin{bmatrix} e \\ w \end{bmatrix} \geq 0 \quad (13)$$

for $\alpha > 0$, where

$$r_{11} = -P(A-KC)-(A-KC)^T P - \alpha P(I+KK^T)P - \delta C_z{}^T C_z - \alpha^{-1} A_f{}^T A_f$$

$$r_{12} = -P(B-KD) - \delta C_z{}^T D_z + \frac{\beta}{2} C_z{}^T - \alpha^{-1} A_f{}^T B_f$$

$$r_{22} = -\delta D_z{}^T D_z - \in I + \frac{\beta}{2}(D_z + D_z{}^T) - \alpha^{-1} B_f{}^T B_f$$

By using the Schur complement [15] twice, we obtain

$$S = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ * & s_{22} & s_{23} & s_{24} \\ * & * & s_{32} & s_{34} \\ * & * & * & s_{44} \end{bmatrix} \geq 0 \quad (14)$$

for

$$s_{11} = -PA + YC - A^T P + C^T Y^T - \delta C_z{}^T C_z - \gamma A_f{}^T A_f$$

$$s_{12} = -PB + YD - \delta C_z{}^T D_z + \frac{\beta}{2} C_z{}^T - \gamma A_f{}^T B_f$$

$$s_{13} = P$$

$$s_{14} = Y$$

$$s_{22} = -\delta D_z{}^T D_z$$

$$s_{22} = -\delta D_z{}^T D_z - \in I + \frac{\beta}{2}(D_z + D_z{}^T) - \gamma B_f{}^T B_f$$

$$s_{23} = s_{24} = s_{34} = 0$$

$$s_{33} = s_{44} = \gamma I$$

where $Y=PK$ and α^{-1} is replaced with γ. The LMI (14) needs to be solved for $P>0$, Y, and $\gamma>0$ in the noisy case and K is found from $K=P^{-1}Y$.

The above development is summarized in the following theorem:

Theorem **:** Given the nonlinear system and measurement scheme in (1) and (2) where $w \in L_2$, the use of the observer (3) leads to the satisfaction of general performance objective (6) for z given by (5) if LMIs (12) and (14) are feasible, respectively for the non-noisy ($w(t) \equiv 0,\ t \geq 0$) and noisy cases, for $P>0$, Y, and $\gamma>0$. The necessary gain is found from $K=P^{-1}Y$.

Remark: Note that the first block inequality in LMIs (12) and (14) automatically imply detectability.

IV. SIMULATION STUDIES

The following section contains some initial simulation results on the observer designs proposed in this work.

Example 1. This example is provided to illustrate the freedom in the choice of the parameters in (2) for a given nonlinearity and its effort on the response of the observer. Consider the system and the measurement equation

$$\dot{x} = f(x)$$

$$y = x + w$$

where

$$f(x) = \begin{cases} -1, & x < -2 \\ 0.5x, & -2 \leq x \leq 2 \\ 1, & x > 2 \end{cases}$$

The procedure followed was by substituting different values of A in (2), the minimum A_f value was solved for from (2). LMI (12) is solved for the estimation error boundedness case. The gain K values corresponding to A=0, A=0.35, and A=0.7 were found to be respectively 1.2414, 1.8328, and 5.9052. The norm of the error is plotted for these three cases in Fig.1. It seems like there is a direct relationship between the magnitude of A and the speed of the error response.

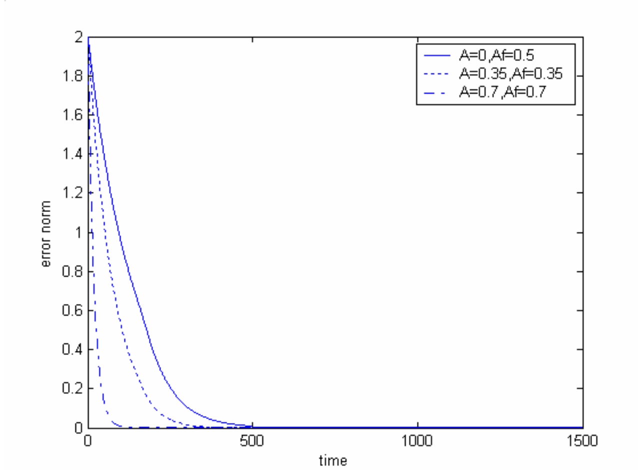

Fig. 1. Plot of $\|e(t)\|$ Boundedness of the Estimation Error - first order system with different As and A_fs.

Next, the sub-optimal H_2 observer problem for the same model is solved.

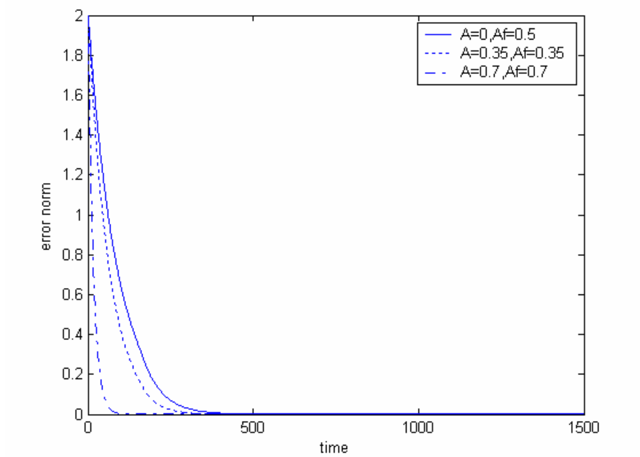

Fig. 2. Plots of $\|e(t)\|$ for the Sub-optimal H_2 observer-first order system with different As and A_fs.

Fig.2 shows the same kind of monotonic relationship that existed in the error boundedness case.

Example 2. This study is conducted to compare the response for three different criteria used in the observer design. The following system and measurement model is considered.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -\sin x_1 \end{bmatrix}, \; y = \begin{bmatrix} 1 & 0 \end{bmatrix} x + w$$

for both non additive noise case, and additive noise case. The sub-optimal H_2 observer ($w(t) \equiv 0, \; t \geq 0$), input strict passivity, and output strict passivity were chosen to show the verification of the proposed design methodology. Table 1. shows the design parameters used in simulation studies.

Table 1. Design Parameters

	Sub-optimal H_2 observer	Input Strict Passivity	Output Strict Passivity
C_z	[1 1]	[1 1]	[1 1]
Dz	0	1	1
A_f	0.1	0.1	0.1
B_f	0	0	0
δ	1	0	0.1
β	0	1	1
∈	0	0.1	0
$w(t)$	0	$0.1e^{(-t)}$	$0.1e^{(-t)}$
$\|e_0\|$	2.091	0	0

For the simulation of the sub-optimal H_2 observer, the observer gain is found to be $K = \begin{bmatrix} 2.3999 \\ 1.8345 \end{bmatrix}$. For the case of the input strict passivity, K is found to be $K = \begin{bmatrix} 4.6661 \\ 3.7618 \end{bmatrix}$, and the gain K for the output strict passivity case is found to be $K = \begin{bmatrix} 2.7169 \\ 2.0397 \end{bmatrix}$.

The estimation error norm plots for each case are given in Fig.s 3-5. Notice the change in the response magnitude and shape for different criteria.

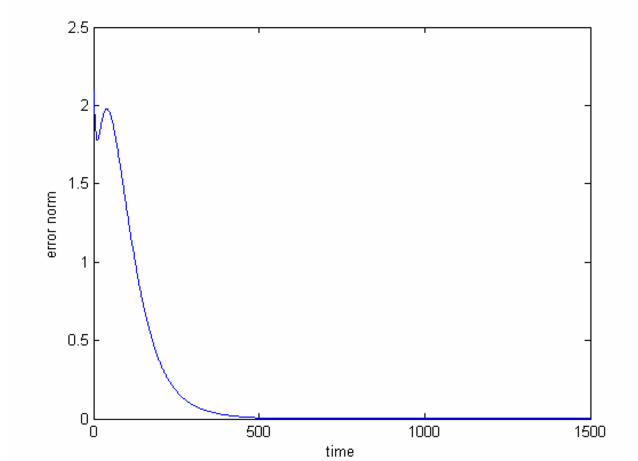

Fig. 3. Plot of $\|e(t)\|$ for the Sub-optimal H_2 observer

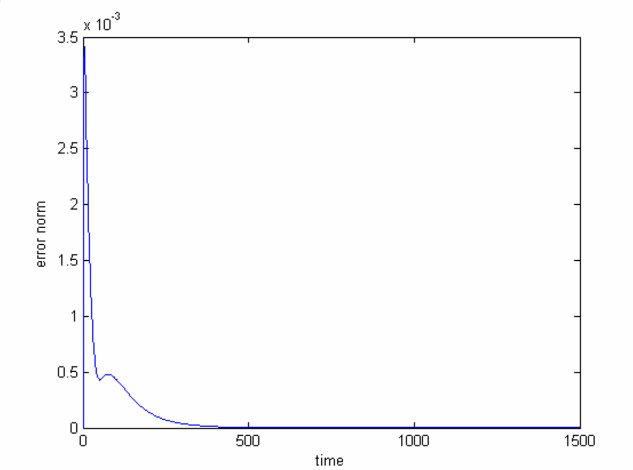

Fig. 4. Plot of $\|e(t)\|$ for the Input strict passivity

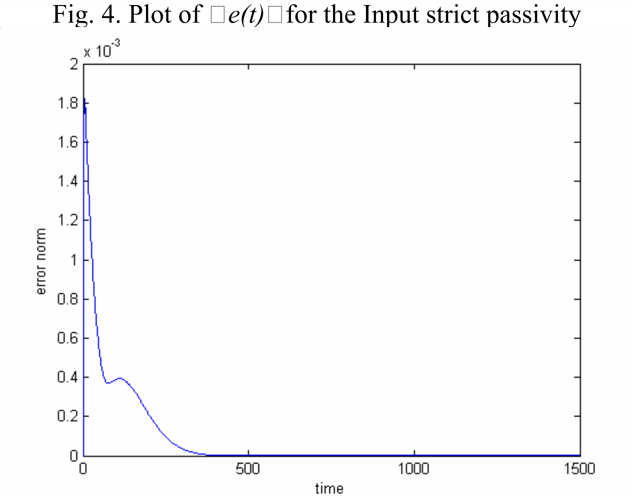

Fig. 5. Plot of $\|e(t)\|$ for the Output strict passivity

V. CONCLUSIONS

An observer design procedure based on linear matrix inequalities has been presented for a class of nonlinear system models. A common framework is provided to design observers according to a variety of performance criteria. The freedom inherent in this design is discussed and initial simulation examples have pointed out to the effectiveness of the proposed technique.

VI. REFERENCES

[1] D. Bestle and M. Zeitz, "Canonical form observer design for nonlinear time-variable systems," *Int. J. Control,* Vol. 38, pp. 419-431, 1983.

[2] A. Isidori, *Nonlinear Control Systems: An Introduction,* Springer Verlag: Berlin, 1985.

[3] A. J. Krener and W. Respondek, "Nonlinear observer with linearizable error dynamics," *SIAM J. Control & Optim*, vol. 23, pp. 197-216, 1985.

[4] B. L. Walcott and S. H. Zak, "State observation of nonlinear uncertain dynamical systems," *IEEE Trans. Autom. Contr.,* vol. 32, pp. 166-170, 1987.

[5] B. L. Walcott, M. J. Corless, and S.H. Zak "Comparative study of non-linear state-observation technique," *Int. J. Control*, vol. 45, pp. 2109-2132, 1987.

[6] E. Yaz and A. Azemi, "Variable structure observer with a boundary-layer for correlated noise/disturbance models and disturbance minimization," *Int. J. Control,* vol. 57, No. 5, pp. 1191-1206, 1993.

[7] E. Yaz and A. Azemi, "Sliding mode observer for nonlinear models with unbounded noise and measurement uncertainties," *Dynamics and Control,* vol. 3, pp. 217-235, 1993.

[8] A. Azemi and E. Yaz, " Sliding-mode adaptive observer approach for chaotic synchronization," *ASME J. Dyn. Sys., Meas., & Contr.,* vol. 122, 758-765, 2000.

[9] W. Baumann and W. Rugh, "Feedback control of non-linear systems by extended linearization," *IEEE Trans. Autom. Contr.,* vol. 31, pp. 40-47, 1986.

[10] G. Bornard and H. Hammouri, "A high gain observer for a class of uniformly observable systems," *Proc. of 30th IEEE CDC*. pp. 1494-1496, 1991.

[11] J. P. Gauthier, H. Hammouri , and S. Othman, " A simple observer for nonlinear system applications to bioreactors", *IEEE Trans. on Automatic Control,* vol. 37, no. 6, pp. 875-880, 1992

[12] F.E. Thau, "Observing the state of non-linear dynamic systems," *Int. J.Control,* vol. 18, pp. 471-479, 1973.

[13] S. R. Kou, D.L. Elliot and T.J. Tarn, "Exponential observer for nonlinear dynamic systems," *Inform. Control*, vol. 29, pp. 504-509, 1980.

[14] M. Vidyasagar, "On the stabilization of nonlinear systems using state detection," *IEEE Trans. Autom. Control*, vol. 25, pp. 504-509, 1980.

[15] E. Yaz and A. Azemi, "Observer design for discrete and continuous non-linear stochastic systems," *Int. J. System Sci*., vol. 24, No. 12, pp. 2289-2302, 1993.

[16] S. Raghavan and J. K. Hedrick, "Observer design for a class of nonlinear systems", *Int. J. of Control*, vol. 59, no. 2, pp/ 515-528, 1994.

[17] R. Rajamani, "Observers for Lipschitz nonlinear systems", *IEEE Trans. on Automatic Control,* vol. 43, no. 3, pp. 397-401, 1998.

[18] S. Boyd, L. E. Ghaoui, E. Feron, V. Balakrishnan, Linear Matrix Inequalities In System and Control Theory, *SIAM Studies in Applied Mathematics. SIAM, Philadelphia,* 1994.

[19] E. Yaz and Y. I. Yaz," LMI-based observer design for nonlinear systems with integral quadratic constraints," *Proc. of 2001 IEEE CDC,* Orlando, FL, pp. 2954-2955, 2001.

[20] C. S. Jeong, E. Yaz, and Y. I. Yaz, "State feedback control design with dissipative performance criteria using LMI," *Proc. of 2002 Allerton Conf. on Communication and Control and Computing,* Monticello, IL, pp. 305-314, 2002.